**On parametrization of Cauchy stress tensor treated as autonomous object using isotropy angle and skewness angle**

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# Abstract

Intrinsic features (eigenproperties) of Cauchy stress tensor are discussed. Novelty notions of *isotropy angle*, and *skewness mode angle* are introduced for improved parametric description of spherical (isotropic) and deviatoric (anisotropic) components of stress tensor. The skewness angle is defined with *pure shear* employed as *comparison reference mode* upon observation that pure shear states can be interpreted as an elementary (atomic) blocks of any macroscopic deviatoric stress state. An original *statistical-physical interpretation* of the stress tensor *orthogonal invariants* is provided. It is proposed micromechanical explanation for observed decrease of the values of stress tensor *anisotropy factor*, measured in terms of *diameter of tensor orbit*, with stress deviator diverging from pure shear mode. There are delivered explicit reasons explaining why *biaxial experimental layouts* (*simple shear* and/or *planar shear*) are insufficient for comprehensive characterization of materials properties submitted to complex stress states loadings. New explicit formulas for *triaxiality factor* valid for *biaxial stress states* are delivered.

***Keywords*** – Cauchy stress; oriented geometrical object; isotropy angle; skewness angle; isomorphic cylindrical coordinates; pure shear, comparison reference state; anisotropy factor; biaxial tests; simple shear, planar shear, triaxiality factor.

# 1. Introduction

Three dimensional second order tensors were introduced into science and engineering with a landmark lecture of Cauchy in 1822, and in written form in 1823, cf. [4]. The brilliant idea of Cauchy survived in unchanged form till nowadays as his argument on tetrahedron element and forces acting on it can be found in practically unchanged form in any and all respectable textbook devoted to continuum mechanics, technical physics, strength of materials and many more engineering and science fields. In his original presentation Cauchy was speaking about *pressures* and not *stress tensor* simply because at that time the tensor notion yet, did not exist. From philosophical point of view Cauchy's introduction of stress tensor concept can be treated as a step towards development of *constitutive theory of forces*. The Cauchy idea of describing force interactions through surfaces can be understood as certain (averaged) continuum model description of the force interactions between molecules – transition from molecular interactions towards continuum media interactions. Interesting exposure on physical grounds underlying stress tensor concept from the above mentioned perspective can be found in chapter 3 of Maugin' book [13], and stimulating historical notes on creation and unfolding of stress tensor idea can be found in chapter 4 of Truesdell book [30]. The constitutive theory of forces is practically undeveloped in comparison to the maturity of *constitutive theory of materials*. Eugster and Glocker have recently published a work remaining within the stream of development of constitutive theory of forces, cf. [6]. The mathematical grounds of *tensor analysis* with all the fundamental underlying formal apparatus was originally developed by Ricci-Curbastro in the years 1888-1892, cf. Ref (19) in Tonolo [29]. The motivation behind this development was completely different from this of Cauchy, namely it was investigation on *invariance of quadratic forms* and Curbastro called the technique *absolute differential calculus*. Ricci-Curbastro can be regarded as father of tensor analysis as it is *invariance feature* with respect to coordinate frames, which is the essence and profound sense of tensorial objects. It is this feature, which decided on versatile usefulness of tensors, and their becoming nowadays the language of all advanced technical sciences; see e.g. Itskov [9] or Spencer [28]. The term *tensor* in its contem­porary meaning was coined by Woldemar Voigt in his work from 1898, cf. e.g. pages v-vi in [31].

Tensors gained its today omnipresence in science and technology – attained the status of a common language, because they proved to be excellent modeling objects. They enable reliable description of many features of real physical objects: *state* (e.g. temperature, velocity, stress, strain), but equally well their *properties* (e.g. such as thermal expansion, piezoelectricity, elastic behavior – this with the aid of second, third and fourth order tensors, respectively), and many other traits (e.g. mechanical, thermal and/or electromagnetic *loadings*). Tensors and their *eigenproperties* – represented by their various *invariants*, by embodying modeling idealization of physical reality indirectly enable its deeper understanding. When the tensor of specific type is found to reliably describe and/or predict specific physical phenomenon, then it is reasonable to infer that the specific eigenproperties of the tensor deliver somewhat clarified picture of specific characteristics of real physical situation, not blurred by disturbing it secondary effects. The feedbacks in the process of model development and its experimental validation can bring about more precise isolation of key characteristic features of the phenomenon itself and in return also its model.

The present work is focused on Cauchy stress tensor treated as a generic instance of any 3 dimensional second order symmetric Eulerian tensor, more precisely Cartesian one because in prevailing part of the present work Cartesian orthonormal bases are used. In fixed tensorial basis generated by three versors of a basis of 3-dimensional Eulerian space – usually accepted as coordinates frame, the second order symmetric tensor is fully characterized by its representation, i.e. components of 3x3 symmetric matrix. These components in an involved manner contain information on six linearly independent parameters, out of which there can be constructed a set of three linearly independent invariants – with respect to change (rotation) of coordinates frame, and complementary set of three parameters being Euler angles describing orientation of the tensor object – its triad of principal directions, with respect to coordinates frame. The Euler angles are not invariants as they do change with alteration (rotation) of coordinates frame. Actually, infinitely many invariants can be constructed for second order symmetric tensor but only sets of maximum *three* of them can be linearly independent. Actual physical interpretations of intrinsic features (invariants) of second order symmetric tensor can be quite diverse. For example, when the tensor is used as modeling abstraction of a parcel these could be interpreted as ("length", "width", "height"), when on the other hand the tensor is used as modeling abstraction of an animal, these can find interprettation as ("hue", "brightness", "saturation") of its fur. Higher order symmetric tensors can deliver more precise modeling description of real life objects. For example, fourth order tensor with internal symmetries characteristic for Hooke's tensor is characterized by 18 invariants (eigenproperties).

In mechanical engineering trace of stress tensor has very important physical interpretation of representing pressure . In the case of small strains tensor its trace only approximately describes volumetric changes of the material  , while the precise characterization of material volumetric changes is assured by determinant of a tensor of deformation gradient . Qualitative decomposition of parameters fully characterizing second order symmetric tensor, in fixed basis, into two triple sets composed of invariants and Euler angles delivers grounds for the idea that it is natural and convenient to interpret second order symmetric tensor – in particular stress tensor, as *oriented object* (entity), which besides orientation in space is characterized by some number of intrinsic features – eigenproperties. These intrinsic features are described by some conveniently selected set of the tensor invariants. The proposed approach to treat tensors as *autonomous directed entities* seems much more intuitive and useful in dealing with tensors as modeling objects representing real physical objects then standard mathematical understanding of the tensor defined to be some kind of *algebraic structure*.

