An alternative approach of initial stability analysis of Kirchhoff plates resting on internal supports by the Boundary Element Method

Michał GUMINIAK*

Poznan University of Technology, ul. Piotrowo 5, 60-965 Poznan, Poland

E-mail: michal.guminiak@put.poznan.pl

tel.: +48 61 665 24 75, fax: +48 61 665 20 59

Abstract. An initial stability of Kirchhoff plates supported on boundary and resting on the internal supports is analysed in the paper. The internal support is understood to be a column and linear continuous constraints. Proposed approach avoids Kirchhoff forces at the plate corner and equivalent shear forces at a plate boundary. Two unknown variables are considered at the boundary element node. The governing integral equations are derived using Betti theorem. The integral equations have the form of boundary-domain integral equations. The constant type of boundary element are used. The singular and non-singular formulation of the boundary-domain integral equations with one and two collocation points associated with a single boundary element located slightly outside of a plate edge are presented. To establish a plate curvature by double differentiation of basic boundary-domain integral equation, a plate domain is divided into rectangular sub-domains associated with suitable collocation points. According to the another approach a plate curvature is also established by considering three collocation points located in close proximity to each other along line pararel to one of the two axes of global coordinate system and establishment of appropriate difference operators.

Key words. The Boundary Element Method, Kirchhoff plates, initial stability, fundamental solution

1. INTRODUCTION

The Boundary Element Method (BEM) can be applied in wide aspect of engineering analysis of the structures. Burczyński [1] described in a comprehensive manner the Boundary Element Method and its application in a variety of fields, the theory of elasticity together with the appropriate solutions and a discussion of the basic types of boundary elements. The main advantage of BEM is its relative simplicity of formulating and solving problems of the theory of potential and elasticity. The application of the Boundary Element Method to a plate analysis has the particular advantages. Many authors used the BEM to solve static, dynamic and initial stability problems of thin plates. There are well known publications of Altiero and Sikarskie [2], Bèzine and Gamby [3], Stern [4] and Hartmann and Zotemantel [5] who applied BEM to solve thin plate bending problem. Abdel-Akher and Hartley [6] presented evaluation of boundary and boundary-domain integralf of fundamental functions used in plate analysis. A number of contributions devoted to the analysis of plates were presented by: Debbih [7, 8], Beskos [9], Wen, Aliabadi and Young [10], Katsikadelis [11, 12], Katsikadelis and Yotis [13], Katsikadelis, Sapountzakis and Zorba [14], Katsikadelis and Kandilas [15], Katsikadelis and Sapountzakis [16]. Wrobel and Aliabadi [17] described an application of BEM in the thick plate analysis together with procedures for calculating singular and hypersingular integrals in wide aspect. Very interesting approach was presented by Litewka and Sygulski [18, 19] who applied the Ganowicz [20] fundamental solutions for Reissner plates to static analysis of plates. Noteworthy is publication of Shi [21] who applied BEM formulation for vibration and initial stability problem of orthotropic thin plates. Ptaszny [22] applied Fast multipole boundary element method for the analysis of plates with many holes. Rashed [23] applied the coupled BEM–flexibility force method to static analysis of thin plates resting on internal column supports. The maior drawback of this approach is the necessary condition of boundary supports which satisfies kinematic constraints. In order to simplify the calculation procedures Guminiak and Sygulski [24] proposed a modified formulation of the boundary integral equation for a thin plate. This approach was applied for static, dynamic and stability analysis of thin plates and it is presented together with several numerical examples in papers, e.g. [25–30]. Guminiak [31] applied difference equation model of establishment of curvatures connected to the aforementioned modified BEM approach to solve initial stability problem of thin plates providing also the wide review of literature devoted to the BEM application in plate analysis. Myślecki [32, 33] proposed BEM to static analysis of plane girders and BEM combined with approximate fundamental solutions for problem of plate bending resting on elastic foundation. Author used non–singular approach of boundary integral equations wherein the derivation of the second boundary integral equation was executed for additional collocation points located outside of a plate domain. The same approach of derivation of boundary integral equation was applied by Myślecki and Oleńkiewicz [34, 35] to solve free vibration problem of thin plates. Particularly notheworthy is work of Katsikadelis [36, 37] in which author applied BEM in a wide aspects of engineering analysis of plates. In this work the conception of the Analog Equation Method (AEM) is widely presented as the tool that allows fully overcomes the main drawback of direct BEM which is limitation only to linear problems. The AEM is based on the principle of the analog equation of Katsikadelis for differential equations [38]. This conception was established to analysis of plate buckling by Nerantzaki and Katsikadelis [39] and Chinnaboon, Chucheepsakul and Katsikadelis [40]. Similarly Babouskos and Katsikadelis [41, 42] solved problem of flutter instability of dumped plate subjected by conservative and non-conservative loading.

In present paper, an analysis of initial stability of internally supported thin plates by the direct BEM will be presented. The analysis will focus on the modified, simplified [31] formulation of thin plate bending. The Bèzine [3] technique will be established to directly derive boundary-domain integral equation.

2. MODELLING OF INTERNAL SUPPORTS

Internal constrains can have the character of support at selected points, column or continuous linear supports. Using direct Bèzine technique it is necessary to expand two boundary-domain integral equations [3] to include additional elements where the unknown values are the suitable reaction as: concentrated forces (Fig. 1), forces distributed over the column cross-sections and distributed along the continuous linear constraints. The internal column support can be modelled as the surface with one collocation point with constant distribution of the reaction (Fig. 2a, 2b, 2c). If the column support has a large dimensions in reference to plate dimensions, several sub-surface can be introduced on the column surface (Fig. 2d). To calculate elements of the characteristic matrix, it is necessary to integrate suitable fundamental functions on the column surface or sub-surfaces. I case of the column of arbitrary shape (Fig. 2b, 2c) the formulas derived by Abdel-Akher and Hartley [6] can be used. The internal linear continuous supports can be modelled as the set of the sections (elements) of the constant type (Fig. 3). Because the fundamental solution for thin plate has a singularity of the second order, the collocation point of internal single element can be located at the centre of them. Using another approach, the internal continuous support can be treated as a column rectangular support with one edge dimension much smaller than the second, perpendicular (Fig. 4).

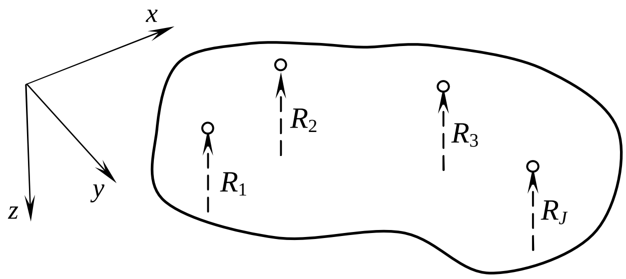

Fig. 1. A plate internally supported at the selected points.

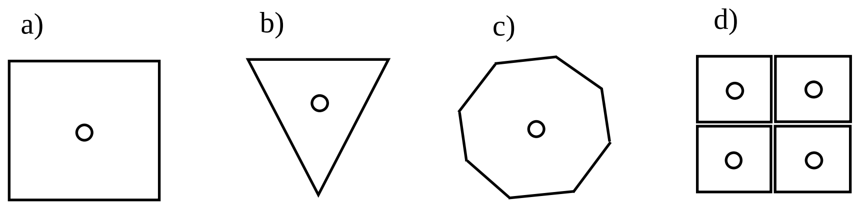

Fig. 2. Definitions of the column support.

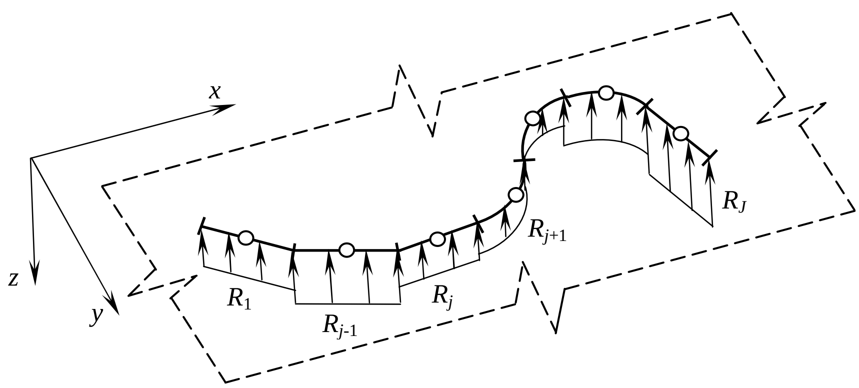

Fig. 3. A plate resting on linear continuous internal support.