Frequently, it can be acquired impression that eigenproperties (invariants) of the stress tensor are the most important features. For example, this premise finds reflection in the deep-rooted at present concept of *effective (equivalent) stress* notion , which is a function of stress tensor only – the quantity universally at present used in engineering sciences cf. e.g. Leckie and Dal Bello [10]. Nevertheless, interaction of stress tensor object with other objects creates qualitatively different situations. For example, upon interaction of stress tensor (loading) with Hooke's tensor representing elastic material (external system), the three Euler angles of the stress tensor (not invariant) and three Euler angles of Hooke' tensor (not invariant) transform together into definite three invariants. The effect can be better intuitively understood when one realizes that stress tensor orientation with respect to Hooke's tensor orientation does not depend on these objects individual orientations with respect to some freely adopted common coordinates frame. Thus, *in interaction with external system* *six parameters* (components) fully characterizing any second order symmetric tensor can be uniquely transformed into *six invariants*.

This observation gives grounds for assessment that much more versatile and perhaps accurate general measure of some *limit condition –* start of plastic yield flow, cracking, damage or phase transition, is not equivalent stress but some expression involving combination of stress tensor and some fourth order tensor  characterizing material submitted to stress action, for example quadratic form . The expression of this type was introduced by R. von Mises already in 1928 – cf. formula (1) in [15], but it receives relatively little attention. The  might be coined a name *weighted equivalent stress* or *Misses stress intensity*, since the stress tensor does not appear in it as stand alone, sovereign entity but it is pondered in interaction with external surround­ings represented by tensor . Similarly, as it is the case when investigating stability problems.

Here, new generic parametrization of stress tensor invariants is introduced composed of stress modulus, isotropic angle and skewness angle . It enables simplification of some useful formulas for example the one for anisotropy factor. New statistical interpretation of stress tensor invariants enabled introduction of the concept of internal entropy of stress tensor and revealed existence of connections between internal entropy and anisotropy degree of stress tensor.

Discussion on practical applications of newly introduced concepts and interpretations in experimental mechanics is also delivered. In particular new very simple explicit formulas are presented for triaxiality factor expressed in terms of skewness angle.

# 2. Tensors – definitions, important bases and representations

Tensors proved to be very convenient modeling objects of reality. Due to that it is important to well understand what they actually are, and what are their properties because then real physical situation modeled with the use of tensors can be better understood. There exist at least several definitions of tensor notion.

*2.1 Definitions and understanding of a tensor notion*

– *Algebraic definition of tensor*;

The most developed and mathematically precise definition of a tensor is algebraic definition. In engineering sciences so called Euclidean tensors are used the most frequently. Definition of Euclidean tensor can be formulated as follows,

An *Euclidean tensor* of order *q* and dimension *n* is and *algebraic structure*, an *element of linear tensorial space* , which is generated by *q*-fold tensorial product of *n* dimensional Euclidean vector spaces . When the same basis is accepted in all spaces  then any tensor  belonging to  can be presented in the form,



where a set of  versors  is a basis of Euclidean vector space , a set of  simple tensors  (-fold tensorial products of versors ) is a basis of space ,  numbers  are called components of tensor  – its *representation* in basis . In the case of second order tensors examined here  and , as we are primarily interested in modeling real physical space . A pair composed of a point *O* belonging to *Euclidean point space*  and a set of basis versors  of associated with it Euclidean vector space  is called *coordinates frame (coordinates system)* . In shortcut, the coordinates frame is frequently noted by the set of basis vectors  only. It is *three dimensional point Euclidean space * with coordinates frame , which is accepted as convenient *model of real physical space* in continuum mechanics[[1]](#footnote-1). When the basis  composed of mutually orthonormal versors is adopted then customarily *Eulerian tensors* are called *Cartesian tensors*.

It is impossible to introduce here explicitly and in detail all the necessary apparatus of tensorial calculus due to space limitations. The precise mathematical definitions can be found in many popular textbooks on continuum mechanics and/or tensorial calculus, see e.g. chapter 1 in Ogden [20] for compact introduction into tensor theory, and for example Itskov [9] or Ostrowska-Maciejewska [21] for comprehensive and mathematically precise expositions.

The most important information to acquire from *algebraic definition* of tensors (elements of a linear space) is that *tensor makes integrity* *of basis and components* (representation of the tensor in the basis). When the basis is fixed, isomorphism exists between tensor and its representation (components) in this basis, i.e. the tensor can be equated with its components. The *tensor components transform* in specific (linear) manner *with linear change (rotation) of the basis*.

Actually, the algebraic definition of a tensor is not very transparent, neither appealing as it requires extensive preliminary knowledge from linear algebra. It is very hard for conceptual, intuitive understanding and thus uses in modeling real life phenomena. Its biggest advantage is quantitative precision necessary and required for obtaining any predictive numerical results when using tensors for modeling purposes.

Let us point out some other more convenient conceptually and as it seems more attractive for modeling purposes definitions of tensor notion.

– *Operational* *definition of tensor;*

A tensor  is a *linear operator* transforming tensor  of order   into a tensor  of order  . It is realized by full contraction of tensors  and  .

For example, fourth order tensor  upon its multiplication (contraction) with second order tensor  transforms this tensor into some other second order tensor . It is clear that correct methodologically execution of whatever linear transformation anyway requires all the algebraic apparatus of tensor calculus. However, in certain tasks it is much easier to perceive the tensor conceptually as linear operator with specific properties then hard for embodiment algebraic structure.

It is advocated here still another approach (interpretation) of a tensor, which somehow returns to the roots of the tensor concept.

– *Geometric definition of tensor*;

A tensor is an *oriented geometrical object*, i.e. the geometrical object for which orientation can be identified with respect to some fixed *reference frame[[2]](#footnote-2)*, and simultaneously characterized by a certain number of features described by its invariants, i.e. properties not changing with rotation of reference frame. The mathematically precise definition and accessible elucidation of the concept of *geometrical object* can be found, for example, in chapter II of Gołąb book [7].