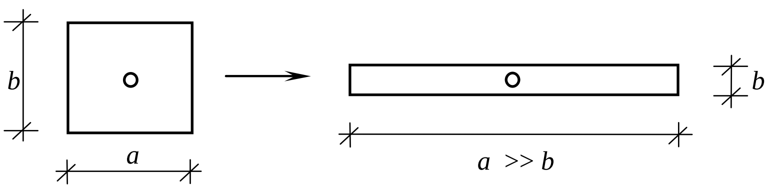

Fig. 4. Internal continuous supports: single element of the constant type.

3. INTEGRAL FORMULATION OF PLATE BENDING AND INITIAL STABILITY PROBLEM CONSIDERING INTERNAL SUPPORTS

The differential equation governing of plate initial stability has the form [43, 44]:

$$D \cdot \nabla^4 w = -\overline{p} \tag{3.1}$$

where $\overline{p}$ is the substitute load, which has the form:

$$\overline{p} = N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2} \tag{3.2}$$

In the majority of contributions devoted to the application of BEM to the thin (Kirchhoff) plate theory, the derivation of the boundary integral equation involves the known boundary variables of the classic plate theory, i.e. the shear force and the concentrated corner forces. Thus, on the plate boundary there are considered the two physical quantities: the equivalent shear force V_n, reaction at the plate k^{th} corner R_k, the bending moment M_n, the corner concentrated forces and two geometric variables: the displacement w_b and the angle of rotation in the normal direction φ_n. The boundary integral equation can be derived using the Betti's theorem. Two plates are considered: an infinite plate, subjected to the unit concentrated force and a real one, subjected to the real in plane loadings N_x, N_{xy} and N_y. The plate bending problem is described in a unique way by two boundary-domain integral equations. The first equation has the form:

$$\begin{aligned}
&c(\mathbf{x}) \cdot w(\mathbf{x}) + \int_\Gamma \left[V_n^*(\mathbf{y},\mathbf{x}) \cdot w_b(\mathbf{y}) - M_n^*(\mathbf{y},\mathbf{x}) \cdot \varphi_n(\mathbf{y}) \right] \cdot d\Gamma(\mathbf{y}) - \sum_{k=1}^{K} R^*(k,\mathbf{x}) \cdot w(k) = \\
&= \int_\Gamma \left[V_n(\mathbf{y}) \cdot w^*(\mathbf{y},\mathbf{x}) - M_n(\mathbf{y},\mathbf{x}) \cdot \varphi_n^*(\mathbf{y},\mathbf{x}) \right] \cdot d\Gamma(\mathbf{y}) - \sum_{k=1}^{K} R_k \cdot w^*(k,\mathbf{x}) + \\
&- \int_{\Omega_r} q_r \cdot w^*(r,\mathbf{x}) \cdot d\Omega_r - \int_{\Gamma_l} q_l \cdot w^*(l,\mathbf{x}) \cdot d\Gamma_l + \\
&+ \int_\Omega \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2} \right) \cdot w^*(\mathbf{y},\mathbf{x}) \cdot d\Omega(\mathbf{y})
\end{aligned} \tag{3.3}$$

where the fundamental solution of this biharmonic equation

$$\nabla^4 w^*(\mathbf{y},\mathbf{x}) = \frac{1}{D} \cdot \delta(\mathbf{y},\mathbf{x}) \tag{3.4}$$

which is the free space Green function given as

$$w^*(\mathbf{y},\mathbf{x}) = \frac{1}{8\pi D} \cdot r^2 \cdot \ln(r) \tag{3.5}$$

for a thin isotropic plate, $r = |\mathbf{y} - \mathbf{x}|$, δ is the Dirac delta, $D = \frac{E\,h^3}{12\,(1-\nu^2)}$ is the plate stiffness, **x** is the source point and **y** is a field point. The coefficient $c(\mathbf{x})$ is taken as:

$c(\mathbf{x}) = 1$, when **x** is located inside the plate domain,

$c(\mathbf{x}) = 0.5$, when **x** is located on the smooth boundary,

$c(\mathbf{x}) = 0$, when **x** is located outside the plate domain.

The second boundary-domain integral equation can be obtained replacing the unit concentrated force $P^* = 1$ by the unit concentrated moment $M_n^* = 1$, which is equivalent to the differentiation of the first boundary integral equation (3.3) with respect to the co-ordinate *n* at a point **x** belonging to the plate domain and letting this point approach the boundary and taking *n* coincide with the normal to it

$$\begin{aligned}
&c(\mathbf{x}) \cdot \varphi_n(\mathbf{x}) + \int_\Gamma \left[\overline{V}_n^*(\mathbf{y},\mathbf{x}) \cdot w_b(\mathbf{y}) - \overline{M}_n^*(\mathbf{y},\mathbf{x}) \cdot \varphi_n(\mathbf{y})\right] \cdot d\Gamma(\mathbf{y}) - \sum_{k=1}^{K} \overline{R}^*(k,\mathbf{x}) \cdot w(k) = \\
&= \int_\Gamma \left[V_n(\mathbf{y}) \cdot \overline{w}^*(\mathbf{y},\mathbf{x}) - M_n(\mathbf{y}) \cdot \overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right] \cdot d\Gamma(\mathbf{y}) - \sum_{k=1}^{K} R_k \cdot \overline{w}^*(k,\mathbf{x}) + \\
&- \int_{\Omega_r} q_r \cdot \overline{w}^*(r,\mathbf{x}) \cdot d\Omega_r - \int_{\Gamma_l} q_l \cdot \overline{w}^*(l,\mathbf{x}) \cdot d\Gamma_l + \\
&+ \int_\Omega \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2}\right) \cdot \overline{w}^*(\mathbf{y},\mathbf{x}) \cdot d\Omega(\mathbf{y})
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
&\left\{\overline{V}_n^*(\mathbf{y},\mathbf{x}), \overline{M}_n^*(\mathbf{y},\mathbf{x}), \overline{R}^*(\mathbf{y},\mathbf{x}), \overline{w}^*(\mathbf{y},\mathbf{x}), \overline{w}^*(r,\mathbf{x}), \overline{w}^*(l,\mathbf{x}), \overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right\} = \\
&= \frac{\partial}{\partial n(\mathbf{x})} \left\{V_n^*(\mathbf{y},\mathbf{x}), M_n^*(\mathbf{y},\mathbf{x}), R^*(\mathbf{y},\mathbf{x}), w^*(\mathbf{y},\mathbf{x}), w^*(r,\mathbf{x}), w^*(l,\mathbf{x}), \varphi_n^*(\mathbf{y},\mathbf{x})\right\}.
\end{aligned}$$

The second boundary-domain integral equation can be also derived by direct application of the boundary domain integral equation (3.3) for noew set of the collocation points located in the same normal line outside the plate edge. This double collocation point approach was presented in publication [32–35]. The detailed procedure for the derivation of the fundamental solution, the integral representation of the solution and the two boundary-domain integral equations is presented by Katsikadelis in [36, 37].

The plate bending problem can also be formulated in a modified, simplified way using an integral representation of the plate biharmonic equation. Because the concentrated force at the corner is used only to satisfy the differential biharmonic equation of the thin plate, one can assume, that it could be distributed along a plate edge segment close to the corner [31]. As a result, the boundary integral equations (3.3) and (3.6) will take the form:

$$\begin{aligned}
&c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}\left[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{dw(\mathbf{y})}{ds}-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot d\Gamma(\mathbf{y})=\\
&=\int_{\Gamma}\left[\tilde{T}_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})+\\
&-\int_{\Omega_r}q_r\cdot w^*(r,\mathbf{x})\cdot d\Omega_r-\int_{\Gamma_l}q_l\cdot w^*(l,\mathbf{x})\cdot d\Gamma_l+\\
&+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y})
\end{aligned}\tag{3.7}$$

$$\begin{aligned}
&c(\mathbf{x})\cdot \varphi_n(\mathbf{x})+\int_{\Gamma}\left[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{dw(\mathbf{y})}{ds}-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot d\Gamma(\mathbf{y})=\\
&=\int_{\Gamma}\left[\tilde{T}_n(\mathbf{y})\cdot \overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})+\\
&-\int_{\Omega_r}q_r\cdot \overline{w}^*(r,\mathbf{x})\cdot d\Omega_r-\int_{\Gamma_l}q_l\cdot \overline{w}^*(l,\mathbf{x})\cdot d\Gamma_l+\\
&+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot \overline{w}^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y})
\end{aligned}\tag{3.8}$$

where

$$\begin{aligned}
&\left\{\overline{T}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_{ns}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(r,\mathbf{x}),\overline{w}^*(l,\mathbf{x}),\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right\}=\\
&=\frac{\partial}{\partial n(\mathbf{x})}\left\{T_n^*(\mathbf{y},\mathbf{x}),M_n^*(\mathbf{y},\mathbf{x}),M_{ns}^*(\mathbf{y},\mathbf{x}),w^*(\mathbf{y},\mathbf{x}),w^*(r,\mathbf{x}),w^*(l,\mathbf{x}),\varphi_n^*(\mathbf{y},\mathbf{x})\right\}
\end{aligned}$$

and

$$\tilde{T}_n(\mathbf{y})=T_n(\mathbf{y})+R_n(\mathbf{y}).\tag{3.9}$$

The expression (3.9) denotes shear force for clamped and for simply-supported edges [31]:

$$\tilde{T}_n(\mathbf{y}) = \begin{cases} V_n(\mathbf{y}) & \text{on the boundary far from the corner,} \\ R_n(\mathbf{y}) & \text{on a small fragment of the boundary close to the corner.} \end{cases}$$

In the case of the free edge we must combine the angle of rotation in the tangent direction $\varphi_s(\mathbf{y})$ with the fundamental function $M^*_{ns}(\mathbf{y})$. Because the relation between $\varphi_s(\mathbf{y})$ and the deflection is known: $\varphi_s(\mathbf{y}) = \frac{dw(\mathbf{y})}{ds}$, the angle of rotation $\varphi_s(\mathbf{y})$ can be evaluated using a finite difference scheme of the deflection with two or more adjacent nodal values [31]. In this analysis, the employed finite difference scheme includes the deflections of three adjacent nodes.

4. CONSTRUCTION OF SET OF ALGEBRAIC EQUATION

The plate boundary is discretized by elements of the constant type. Three approaches of constructing the boundary integral equations, applied also in [31], are considered: the first, singular, where the collocation point is located exactly at a plate edge (Fig. 5),

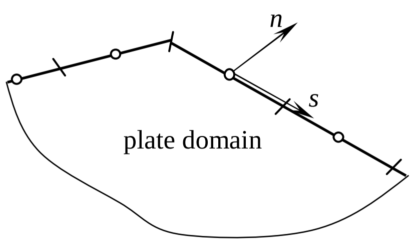

Fig. 5. Collocation point assigned to the boundary element of the constant type.

the second non-singular approach, the boundary-domain integral equations can be formulated using one set of collocation points (Fig. 6a) and third one, where two sets of collocation points (Fig. 6b) located outside of the plate boundary on the line normal to the plate edge are considered.

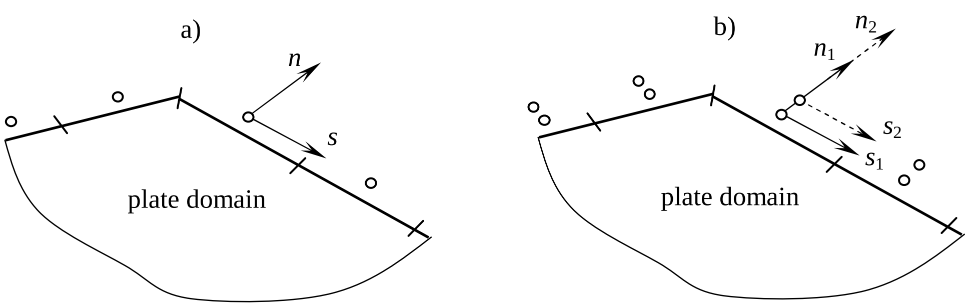

Fig. 6. One collocation point a) and two collocation points b) assigned to the boundary element of the constant type.

It is assumed that a rectangular plate is compressed only by N_x forces. Then, in the boundary integral equations (3.8) and (3.9) takes a stand only the part $N_x \cdot \left(\partial^2 w / \partial x^2\right)$. The unknown variable in internal collocation points is the parameter $\kappa = \partial^2 w / \partial x^2$, the plate curvature in x direction [24, 31]. It is also assumed, that a plate has a regular shape without any holes. The distribution of the normal in plane loading along plate edge perpendicular to the x direction has the constant value. The plate domain Ω is divided into finite number of sub-domains just to define a plate curvature in selected internal collocation points associated with these sub-domains Ω_m. The plate domain Ω is divided into finite number of sub-domains just to define a plate curvature in selected internal collocation points associated with these sub-domains Ω_m. The normal loading $N_x = N_{cr}$ is constant on the length of the single plate edge (Fig. 7).

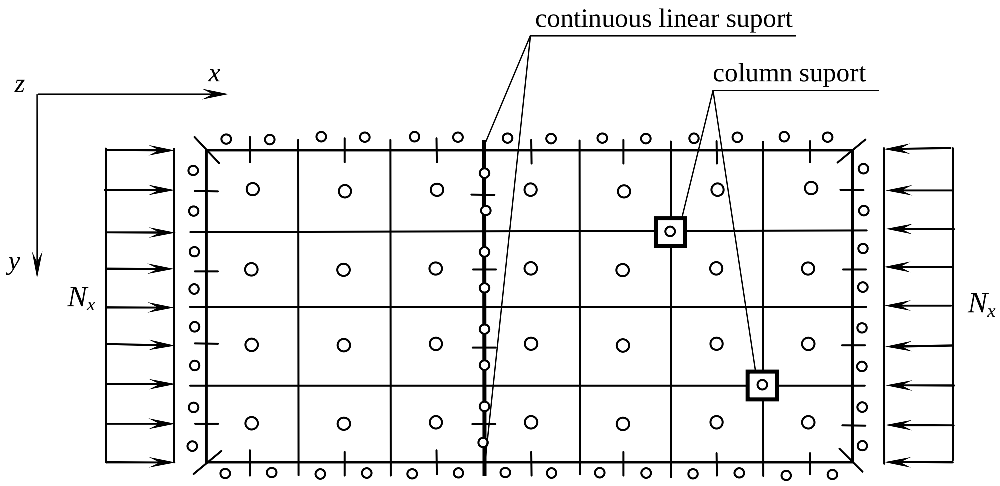

Fig. 7. Distribution of in plane loading.

The set of algebraic equation can be written in the form:

$$\begin{bmatrix} \mathbf{G_{BB}} & \mathbf{G_{BS}} & \mathbf{G_{Bq}} & -\lambda \cdot \mathbf{G_{B\kappa}} \\ \mathbf{\Delta} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G_{qB}} & \mathbf{G_{qS}} & \mathbf{G_{qq}} & -\lambda \cdot \mathbf{G_{q\kappa}} \\ \mathbf{G_{\kappa B}} & \mathbf{G_{\kappa S}} & \mathbf{G_{\kappa q}} & -\lambda \cdot \mathbf{G_{\kappa\kappa}} + \mathbf{I} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \varphi_S \\ \mathbf{q} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{4.1}$$

where $\lambda = N_{cr}$ and

$\mathbf{G_{BB}}$ and $\mathbf{G_{BS}}$ are the matrices of the dimensions of the dimension $(2N \times 2N)$ and of the dimension $(2N \times S)$ grouping boundary integrals and depend on type of boundary, where *N* is the number of boundary nodes (or the number of the elements of the constant type) and *S* is the number of boundary elements along free edge;

$\mathbf{G_{Bq}}$ is the matrix of the dimension $(2N \times L)$ grouping integrals over the internal support (column or linear) sub-domains, where: $\Omega_L = \Omega_r$ for column supports and $\Omega_L = \Gamma_l$ for continuous linear supports;

$\mathbf{G_{B\kappa}}$ is the matrix of the dimension $(2N \times M)$ grouping integrals over the internal sub-domains Ω_m;

$\mathbf{\Delta}$ is the matrix grouping difference operators connecting angle of rotations in tangent direction with deflections of suitable boundary nodes if a plate has a free edge.