In this approach second order symmetric tensor can be envisioned as an autonomous entity, which direction in space is prescribed by orientation of triad of its principal directions with respect to prescribed *laboratory (reference) frame* and possessing features described by three linearly independent invariants. Again for actual quantitative calculations all the algebraic tensorial apparatus is still necessary when conceptually treating tensors as oriented objects. This definition, at first sight, seems to be quite indistinct but in essence it well reflects, natural characteristics possessed by many real physical objects. The situation is similar to the one from classical geometry, in which *point*, *line* and *plane* are actually undefined prime notions but everybody knows what they are, and/or how to understand them.

The usefulness of specific definition and/or conceptual understanding of tensors depend on the actual area, tasks and targets of their use. However, in order to grasp well intuitive and profound understanding of the tensor object as modeling tool and in this manner, e.g. indirectly comprehend some physical rule/law expressed by the tensorial relation, it is argued that *geometrical definition/interpretation* of the tensor is very convenient. This is so, first of all because one does not have to concentrate attention on technical (secondary) issues such as numerous components, their indices and contexts resulting from adopted specific basis (coordinates frame).

*2.2 Instances of convenient representations (bases; coordinate frames) and reference frames of second order symmetric tensors*

It is important to carefully distinguish different notations used for tensors as information about them is differently distributed between the basis and the components depending on the notation. The basis of second order tensors space  can be constructed from nine simple tensors − dyads . Frequently, for the vectors  elements of orthonormal basis of 3-dimensional Euclidean vector space  are selected . In the case of symmetric second order tensors its matrix representation components fulfill condition of symmetry , cf. 1. The stress tensor is a symmetric second order tensor (). The following notations are very commonly used to express stress tensor,



where  denote principal values and  are principal directions (eigenvectors) of the tensor  . The following denotations are used in 2,





in the last row of the structure of tensors  in basis  is shown graphically.

The leftmost notation in is a standard notation for 3 dimensional second order tensors written in 9 dimensional orthonormal basis . The middle notation, also known as Kelvin notation, results from writing second order symmetric tensor in 6 dimensional orthonormal symmetric basis . In the rightmost notation tensor  is written in a very special manner, in which all not diagonal terms of tensor representation are zero, i.e. . Information about the tensor object is differently distributed between the basis and components in different notations. In notation 1 the basis can be completely freely selected, so all information about the specific tensor is contained in its components. In notation 2 in view of known symmetry of the tensor  the size of the basis was reduced from nine to six elements and vector (simpler) recording of components was introduced. But again symmetrical tensorial base can be freely selected and thus all the information about the tensor is contained in its components. In the notation 3 the components does not bear all the information about the tensor as part of it is contained in specially selected basis composed of principal directions  – always orthogonal, cf. e.g. Ogden [20]), which cannot be freely selected but are specific functions of the general basis  . The triple of principal axes  is rotated with respect to triple  by three (Euler) angles . All notations assure that the value of norm of tensor   is preserved when computed using standard for the specific notation operational rules. This property is not preserved in the case of frequently used in numerical computations Voigt notation, for which . Notations are valid regardless of the tensor  interaction with some external settings.

Let us also present one more notation (decomposition) of second order tensor when its interaction with some external object is taken into account and which seems not to be widely known nor sufficiently exploited. Let us assume that it is prescribed fourth order symmetric tensor  which may be recognized as physically representing properties of some linear elastic material, i.e. Hooke's tensor. Then, the problem for eigenvalues (eigenstresses) of this tensor can be solved, i.e. roots can be found of so called *characteristic equation* for . The characteristic equation of fourth order symmetric tensor takes the form of six order polynomial equation with real coefficients. In the most general case the solution of characteristic equation is composed of 6 different eigenvalues  and corresponding to them six different eigentensors (). It has been proved by Rychlewski cf. for example [23], that eigentensors  of  corresponding to different in value eigenvalues  are mutually orthogonal and generate the whole space of second order symmetric tensors, i.e. set  make a basis of this space, and the set  can be adopted as basis of ,



where  are so called *Kelvin moduli* ( in view of physical requirement of semipositiveness of Hooke's tensor),  are corresponding to Kelvin moduli eigentensors,  denotes unit tensor in the space of fourth order symmetric tensors . The decomposition 3 of tensor  − with only diagonal components having nonzero values, is called *spectral decomposition* of Hooke's tensor.

Taking advantage of the following decomposition of stress tensor can be obtained,



Notation 1 is also called *energy-orthogonal decomposition* of stresses space for a *given elastic body* , in view of 4. This decomposition is *unique* when all the Kelvin moduli  have different values. For fixed tensor , the six components  are scalar invariants of stress tensor , in standard sense that they do not change when the reference frame  is changed (rotated). It may also be said that  are invariants of  in interaction with  . The parts  can be classified as tensorial invariants of . The decomposition 1 changes when tensor  changes. Detailed exposition of this issue can be found in original papers of Rychlewski, see e.g. [23], [24].

It was already mentioned that basis of tensor space can be freely selected. Natural desire exists to select such basis of tensor space  as to make analytical and/or numerical computations executed on components of tensors, possibly simple and/or effective. Let us list some convenient tensorial bases − coordinates frames, useful in the further discussion,

i) Laboratory basis.

Such a basis set of dyads , and corresponding to it coordinates frame − set of versors , is selected for example in view of convenience in expressing imposed boundary conditions, prescribing loadings or constraints.

ii) Symmetry basis.

Such basis and corresponding to it coordinates frame is selected to be collinear with axes of some kind of material symmetry or geometrical shape/layout of examined engineering structure/device. For example, it is preset to be collinear, consistent with natural axes of symmetry of (anisotropic) material of which engineering device is made.

iii) Principal axes basis.

Such a basis set of dyads , and corresponding to it coordinates frame − set of principal directions vectors , is selected when it is subject matter justified or convenient to work with principal values of second order symmetric tensor. In 1920 Haigh [8] and independently Westergaard [32] in search of the best manner to describe strength of materials intuitively assumed that for isotropic, linear elastic materials Euler angles of stress tensor (loading) should not influence material strength and can be neglected. Basing on this conjecture, they proposed to introduce three dimensional *principal values vector space* with orthogonal coordinates frame composed of principal directions of stress tensor, cf. notation 3. The principal values of stress tensor  are Cartesian coordinates of points in this space. The *principal values space* was coined the name *Haigh-Westergaard space*, see e.g. p. 14 in Maugin [14].

iv) Eigentensors (eigenstresses) basis.

Orthonormal, eigenstresses bases composed of sets  − cf. , are convenient for example in constructing models of materials effort (plastic yield, cracking, etc.). Let us now recall a definition of *isometric* bases. Two orthonormal bases are called *isometric* with respect to a group of *proper orthogonal rotations*, if such an orthogonal tensor  exists () – cf. e.g. [21], that

.

It is worth noting that *all* orthonormal bases in 3D Euclidean space are isometric.

An important property of orthonormal bases  is that, in general, they are *not* *isometric* with orthonormal basis  − cf. , i.e. there does not exist such an orthogonal tensor , which will transform basis  into basis . Let us take, for example, a set of second-order tensors,



It is easy to verify by making direct calculations that tensors  are the *eigenstates of the isotropic Hooke tensor* , i.e. they are solution of the eigenvalue problem  cf. , with eigenvalues ,. The set  makes an orthonormal basis of symmetric second order tensors space  and it *is not isometric* with orthonormal basis , cf. . The tensor  is a spherical tensor and the other tensors  are deviators.

The isotropic[[3]](#footnote-3) tensors have *identical* *representation in all isometric orthonormal bases* but isotropic tensors *do not have the same representation in all orthonormal bases*.

# 3 Various sets of useful second order symmetric tensor invariants (eigenproperties)

*3.1 Characteristics (eigenproperties) of second order symmetric tensor and their physical interpretations*

The second order symmetric tensor is fully characterized by six parameters – its components in some basis, as it was recalled in the previous section. Usually the values of components are known from experimental or numerical tests. Infinite number of invariants can be constructed out of them but always maximum three are linearly independent. The common set of second order tensor invariants most frequently encountered in mathematical studies are so called *basic invariants*, see e.g. Spencer [28],



where  denotes *modulus* *(norm) of a tensor*,  denotes unit tensor (also called indentity tensor) in second order tensors space − in notation 2 diag[1,1,1,0,0,0] or in notation 3 [1,1,1]. The dot symbol denotes full (double) contraction of second order tensors .

The popularity of basic invariants − in some publications also called *main invariants*, comes from *computational (numerical) effectiveness* of their determination, which requires only multiplication of tensor matrix representation for which very effective numerical algorithms exist.

In continuum mechanics alternative set of linearly independent invariants so called *principal values*  of second order symmetric tensor gained popularity and is in widespread use. The reason for that is their physical interpretation, e.g. in the case of stress tensor they proved to be handy in relatively simple description of the *effort state* of engineering material submitted to specific stress loading. The principal values are determined as roots of so called *characteristic equation* for principal values,



where  are principal directions (eigenvectors) of a tensor , second order unit tensor in notation 1 takes the form , and 3 . Exclamation symbol  means no summation over index . It is adopted naming convention here that . The following denotations were introduced in ,

,

symbol det(…) denotes determinant operation,  is mean value of principal values, and  is permutation symbol. Upon, substituting principal values  consecutively into characteristic equation 2, multiplying it by corresponding eigenvector , taking advantage of the relations 3 for powers of , and summing up three such obtained identities, the well known Cayley-Hamilton equation can be recovered,

,

The set of three coefficients appearing in characteristic equation for determination of principal values of second order tensor  are called *principal invariants*. In the case of symmetric second order tensor all principal values are real, and when they are unique then three principal directions make an orthogonal triad, cf. e.g. Odgen [20].

*3.2 Decomposition of second order tensor into spherical (isotropic) and deviatoric (anisotropic) parts and their attributes*

Any second order tensor  can be decomposed into *spherical* and *deviatoric* parts – they are mutually orthogonal, cf. also Figure 1,



where  denotes spherical part and  denotes deviatoric part of a tensor .

The decomposition means that sum of any two spherical tensors  always gives spherical tensor and sum of any two deviatoric tensors always gives deviatoric tensor. Thus, space of second order symmetric tensors can be divided into two separate (orthogonal) subspaces. Decomposition 1 at the same time makes division of the tensor into *isotropic part*  and *anisotropic part* . The spherical part of the tensor is isotropic in conventional sense, i.e. it does no change under application of any orthogonal tensor , where  is a group of all orthogonal tensors, cf. Supplement at the end of the present work. A set  of all tensors that can be obtained by transformation of  with any orthogonal tensor  is called an *orbit* of tensor ,



In analogy to characteristic equation for principal values of complete stress tensor cf. 2, there can be formulated characteristic equation for eigenvalues of tensor deviator  only, coefficients of which makes a set of *principal invariants of tensor deviator*  defined as follows,



where  denotes tensor deviator. The opposite sign in definition of second invariant of deviator  in comparison to definition of second invariant of full tensor  assures that it is always nonnegative. The  invariant gained widespread use due to its physical interpretation of *shear stress intensity* measure.

Two tensors  are *coaxial* when they have the *same principal directions*. It can be shown that necessary and sufficient condition for coaxiality of two tensors is that their single *contraction products* *commute* (). It can be shown by direct calculation that the tensor and its deviator products commute . Hence, they have the same principal directions .

The characteristic equation for principal values of deviator 1 can be solved upon substitution for  and taking advantage of the following trigonometric identity , to obtain explicit formulas for stress deviator principal values , cf. also page 92 in Malvern [12]. Then, stress tensor and its principal values can be expressed in the following form,



where  denotes so called *effective stress*,  is called *Lode angle*, and the term  is called *normalized third invariant of deviator,* .

Let us return to stress tensor anisotropy feature finding source in its deviatoric part only. Rychlewski in [25], [26] proposed that in order to quantitatively evaluate magnitude of the tensor anisotropy, it is appropriate and convenient to employ the concept of diameter of tensor orbit, cf. 3. He defined the *diameter of tensor orbit* as maximum distance between any two members in the orbit of tensor  − in the sense of tensorial norm cf. 2. It can be expressed as follows,



where  denotes diameter of the tensor orbit,  denotes distance generated by usual tensorial norm,  denote any two tensors in the tensor orbit,  is any orthogonal tensor.

Rychlewski proposed the measure of tensor anisotropy, which he called *degree of anisotropy*, to be expressed with the following formula, cf. relation (6) in [26],



Here, Rychlewski's degree of anisotropy is called *anisotropy factor* and is denoted by . The definition of anisotropy factor is actually applicable to tensors of any degree.

Rychlewski has proved that diameter of orbit of second order symmetric tensor is equal to  and next taking advantage of this, by far not obvious result, he showed that anisotropy factor can be expressed in the following form, cf. also formulas (32), (37) in [26],



where  denotes maximum shear stress of the tensor **. It is clear from the above that anisotropy factor  is still another *invariant* of tensor **, and taking it formally makes a fundamental measure of sensitivity of the tensor  to rotations.