$\mathbf{G_{qB}}$ and $\mathbf{G_{qS}}$ are the matrices of the dimension $(L \times 2N)$ grouping the boundary integrals of the appropriate fundamental functions, where *L* is the number of the internal collocation points associated with internal supports and *N* is the number of the boundary nodes;

$\mathbf{G_{qq}}$ is the matrix of the dimension $(L \times L)$ grouping integrals over the internal support sub-domains

$\mathbf{G_{q\kappa}}$ is the matrix of the dimension $(L \times M)$ grouping integrals over the internal sub-domains Ω_m;

The fourth matrix equation $(4.1)_4$ in the set of equation (4.1) is obtained by construction the boundary integral equations for internal collocation points associated with internal sub-domains Ω_m. According the typical approach, in this equation, the plate curvature can be derived by double differentiation of boundary integral equation (3.7) or by constructing one integral equation with respect to central collocation point '1' (Fig. 8) belonging to each internal sub-surface;

$\mathbf{G_{\kappa B}}$ is the matrix of the dimension $(M \times 2N)$ grouping the boundary integrals of the second derivatives with respect to the co-ordinate *x* of the appropriate fundamental functions, where *M* is the number of the internal collocation points and *N* is the number of the boundary nodes;

$\mathbf{G_{\kappa S}}$ is the matrix of the dimension $(M \times S)$ grouping the boundary integrals of the second derivatives with respect to the co-ordinate *x* of the appropriate fundamental functions;

$\mathbf{G}_{\kappa q}$ is the matrix of the dimension $(M \times L)$ grouping integrals over the internal support (column or linear) sub-domains, where $\Omega_L = \Omega_r$ for column supports and $\Omega_L = \Gamma_l$ for continuous linear supports;

$\mathbf{G}_{\kappa\kappa}$ is the matrix of the dimension $(M \times M)$ grouping the integrals of the second derivatives with respect to the co-ordinate *x* over the internal sub-surfaces $\Omega_m \in \Omega$.

In accordance with the simplified approach, the plate curvature can be also establish by addition two internal collocation points ('2' and '3') [31]. Due to this conception it is necessary to construct three integral equation considering three collocation points ('1', '2' and '3') and using equation (3.7) in unchanged form. These two approaches are illustrated in Fig. 8.

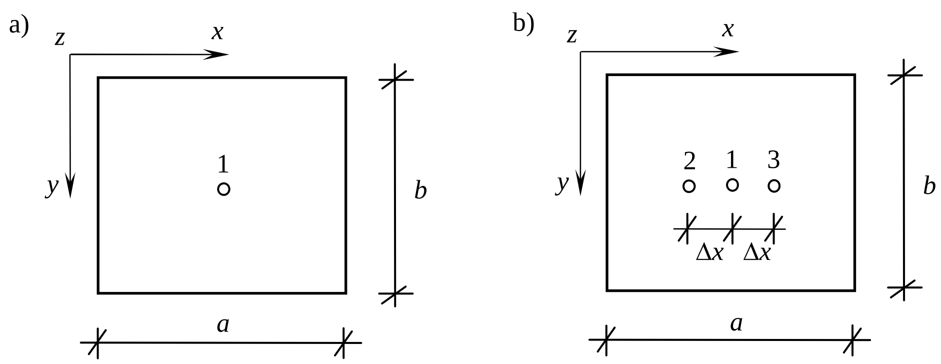

Fig. 8. Definition of the curvature in central collocation point '1' [31].

According the second approach the plate curvature at central point '1' is calculated by constructing difference quotient

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_x = \frac{\Delta^2 \mathbf{w}}{\Delta x^2} = \frac{\mathbf{w}_2 - 2 \cdot \mathbf{w}_1 + \mathbf{w}_3}{(\Delta x)^2} \tag{4.2}$$

hence elements of the matrices $\mathbf{G}_{\kappa B}$, $\mathbf{G}_{\kappa S}$, $\mathbf{G}_{\kappa q}$ and $\mathbf{G}_{\kappa\kappa}$ can be evaluated using three boundary integral equations based only on the boundary integral equation (3.7). Elimination of boundary variables $\mathbf{B}$, φ_S and internal support vector $\mathbf{q}$ from matrix equation (4.1) leads to the standard eigenvalue problem:

$$\{\mathbf{A} - \tilde{\lambda} \cdot \mathbf{I}\} \cdot \boldsymbol{\kappa} = \mathbf{0} \tag{4.3}$$

where $\tilde{\lambda} = 1/\lambda$,

$$
\begin{aligned}
\mathbf{A} = \mathbf{G}_{\kappa\kappa} - \tilde{\mathbf{G}}_{\kappa B} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{B\kappa} + \\
-\left[\mathbf{G}_{\kappa q} - \tilde{\mathbf{G}}_{\kappa B} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{Bq}\right] \cdot \left[\mathbf{G}_{qq} - \tilde{\mathbf{G}}_{qB} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{Bq}\right]^{-1} \cdot \left[\mathbf{G}_{\kappa q} - \tilde{\mathbf{G}}_{qB} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{B\kappa}\right]
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
\tilde{\mathbf{G}}_{BB} &= \mathbf{G}_{BB} + \mathbf{G}_{BS} \cdot \mathbf{\Delta} \\
\tilde{\mathbf{G}}_{qB} &= \mathbf{G}_{qB} + \mathbf{G}_{qS} \cdot \mathbf{\Delta} \\
\tilde{\mathbf{G}}_{\kappa B} &= \mathbf{G}_{\kappa B} + \mathbf{G}_{\kappa S} \cdot \mathbf{\Delta}
\end{aligned}
\tag{4.5}
$$

5. MODES OF BUCKLING

The elements of the eigenvector $\boldsymbol{\kappa}$ obtained after solution of the standard eigenvalue problem (4.3) present the plate curvatures. The set of the algebraic equation indispensable to calculate the eigenvector $\mathbf{w}$ elements has a form:

$$
\begin{bmatrix}
\mathbf{G}_{BB} & \mathbf{G}_{BS} & \mathbf{G}_{Bq} & \mathbf{0} \\
\mathbf{\Delta} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{G}_{qB} & \mathbf{G}_{qS} & \mathbf{G}_{qq} & \mathbf{0} \\
\mathbf{G}_{wB} & \mathbf{G}_{wS} & \mathbf{G}_{wq} & \mathbf{I}
\end{bmatrix}
\cdot
\begin{Bmatrix}
\mathbf{B} \\ \varphi_S \\ \mathbf{q} \\ \mathbf{w}
\end{Bmatrix}
=
\begin{Bmatrix}
\lambda \cdot \mathbf{G}_{Bw} \cdot \boldsymbol{\kappa} \\ \mathbf{0} \\ \lambda \cdot \mathbf{G}_{qw} \cdot \boldsymbol{\kappa} \\ \lambda \cdot \mathbf{G}_{ww} \cdot \boldsymbol{\kappa}
\end{Bmatrix}
\tag{5.1}
$$

In the set of the equation (5.1) the first, second and third equations $(5.1)_1$, $(5.1)_2$ and $(5.1)_3$ are obtained from the first, second and third equations of (4.1) and the fourth equation $(5.1)_4$ is gotten by construction the boundary integral equations for calculating the plate deflection in internal collocation points. and internal support vector $\mathbf{q}$ from equation (5.1) gives the elements of the wanted displacement vector:

$$
\begin{aligned}
\mathbf{w} = \lambda \cdot \Big[\mathbf{G}_{ww} - \tilde{\mathbf{G}}_{wB} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{Bw} + \\
-\left[\mathbf{G}_{wq} - \tilde{\mathbf{G}}_{wB} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{Bw}\right] \cdot \left[\mathbf{G}_{qq} - \tilde{\mathbf{G}}_{qB} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{Bq}\right]^{-1} \cdot \left[\mathbf{G}_{wq} - \tilde{\mathbf{G}}_{qB} \cdot \tilde{\mathbf{G}}_{BB}^{-1} \cdot \mathbf{G}_{Bw}\right]\Big] \cdot \boldsymbol{\kappa}
\end{aligned}
\tag{5.2}
$$

and

$$
\tilde{\mathbf{G}}_{wB} = \mathbf{G}_{wB} + \mathbf{G}_{wS} \cdot \mathbf{\Delta} \tag{5.3}
$$

6. NUMERICAL EXAMPLES

The initial stability problem of a rectangular plates resting on internal column or linear continuous supports is considered. Considered plates have all boundary simply-supported or supported two opposite edges. In plane loading N_x is acting along a supported edge. The critical value of the in plane loading is investigated. Each of plate edge is divided by the boundary elements of the constant type with the same length. Internal continuous linear supports is divided by the sections (elements) of the constant type with the same length. A column has a square cross-section associated with one collocation point and the side length mich smaller than the shorter side of a plate. The set of the internal collocation points in which the curvature vector $\boldsymbol{\kappa}$ is established, associated with internal sub-surfaces is regular.

Quasi-diagonal terms of the matrix $\mathbf{G_{BB}}$ in equations (4.1) and (5.1) are calculated analytically and the rest of them using 12–points Gauss quadrature. All the terms of the matrices $\mathbf{G_{Bq}}$, $\mathbf{G_{B\kappa}}$, $\mathbf{G_{Bw}}$, $\mathbf{G_{qq}}$, $\mathbf{G_{q\kappa}}$, $\mathbf{G_{\kappa q}}$, $\mathbf{G_{\kappa\kappa}}$, $\mathbf{G_{qw}}$, $\mathbf{G_{wq}}$ and $\mathbf{G_{ww}}$ in equations (4.1) and (5.1) are evaluated analytically. The rest of the terms in matrices $\mathbf{G}_{ik}$ are calculated numerically by applying 12–points Gauss quadrature.

To compare obtained results with previous ones [29, 30], the following material properies are assumed: for plates resting on internal column supports the Young modulus E = 1.0 kPa and the Poisson ratio v = 0.3; for plates resting on internal continuous linear supports Young modulus E = 30.0 GPa and the Poisson ratio v = 0.167.

The following notations are assumed:

BEM I – singular formulation of governing boundary-domain integral equations (3.7) and (3.8) with the second equation obtained by single differentiation of equation (3.7), the vector of curvatures is established by double differentiation of the first governing boundary-domain integral equation (3.7);

BEM II – non-singular formulation of governing boundary-domain integral equations (3.7) and (3.8), with the second equation (3.8) obtained by differentiation of the equation (3.7), the vector of curvatures is established by double differentiation of the first governing boundary-domain integral equation (3.7). The collocation point of single boundary element is located outside, near the plate edge. For one collocation point: $\varepsilon_1 = \tilde{\delta}_1 / d$ where $\tilde{\delta}_1$ is distance of collocation point from the plate edge and d is the boundary element length;

BEM III – non-singular formulation of governing boundary-domain integral equation (3.7) with the second boundary-domain integral equation obtained for the set of additional collocation points with the same fundamental solution w^*, the vector of curvatures is established by constructing difference quotient (4.2) and fundamental solution w^*. Localization of two collocation points for single boundary element is determined by: ε_1 and $\varepsilon_2 = \tilde{\delta}_2 / d$. For three collocation point belonging for each internal sub-domain element: $\varepsilon_\Delta = \Delta x / a$.

FEM – regular finite element mesh and two types of element S4R (four node with three degree of freedom per node) and S8R (eight node with three degree of freedom per node) of ABAQUS program with reduced integration were assumed into comparative analysis.

The critical force N_{cr} is expressed using non-dimensional term:

$$\tilde{N}_{cr} = \frac{N_{cr}}{D} \cdot l_x \cdot l_y \tag{6.1}$$

6.1. The plate simply-supported on two opposite edges with two remaining edges free resting on two internal column supports under uniformly constant normal loading

Static and loading scheme is shown in the Fig. 9. The plate dimensions are: $l_y = 0.25 \cdot l_x$ and the internal support dimensions: (square $b \times b$), $b = 0.02 \cdot l_y$. Two boundary and domain discretization are adopted:

(a) number of boundary elements is equal to 96, number of internal collocation points is equal to 144 and internal sub-surface dimension (square $a \times a$) is $a = 1/6 \cdot l_y$;

(b) number of boundary elements is equal to 120, number of internal collocation points is equal to 400 and internal sub-surface dimension (square $a \times a$) is $a = 0.05 \cdot l_y$.

The results of calculation are presented in Tables 1 – 3. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Tables 2 and 3. The first buckling mode is shown in the Fig. 10.

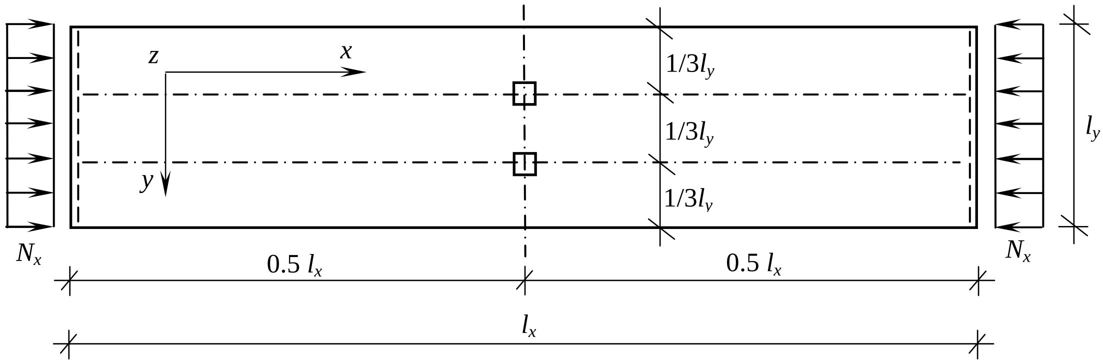

Fig. 9. The plate simply-supported on two opposite edges with two remaining edges free resting on two internal column supports under uniformly constant normal loading.

Table 1. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\tilde{N}_{cr}$	BEM II(a) [29]	BEM III(a) present	BEM II(b) [29]	BEM III(b) present	FEM S4R [29]	FEM S8R [29]
1	8.8260	9.4300	8.9410	9.7432	8.5580	8.9490
2	19.6310	20.7885	19.6740	21.2720	17.8870	17.5840
3	36.8010	39.1424	37.0280	40.2456	35.7170	34.6450
4	56.7990	60.0224	56.8830	61.4581	55.4510	52.8860

Table 2. Critical force values. Solution BEM III(a) for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	9.4304	9.4300	9.4300	9.4289	9.4300
2	20.7870	20.7884	20.7885	20.7869	20.7884
3	39.1460	39.1426	39.1424	39.1438	39.1424
4	60.0066	60.0224	60.0224	60.0320	60.0228

Table 3. Critical force values. Solution BEM III(b) for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	9.7431	9.7432	9.7432	9.7437	9.7458
2	21.2721	21.2720	21.2720	21.2741	21.2812
3	40.2448	40.2455	40.2456	40.2527	40.2732
4	61.4605	61.4581	61.4581	61.4730	61.5156

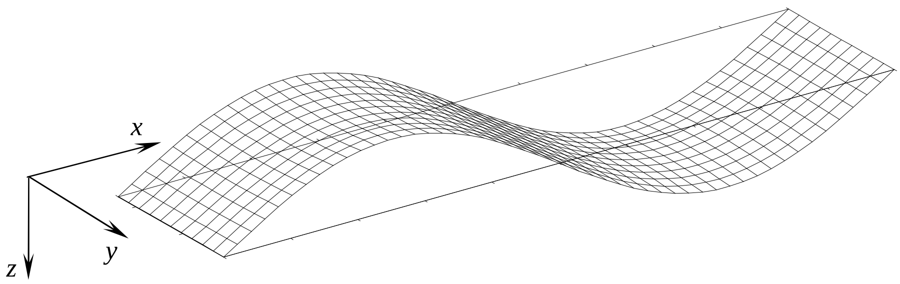

Fig. 10. The first buckling mode.