When passing from vector space − natural model of real 3D physical space, to spaces of higher order tensors many "obviously true" intuitive feelings from standard vector analysis fails. For example, while full tensor and its deviator are collinear − in tensorial sense, their vector representations utterly correctly obtained from methodological standpoint are not parallel, cf. Figure 1. The collinearity of tensors *does not* translate into parallelism of vectors. The same failure of intuitive feelings one experiences when dealing with measure of tensor orbit diameter . The origins of this second difficulty can be at least partially, intuitively grasped when one realizes that e.g. the distance between unit tensor and any rotated unit tensor is zero − in the sense of tensorial norm, more generally between any isotropic tensor and its orthogonally rotated instance. The direct cause of this reality is that in any rotated coordinates frame unit tensor  has the same values of components (representation), namely . This effect is difficult to agree on with our everyday experience because there *do not exist* isotropic vectors – actually the only vector which components do not change under rotation of basis is zero vector.

The invariants  can be expressed in terms of general components of stress tensor and in terms of its principal values as follows − cf. also ,

 

Fig. 1 a) Graphical illustration of structure of direct sum (orthogonal) decomposition of second order symmetric tensor into spherical (isotropic) part and deviatoric (anisotropic) part  in Haigh-Westergaard (H-W) principal values space, b) Graphical illustration of elements involved in isomorphic – Murzewski, cylindrical coordinates, cf. Sect. 3.3.





The following relations are valid for *basic*, *principal* and *deviatoric* invariants , , ,



Relations 2,3 can be straightforwardly obtained upon double (full) contraction of Cayley-Hamilton equation with unit tensor − taking trace of it. The Cayley-Hamilton equation written down for deviator leads directly to the relation .

*3.3 Sets of coordinates based on stress tensor invariants*

Every set of three linearly independent invariants may be adopted as a system of coordinates in 3D stress principal values vector space – *Haigh-Westergaard (H-W) Space*, in place of Cartesian system of principal values coordinates. Such action actually involves modification of not only coordinates but also the basis vectors (curvilinear in general), which after that are usually not anymore stress principal directions. A very popular set of coordinates system in H-W space are (cylindrical) coordinates − pressure, effective stress, Lode angle. They are used to present plastic flow yield, damage, failure or phase transition critical surfaces for different materials in octahedral  and/or meridional  cross sections – frequently  coordinates are also encountered. The problem with the mentioned coordinates is that they distort 2D projections shapes of actual 3D shape of the critical surfaces.

According to the present author literature survey, Murzewski was the first researcher who in his work from 1960 − cf. [16], consciously introduced *isomorphic cylindrical coordinates* in Haigh-Westergaard space, i.e. coordinates preserving correct shapes (distances and angles) of critical surfaces in H-W space cross sections. This system of coordinates is as follows − cf. Fig. 1,



Murzewski in his work [16] used the following denotations .

Stress tensor and its principal values expressed in Murzewski coordinates take the form,



where second order tensors  are defined in 3. It is worth pointing out that in H-W space *infinite number of non-coaxial tensors* having the same principal values but different orientation of principal axes reduces to a *single point representation[[4]](#footnote-4)*, what actually means that a kind of strong filter operates in H-W space.

*Tensorial decomposition* of tensor  into spherical (isotropic) and deviatoric (anisotropic) parts 1 should be carefully distinguished from *vectorial decomposition* of *octahedral traction*  into *octahedral normal stress* **** and *octahedral shear stress* ****, which values can be expressed in terms of principal values as follows,



An alternative isomorphic coordinates for meridional cross sections in Haigh-Westergaard space is pair of variables . These coordinates were already used in 1929 by Burzyński − with denotation , in his work devoted to formulation of extended plastic yield strength criterion for linearly elastic, isotropic solids taking into account influence of pressure (first invariant), cf. formula 12 in [3]. Burzyński in this work correctly disregarded Lode angle influence on such criterion, as it cancels out from expression for elastic energy in the case of isotropic, linearly elastic solids, what is clearly expounded here below.

The stress tensor invariants ,  and  present in have well known clear physical interpretations of pressure (with negative sign), shear magnitude (norm) and *total shear effort* (intensity) of the material. The Lode angle  as yet *does not have clear physical interpretation*. From mathematical standpoint, Lode angle describes angle between projection of stress tensor (vector) and projection of principal axis I versor (corresponding to the greatest principal value of stress tensor) on octahedral (deviatoric) plane, cf. Figure 1. It is easy to show by direct calculation that the following formulas are valid,

,

The above expressions show that ordering of principal values actually does not have influence on effective definition and/or interpretation of Lode angle.

Formulas for Lode angle in terms of principal values of deviator are inconvenient because principal values of deviator have to be computed first – a costly operation, before Lode angle value can be determined. Formulas for Lode angle expressed in terms of stress invariants − cf. 7, are much more convenient numerically. The present author in his historical survey found Novozhilov's paper from 1951 to be the earliest publication in which third invariant of deviator is explicitly expressed in terms of trigonometric function, cf. formula (1.13) in [19]. Actually, Novozhilov angle  is defined not with cosine but with sine function . The  is just negative of skewness angle introduced here below with 1 .

Summarizing the present state of the art review regarding eigenproperties of second order symmetric tensors, it can be stated that the tensor treated conceptually as oriented object exhibits fixed characteristics described by the tensor invariants. A separate feature is orientation of the tensor in space − laboratory frame, described with its Euler angles. Depending on the internal symmetries of the tensor, its specific orientation can lead to appearance or not of certain (directional) effects and can influence their magnitude. Such directional sensitivity of a tensor becomes fully exposed only upon its interaction with other tensorial objects, which takes place through Euler angles characterizing uniquely mutual relative orientation of two tensors. Similar situation exists in the case of colliding objects, when it is important whether one object collides with its front or with its back with the other object and under what angle. It is worth to indicate that isotropic tensors are insensitive to their directional orientation in space, if at all it can be said that they have one. Examination of very interesting aspects of the influence of symmetry of causes on symmetry of effects can be found in the Rychlewski's book [27].