6.2. The plate clamped on two opposite edges with two remaining edges free resting on two internal column supports under uniformly constant normal loading

Static and loading scheme is shown in the Fig. 11. The plate properties were assumed identically as in Example 6.1.

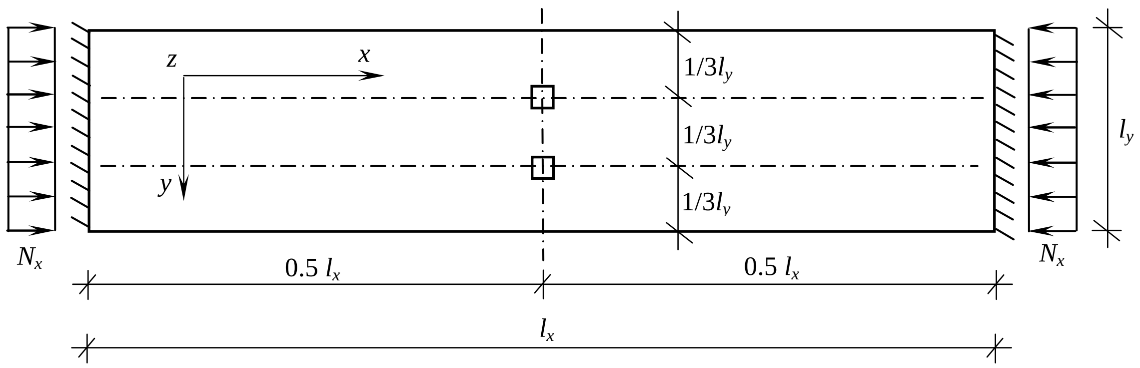

Fig. 11. The plate simply-supported on two opposite edges with two remaining edges free resting on two internal column supports under uniformly constant normal loading.

The results of calculation are presented in Tables 4 – 6. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Tables 5 and 6. The first buckling mode is shown in the Fig. 12.

Table 4. Critical force values. ε_1 = 0.01, ε_2 = 0.1, ε_Δ = 0.01.

$\tilde{N}_{cr}$	BEM II(a) [29]	BEM III(a) present	BEM II(b) [29]	BEM III(b) present	FEM S4R [29]	FEM S8R [29]
1	18.6948	20.3522	18.952	21.0271	18.0400	17.7350
2	37.4158	40.0875	37.669	41.2555	36.0710	34.9630
3	56.7005	60.5224	57.006	62.1079	55.6600	53.0770
4	77.9972	79.8567	77.895	83.5699	77.2380	72.2580

Table 5. Critical force values. Solution BEM III(a) for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	20.3504	20.3522	20.3522	20.3522	20.3522
2	40.0962	40.0875	40.0875	40.0875	40.0875
3	60.5097	60.5221	60.5224	60.5224	60.5228
4	79.8718	79.8567	79.8567	79.8567	79.8561

Table 6. Critical force values. Solution BEM III(b) for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	21.0270	21.0271	21.0271	21.0286	21.0340
2	41.2552	41.2553	41.2555	41.2616	41.2806
3	62.1090	62.1076	62.1079	62.1224	62.1641
4	83.5686	83.5699	83.5699	83.5910	83.6551

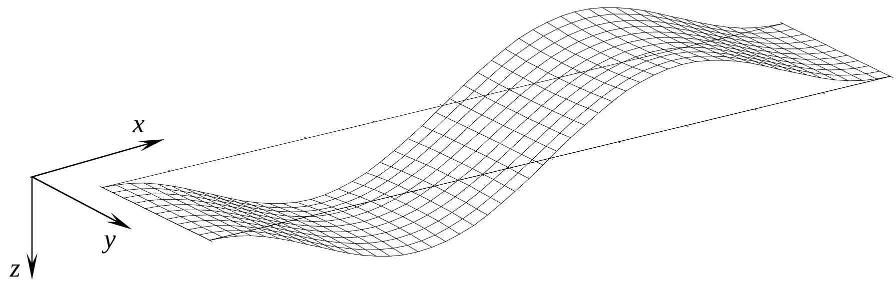

Fig. 12. The first buckling mode.

6.3. The plate simply-supported on two opposite edges with two remaining edges free resting on linear continuous internal support under constant normal loading

Static and loading scheme is shown in the Fig. 13.

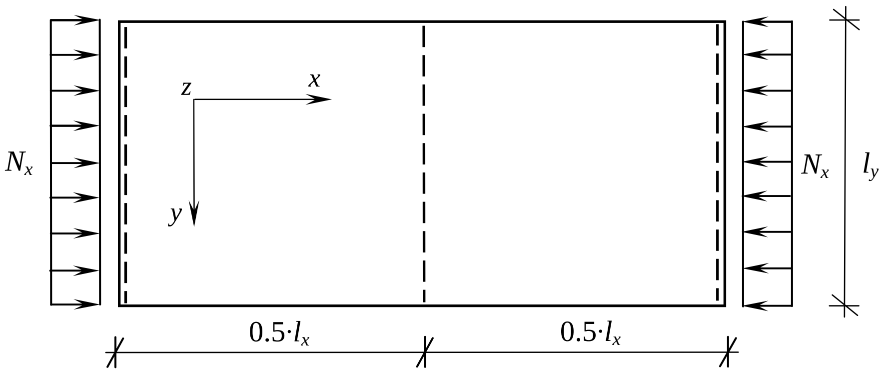

Fig. 13. The plate, simply-supported on two opposite edges with two edges free and one internal continuous support under constant normal loading.

Each plate edge divided into number of 40 boundary elements of the same length. Number of internal linear continuous elements of the same length is equal to 40 and number of internal sub-surfaces used to describe the plate curvature is equal to 200. The plate geometry is defined as: l_x = 2.0·l_y = 20.0 m, h_p = 0.2 m. The results of calculation are presented in Tables 7 and 8. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Table 8. The first buckling mode is shown in the Fig. 14.

Table 7. Critical force values. ε_1 = 0.01, ε_2 = 0.1, ε_Δ = 0.01.

$\tilde{N}_{cr}$	BEM II [30]	BEM III present	FEM S4R [30]
1	19.3976	21.3618	19.4324
2	40.2226	43.4987	40.7566
3	58.6534	59.8541	58.3006
4	79.0824	82.8516	77.3624

Table 8. Critical force values. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$, ε_1 = 0.01, ε_2 = 0.1.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	21.4618	21.4618	21.4618	21.4596	21.4539
2	43.4985	43.4987	43.4987	43.4983	43.4993
3	59.8548	59.8548	59.8541	59.8353	59.7845
4	82.8509	82.8516	82.8516	82.8332	82.7867

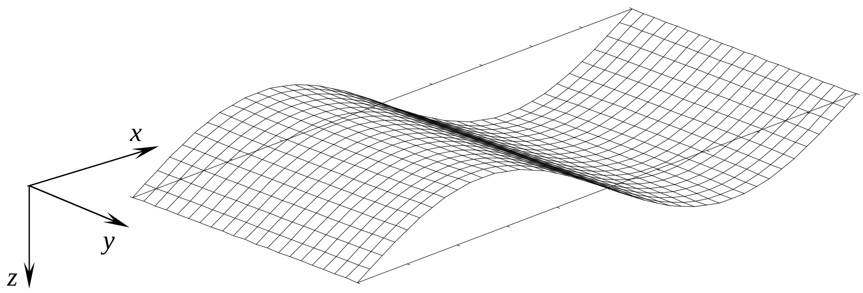

Fig. 14. The first buckling mode.

6.4. The plate clamped on two opposite edges with two remaining edges free resting on linear continuous internal support under constant normal loading

Static and loading scheme is shown in the Fig. 15. The plate properties were assumed identically as in Example 6.3.