# 4. New structural parametrization of second order tensor eigenproperties – the concept of isotropy angle and skewness angle

Great variety of tensor eigenproperties parameterizations based on invariants is possible, the usefulness and applicability of specific parameterization set depending on particular area of interest. Let us introduce a new set of invariant parameters characterizing second order symmetric tensors. The set seems to be especially convenient because it leads to simplification of formulas expressing tensor properties and thus makes more lucid characterization of described by it real physical phenomena. The new generic structural parameterization transparently and clearly corresponds to internal structure of the tensor object and can also be conveniently used for constructing derivative sets matched to specific applications.

Let us first introduce the concept of *isotropy angle* defined as follows,



 

Fig. 2 a) Illustrative drawing of direct sum decomposition, in Haigh-Westergaard (H-W) principal values space, of second order symmetric tensor into spherical (isotropic) and deviatoric (anisotropic) parts . Segments drawn with green color in octahedral planes mark respective projections of hypothetical critical surface for some *isotropic material* (three projections of principal axes on octahedral plane make symmetry axes of the isotropic material critical surface), e.g. plastic flow yield surface , b) Isomorphic cylindrical coordinates of tensor deviator drawn in octahedral plane;  is skewness angle,  denotes Lode angle.

Graphical representation of isotropy angle is shown in Figure 2a). The isotropy angle enables extraction of spherical (isotropic) part and deviatoric (anisotropic) part of the tensor in very straightforward and convenient manner. The sine and cosine functions of isotropy angle can also be treated as convenient normalized factors (indexes) describing magnitude of spherical and/or deviatoric parts relative to overall magnitude of the tensor (its modulus).

Let us further introduce the concept of *skewness angle* defined as follows,



where  denotes so called Lode parameter introduced by Lode in 1925, cf. e.g. [11].

The skewness angle in mathematical terms describes departure of the actual tensor deviator from corresponding reference pure shear, i.e. deviator with the same modulus but with third invariant equal to zero , cf. Figure 2b). The reasons why pure shear is considered to be appropriate comparison reference, what physical interpretation can be assigned to skewness angle, and why "skewness" is appropriate name for the mode angle  is elucidated in the next section. The skewness angle has very simple (linear) connection with universally at present used in multiaxial stress studies Lode angle, cf. 5. The isotropy angle and skewness angle can be easily normalized to the range , dividing them by  and  respectively. Let us introduce *generic structural parameterization* of second order symmetric tensor in the form,



This set of parameters delivers convenient − a kind of military, structural characterization of the tensor object. The  immediately tells about the overall strength of the entity,  allows to quickly evaluate what is the magnitude of general purpose (non-oriented) forces  and the magnitude of special (oriented) resources , while  informs about a kind of versatility of the oriented resources of the tensor entity. The newly proposed generic structural parameterization can be conveniently adapted for the purposes of specific areas of application to form derivative parameterizations. In particular, already introduced various sets of invariants can be expressed in terms of newly proposed set, e.g. Murzewski isomorphic coordinates 1 or principal values of stress tensor 1, upon uncomplicated operations can be expressed as follows,



Substituting  for  upon 2, and  upon 4 into 1, simple manipulations lead to extremely simple and elucidating formula for anisotropy factor,

.

The first term in formula clearly shows that anisotropy degree of second order symmetric tensor grows with growing fraction of its deviatoric part, reaching maximum for tensors being pure deviators (). The second term shows that *the most anisotropic deviators* are *pure shears* (). The anisotropy factor decreases with deviatoric part departing from respective comparison pure shear mode. In the case of pure deviators it drops from 1 to a minimum value of  reached at uniaxial tension or at uniaxial compression .



Fig. 3 a) Graphical illustration of variation of anisotropy factor in dependence on isotropy angle  and skewness angle , b) Graphical illustration of variation of anisotropy factor in octahedral *π* plane .

Proposition on how to explain the reasons for this rather puzzling behavior of anisotropy factor diminishing with departure from pure shears is discussed in section 5 below.

# 5 Special character of pure shear mode – elementary unit of microstructure of deviators

In order to recognize the physical interpretation of the skewness angle defined through normalized third invariant of deviator  − cf. , let us give some thought to the problem of what is the most elementary (atom) non-trivial form of second order tensor. It right away comes to mind that it is the tensor, which has single nonzero entry on diagonal in its matrix representation , the option of single nonzero off-diagonal component being excluded due to symmetry requirement. Such representation has for example the uniaxial tension (extension) and/or uniaxial compression tensors. However, upon further reflection it can be realized that uniaxial tension tensor is not as simple as it seems, and in fact several elemental (atom) components can be distilled from it along the lines of deviatoric decomposition 1. The most elementary component of uniaxial tensor that can actually be identified as irreducible to more simple modes, is the spherical tensor, *spherical elementary mode*, having three identical in value diagonal components . It can be physically interpreted as describing the simplest 3 dimensional layout of action of forces in physical space, i.e. forces acting uniformly in all three physical directions or alternatively 3D kinematics of displacements taking place uniformly in physical space, describing volume change. The remaining deviator of uniaxial tensor on the other hand proves to be always decomposable, in general in *infinitely many ways*, into two the most elementary irreducible deviator modes involving so called *pure shears*, for example +. The pure shear, *deviatoric elementary mode*, can be physically interpreted as the most simple two dimensional (plane) layout of action of forces in physical space, i.e. forces operating uniformly in all parallel planes having fixed common normal axis (alternatively two dimensional kinematics of displacements taking place uniformly in planes with common normal axis, and proportional to the distance from some fixed plane, similarly like it is in the case of sliding tile of cards).

Let us recall after Blinowski and Rychlewski [2] precise mathematical definition of pure shear. A second order tensor  is called a *pure shear* when the following conditions are fulfilled, some other equivalent definitions can be found in original publication [2],



Fig. 4 Graphical illustration of pure shear tensor  shown in two bases (coordinate frames) rotated by 45 degrees, which result in two very characteristic pure shear tensorial representations.

.

Depending on the selection of the basis, the following two very characteristic, easily recognizable, tensor representations of pure shear can be specified,



where versors  are called *shear directions*, and plane determined by the pairs  or  is called *shear* *plane*. The line along versor  or  is called *shear* *axis*. It is clear that pure shears are planar tensors, cf. Figure 4.

In accordance with the above nomenclature there can be distinguished two very useful classes of pure shears, namely shears with *common shear direction* and shears with *common shear axis*. Two parameter family of pure shears with common shear direction  and family of pure shears with common shear axis  can be expressed in the following mathematical form,



Thus, all possible pure shears having common shear direction  parallel to axis  can be generated with freely selected vector  orthogonal to direction , cf. 1. All pure shears with common shear axis  parallel to axis  can be generated with arbitrarily selected vector  orthogonal to shear axis  and vector  mutually orthogonal to vectors  and , cf. 2.