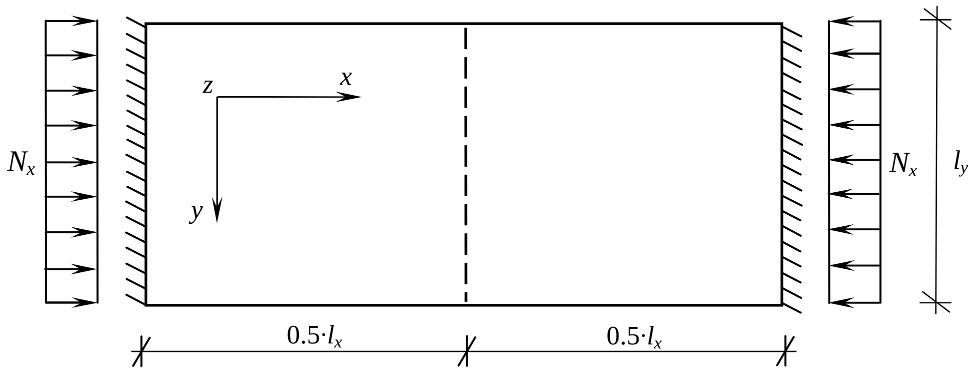

Fig. 15. The plate clamped on two opposite edges with two edges free and one internal continuous support under constant normal loading.

The results of calculation are presented in Tables 9 and 10. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Table 10. The first buckling mode is shown in the Fig. 16.

Table 9. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\tilde{N}_{cr}$	BEM II [30]	BEM III present
1	40.1352	44.6064
2	79.3184	83.1841
3	79.5774	85.9430
4	118.7634	127.3070

Table 10. Critical force values. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	44.6061	44.6064	44.6064	44.6055	44.6057
2	83.1834	83.1841	83.1841	83.1634	83.1115
3	85.9446	85.9431	85.9430	85.9567	86.0016
4	127.3064	127.3072	127.3070	127.2957	127.2769

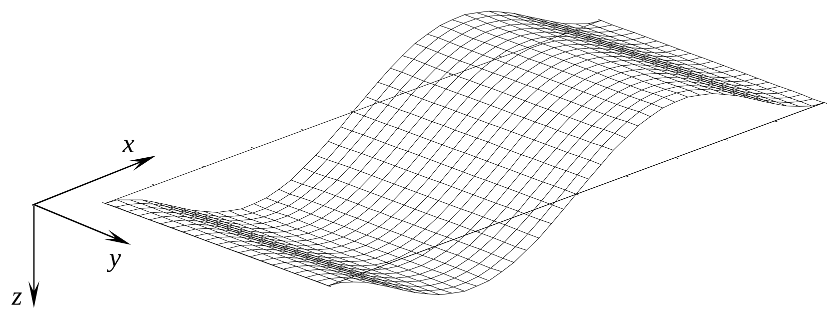

Fig. 16. The first buckling mode.

6.5. The plate simply-supported on all edges resting on linear continuous internal support under constant normal loading

Static and loading scheme is shown on the Fig. 17. The plate properties were assumed identically as in Example 6.3.

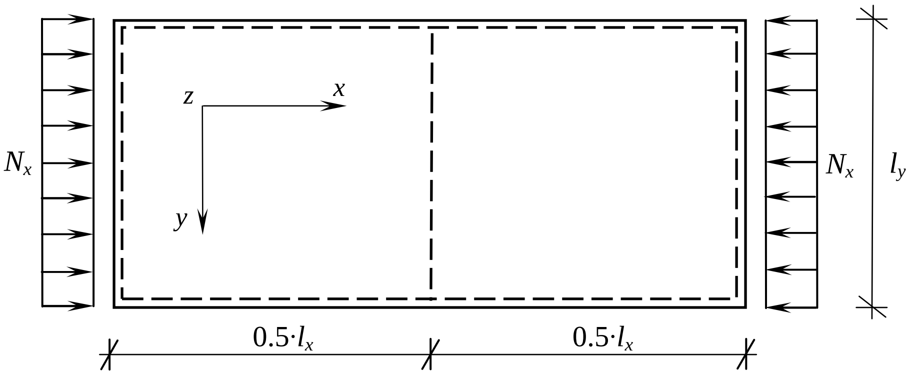

Fig. 17. The plate simply-supported on all edges and one internal continuous support under constant normal loading.

The results of calculation are presented in Tables 11 and 12. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Table 12. The first buckling mode is shown in the Fig. 18.

Table 11. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\tilde{N}_{cr}$	BEM I present	BEM II [30]	BEM III present
1	79.7721	79.7720	79.7708
2	96.8938	96.8938	96.8966
3	125.9391	125.9394	125.9334
4	169.3239	169.3238	169.3156

Table 12. Critical force values. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	79.7708	79.7701	79.7708	79.7767	79.7963
2	96.8938	96.8957	96.8966	96.9121	96.9624
3	125.9347	125.9331	125.9334	125.9752	126.1018
4	169.3183	169.3147	169.3156	169.4012	169.6607

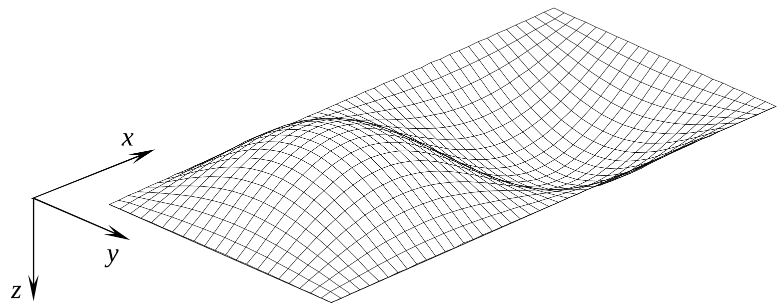

Fig. 18. The first buckling mode.

6.6. The plate simply-supported on two opposite and diagonal edges resting on linear continuous internal support under constant normal loading

Static and loading scheme is shown on the Fig. 19. The plate properties were assumed identically as in Example 6.3.

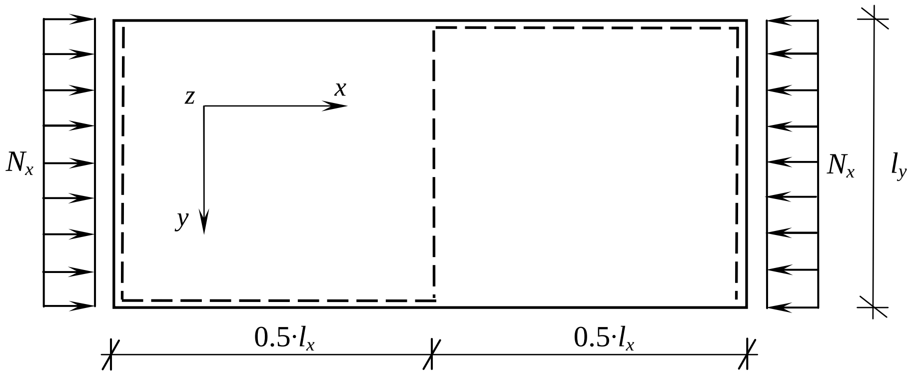

Fig. 19. The plate simply-supported on two opposite and diagonal edges resting on linear continuous internal support under constant normal loading.

The results of calculation are presented in Tables 13 and 14. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Table 14. The first buckling mode is shown in the Fig. 20.

Table 13. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\tilde{N}_{cr}$	BEM II present	BEM III present
1	38.4612	40.0258
2	48.3226	51.0846
3	97.2487	101.4683
4	125.2491	132.6167

Table 14. Critical force values. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	40.0259	40.0258	40.0258	40.0243	40.0213
2	51.0844	51.0846	51.0846	51.0841	51.0849
3	101.4677	101.4680	101.4683	101.4914	101.5642
4	132.6176	132.6165	132.6167	132.6527	132.7688

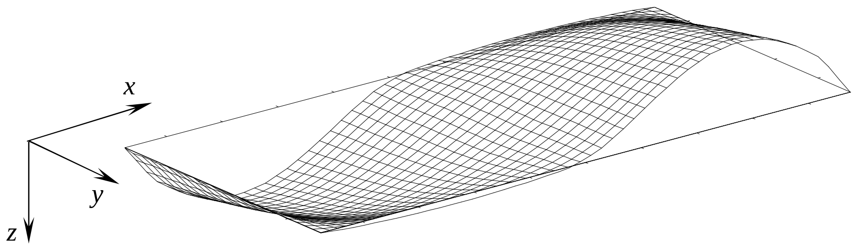

Fig. 20. The first buckling mode.