The pure shears prove to be excellent modeling idealizations of many commonly encountered, actual physical situations. For example *uniform plastic slip* deformation can be understood in modeling terms as a group of pure shears with common axis, while the formation of *compound martensitic twin* can be understood as a pair of two pure shears with common shear direction. Experimental setups leading to pure shear stress or strain are very frequently used in experimental mechanics to determine material properties. This issue is discussed here in more details in section 8.

Blinowski and Rychlewski demonstrated in [2] that population of all pure shears generates complete subspace of all deviators. They also proved that *any deviator*, *in* *infinitely many ways*, can be decomposed into a sum of *two orthogonal pure* shears, cf. formula (3.9) and accompanying text. Hence, pure shears can be regarded as elementary building blocks of deviators subspace. However, pure shears themselves do not create a linear subspace because sum of two pure shears is not always a pure shear. It is worth noting that sum of whatever number of pure shears will never result in spherical tensor. All pure shears have the same "shape" in this sense that any and all pure shears can be obtained from arbitrary, preselected unit pure shear  by its proportional scaling  and rotating . It is interesting to note that strictly non-orthogonal bases – any two basis tensors are not orthogonal, and/or strictly orthogonal bases of 5 dimensional subspace of deviators can be created composed of pure shears only, cf. e.g. page 488 and page 498 in [2]. Blinowski and Rychlewski in their pivotal publication [2], which rather opens up than terminates research devoted to pure shears, reveal many more attractive properties of pure shears.

The above discussion indicates for special role played by pure shears, which can be regarded as two dimensional (plane), irreducible basic modes (atoms) of shearing making component parts of any deviator. This feature justifies their use as comparison reference elements for any other deviatoric mode of second order tensor, and introduction of skewness angle notion as appropriate quantitative parameter characterizing departure of specific given deviator from its comparison pure shear. An attempt to explain in what physical sense is such difference and what are the other consequences of it is exposed in the following section.

# 6 Statistical interpretation of second order symmetric tensor invariants, skewness angle as measure of entropic part of stress tensor anisotropy

It is interesting to note that very simple and straightforward connections exist between principal invariants of deviator  and quantities known as statistical central moments , namely,

.

Please note that also in statistical sense there exists "orthogonal" decomposition of the tensor  into spherical and deviatoric part in this sense that for spherical part only the first central moment is different from zero and all the remaining central moments are equal to zero , while for deviator part it is .

Substitution of the relations 1,2 into the formula for *Fisher-Pearson* *skewness coefficient* – cf. e.g. formula (20...) in Polyanin and Manzhirov [22], and comparing it with expression 1 reveals existence of the following connection,

.

The connection 2, delivers grounds for assigning the name "skewness angle" to the angle defined with formula 1. At the same time this relation supplies lead for revealing one more unexpected physical interpretation of skewness angle of statistical character, which allows for explanation of the mysterious, at first sight, reduction of anisotropy degree of the tensor  with increasing departure of its deviator from pure shear – cf. Figure 3 and accompanying text.

In statistical literature there exist very well known interpretations of central moments. The first is understood, linked with the mean value of the population of objects. The square root of the second central moment is called standard deviation and is understood as describing the magnitude of scatter, or magnitude of non-uniformity, or disorder of the population around its mean. The third central moment normalized with standard deviation is understood as describing the non-symmetry or "skewness" of the population towards the left or right wing of its statistical distribution.

It seems that these classical understandings of central moments must be modified to take into account the specific situation of applying them to deviator of a tensor, i.e. quantity which first central moment is always equal to zero. It seems that the classical interpretations should be shifted by one in view of zeroing of the first central moment of deviator, i.e. the second central moment of the deviator should be interpreted as the mean value, and the third central moment should be interpreted as scatter or disorder of the population about the mean. This proposal finds support in Novozhilov's work [18] in which he demonstrated that second principal invariant, which is linearly proportional to second central moment of deviator () is proportional to average shear stress of tensor  calculated over all directions on unit sphere,

,

where  is average shear stress over all possible directions on unit sphere, also called by Novozhilov *shear stress intensity*,  is shear stress operating on elementary surface  of unit sphere,  are principal stresses,  are direction cosines determining orientation of normal to surface  in relation to principal directions of tensor .

Pursuing the above line of approach, it is shown that the normalized third invariant of tensor deviator  can be assigned an extra interpretation of *standard deviation of "directional dipoles"*, i.e. parameter describing orientational (directional) spread or disorder of population of elementary shears around their average value . For that purpose, let us first indicate that whenever some directional entities has influence on some total (macroscopic) orientational property − an analogy with magnetic and/or electric dipoles immediately comes to ones mind, then the more ordered are directional units the bigger is the overall orientational effect. As we have already indicated in the previous section the tensor deviator can be treated as macroscopic parameter describing overall (average) action of population of pure shears ("directional dipoles"). A good measure of such overall directional effect of action of population of pure shears is maximum shear stress . Novozhilov has shown that the following relations are valid, cf. formulas 2.1-2.6 in [18],

,

Inequalities 3 show that the ratio of maximum shear stress to average shear stress  () attains maximum value − attains maximum directional effect, for pure shear mode and minimum value for uniaxial tension or uniaxial compression.

In view of the presented above argumentation this effect can be explained when one accepts that population of micro shears generating macro pure shear mode is the most ordered directionally and population of micro shears generating macro uniaxial tension/compression mode is the most scattered directionally. It is known from thermodynamics that good measure of the degree of internal ordering of any system is entropy.

***Corollary*** Decrease of the value of anisotropy factor of second order tensor with departure of its deviator from pure shear mode – growth of absolute magnitude of skewness angle, can be attributed to increase of *internal entropy of the tensor*. This last understood as growth of orientational scatter in population of micro pure shears generating specific mode of the tensor deviator.

The above gives grounds to call the term  *entropic part of tensor anisotropy*, while  can be called *deviator modulus part of tensor anisotropy*.

The classical interpretation of the third central moment also does not lose its validity. This is so because bias (skewness) of micro pure shears statistical distribution generating specific macro stress state can be identified to be the reason for shifting direction of its deviator from direction of respective reference pure shear mode  towards either direction of the projection of the first  or the third  principal stress on octahedral plane. Speaking otherwise, although different pure shears are activated when generating uniaxial tension mode and different ones when generating uniaxial compression mode their directional scatter about mean value is the same.