6.7. The plate simply-supported on two opposite edges resting on two linear continuous internal support under constant normal loading

Static and loading scheme is shown on the Fig. 21. The plate properties were assumed identically as in Example 6.3.

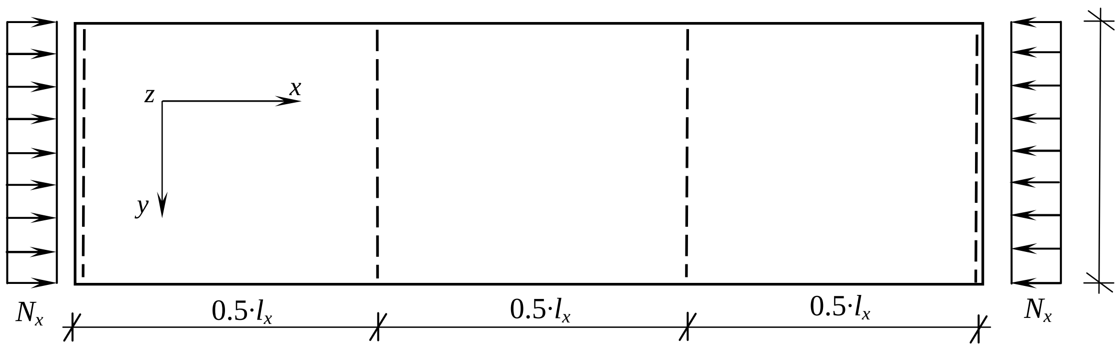

Fig. 21. The plate simply-supported on two opposite and diagonal edges resting on linear continuous internal support under constant normal loading.

Each plate edge divided into number of 45 boundary elements of the same length. Number of internal linear continuous elements of the same length is equal to 40 and number of internal sub-surfaces used to describe the plate curvature is equal to 300. The plate geometry is defined as: $l_x = 3.0 \cdot l_y = 30.0$ m, $h_p = 0.2$ m. The results of calculation are presented in Tables 15 and 16. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Table 14. The first buckling mode is shown in the Fig. 22.

Table 15. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\tilde{N}_{cr}$	BEM II present	BEM III present
1	29.0968	32.4116
2	43.8981	47.8172
3	78.7263	85.0558
4	87.4007	88.6611

Table 16. Critical force values. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x / a$				
	0.0001	0.001	0.01	0.1	0.2
1	32.4118	32.4115	32.4116	32.4152	32.4257
2	47.8162	47.8170	47.8172	47.8245	47.8457
3	85.0538	85.0553	85.0558	85.0776	85.1412
4	88.6600	88.6606	88.6611	88.6751	88.7118

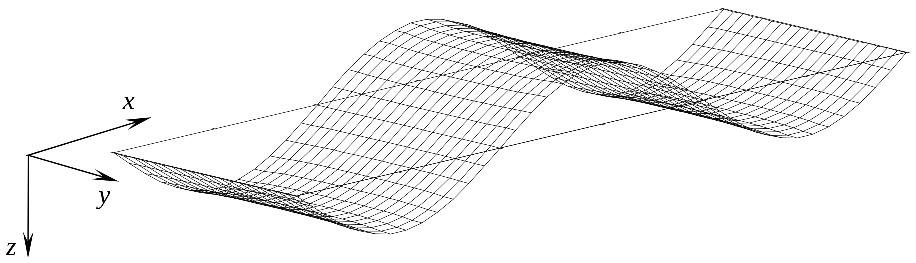

Fig. 22. The first buckling mode.

6.8. The plate simply-supported on all edges resting on two linear continuous internal support under constant normal loading

Static and loading scheme is shown on the Fig. 23. The plate properties were assumed identically as in Example 6.7.

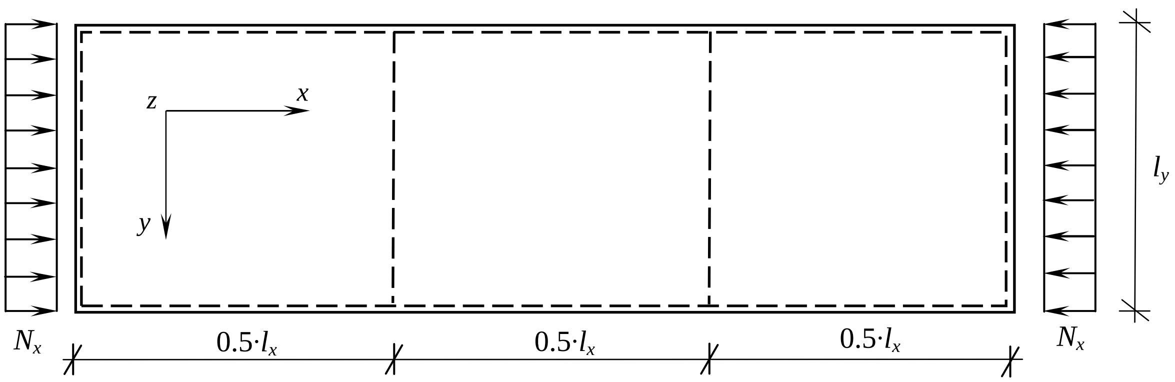

Fig. 23. The plate simply-supported on all edges resting on two linear continuous internal support under constant normal loading.

The results of calculation are presented in Tables 17 and 18. The influence of localization of internal collocation points on critical force values using BEM III approach is presented in Table 14. The first buckling mode is shown in the Fig. 24.

Table 17. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\tilde{N}_{cr}$	BEM I present	BEM II present	BEM III present
1	119.7662	119.7672	119.7701
2	132.2487	132.2499	132.2535
3	161.2000	161.2006	161.2066
4	189.0590	189.0507	189.0539

Table 18. Critical force values. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\tilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	119.7721	119.7701	119.7701	119.7800	119.8105
2	132.2499	132.2535	132.2535	132.2703	132.3207
3	161.1999	161.2061	161.2066	161.2426	161.3524
4	189.0615	189.0531	189.0539	189.11690	189.3086

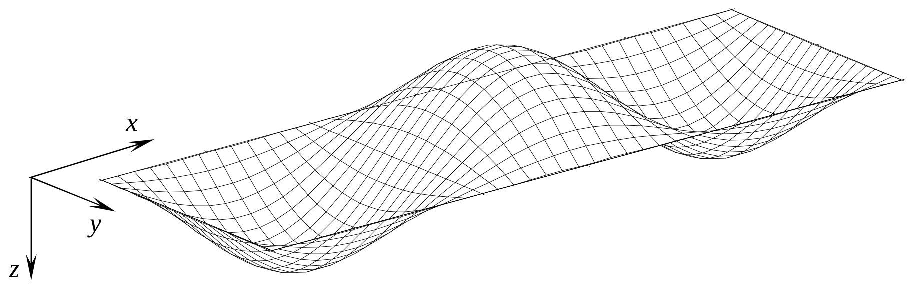

Fig. 24. The first buckling mode.

CONCLUSIONS

An initial stability of thin plates resting on internal supports using the boundary element method was presented. Presented issue was solved with the modified and simplified approach, in which the boundary conditions are defined so that there is no need to introduce equivalent boundary quantities dictated by the boundary value problem for the biharmonic differential equation. The collocation version of boundary element method with singular and non-singular calculations of integrals were employed and the constant type of the boundary element was introduced. The Bèzine technique was used to establish the vector of internal support reaction forces and the vector of curvatures inside a plate domain. A plate domain was divided into rectangular sub-surfaces associated with one collocation poinits in which the vector of curvatures is established. In presented examples considered plates are subjected in plane by constant normal loading. The high number of boundary elements, sections representing internal linear continuous constraints and internal sub-surfaces was not required to obtain sufficient accuracy. The loaded plate edge must be supported which is required in proposed formulation of buckling analysis. A significant increase in the number of boundary elements and internal sub-surfaces does not affect the radical improvement of the results of calculations. The resulting solution is stable for a large range of values $\varepsilon_\Delta = \Delta x/a$.

Presented work relates entirely to the paper [31], in which the buckling problem of a rectangular plates by the Boundary Element Method was investigated. The boundary element results obtained for proposed conception of thin plate bending issue considering internal linear continuous and plane constraints demonstrate the sufficient effectiveness and efficiency.

ACKNOWLEDGEMENTS

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