The present discussion can be concluded with the statement that the value of skewness angle delivers twofold information. Firstly, it informs about the magnitude of *internal entropy* of the tensor finding reflection in the value of anisotropy factor − the greater the internal order the bigger the value of anisotropy factor. Secondly, it informs about skewness of the population of micro pure shears generating examined macro stress mode, what finds reflection in departure of the macro stress deviator direction from the direction of corresponding pure shear mode on the octahedral plane.

# 7. Stress tensor in interaction with external system (environment)

*7.1 Independence of linear elastic isotropic material elastic energy from skewness (Lode) angle*

In the previous sections stress tensor was considered as autonomous object. However, it is examination of its interaction with other tensorial objects, which allow for effective modeling of real physical phenomena. One of the classical problems of mechanics, still open and arousing interest of many researchers is formulation of criteria of material strength. Such criteria are commonly proposed employing the concept of *elastic energy* stored in the material macroelement under consideration. In order to obtain expression for elastic energy of macroelement, interaction between stress tensor and fourth order Hooke’s tensor – describing elastic properties of the material under consideration has to be considered. The most simple case of Hooke's tensor, actually the most commonly encountered in continuum mechanics literature and making starting point for any more advanced studies, is the tensor describing properties of *linear elastic, isotropic material*. This tensor leads to the following linear constitutive relations between stress and strain tensors,



where  denote isotropic stiffness and compliance tensors of elasticity,  denote Lame constants,  is called shear modulus,  are Young and Bulk modules,  denotes Poisson's ratio, are strain and strain deviator, respectively,  is fourth order symmetric unit tensor.

Relations 4,5 correspond to relation 1 upon its decomposition into spherical and deviatoric parts. Taking the dot product (full contraction) of stress  − 3, and strain  − 2, leads to the following expression for elastic energy stored in unit volume of the linear elastic isotropic material ,

.

There can be formulated two important remarks ensuing from formula .

Remark I. *Elastic energy of linear elastic isotropic materials decouples into two parts*, first depending on pressure and second depending on shearing part of stress tensor, only – or speaking alternatively first depending on first invariant of stress ** and second depending on second invariant of stress deviator **, only. The mixed term is not present. When the energy is expressed in terms of small strains tensor then the parts are connected with volumetric and distortional parts of strain tensor, respectively.

Remark 2. *Elastic energy of linear elastic, isotropic material* *does not depend on skewness angle  (Lode angle )* – *it* *does not depend on third principal invariant of stress tensor deviator *.

It is worth to examine in detail, how it occurs that Lode angle ** (skewness angle**) cancels out from the expression for elastic energy. For that purpose the formulas 1-3 are instrumental,



While the formula is very common knowledge, the present author has not encountered in the literature explicitly formulated statement similar to that in Remark 2. Probably because being so obvious, it very often escapes attention or it is somehow forgotten.

In numerous works devoted to more advanced materials research, it is assumed linear elastic, isotropic constitutive relation for the investigated material behavior in order to search in subsequent steps for "elastic energy criterion" of material effort, which contains Lode angle as argument. In view of Remark 2 such approach leads to methodological inconsistencies, at the best. Their removal requires in each specific case clearly formulated and well justified additional assumptions, usually missing at present. What factors then can be identified to be responsible for very often encountered in experimental works dependence of e.g. critical stress of plastic yielding on skewness (Lode) angle, besides at the same time material exhibiting with acceptable approximation linear elastic and isotropic behavior. There can be identified at least three such causes: i) material is actually *not linear elastic*, ii) material is *not isotropic*, iii) so called, *internal constraints* operate in the material – of force or kinematic character, of known or unknown physical origins. The first factor (i) can be identified to be the primary reason, why it is a standard that elastic energy functions proposed for rubberlike and/or polymeric materials as potentials for derivation of their constitutive relations are proposed to be functions of all three principal invariants of strain tensor. Thus, it is rightly assumed that their elastic energy depends on skewness (Lode) angle of strain tensor. Actually, the second factor also plays a role in polymers. Elasticity in polymeric materials is physically generated by change of internal entropy of these materials and not internal energy, cf. e.g. Müller [17] pp. 111-112. Due to that polymeric material, even when isotropic at zero loading, changes its internal symmetry usually into transversely isotropic one when loaded to moderate strains. It returns to original symmetry (isotropy) upon removal of loading. The second factor (ii) does not require additional comments. Typical situation when the third factor (iii) becomes important is in the case of for example composite materials in which there are present some kind of reinforcement elements.

*7.2 Mechanism of formation of six invariants of stress tensor*

Let us return to the eigenstates problem and Rychlewski's energy orthogonal decomposition of stress tensor [24], cf. formula and accompanying text. The discussed above case of linear elastic, isotropy is a special case of interaction of stress tensor with Hooke's tensor, which resulted in fully decoupled decomposition of stress tensor into two parts spherical–deviatoric (pressure–shear). In the most general case of linear elastic, anisotropic material stress tensor can be decomposed into six parts, energy orthogonal in the sense 4. This sextuple is independent from any vector basis − reference frame . Hence, its components actually make *six linearly independent stress tensor invariants*. We may notice that in this manner the very basic feature underlying the tensor notion − i.e. invariance, manifested itself in full light. The statement about six invariants may seem contradicting with earlier statement of only three linearly independent tensor invariants that can be generated, but it is not as can be realized from the commentary below.

A very interesting loop has been encircled. Ricci-Curbastro motivated by the idea of quadratic forms invariance devised objects − and the whole mathematical apparatus, which predicts that in the case of second order symmetric tensor its representation six components transform in a specific manner with change of reference frame. Next, it was identified that from these six components there can always be formed a set of *three linearly independent invariants* independent from the coordinates frame, and a set of another three parameters changing with coordinates frame change. This, when tensor is considered an autonomous object – an analogy of free vector comes to mind. When the tensor is considered in some environment, in interaction with other tensors then *six linearly independent invariants* can be formed out of its representation components – and analogy of anchoring of the vector to a fixed reference point (frame) comes to mind.

Let us consider the following situation in order to better understand in what sense anchoring of the tensor takes place. Take two autonomous tensors, e.g. stress tensor  and not coaxial with it strain tensor  − a typical situation for non-isotropic materials. Each of these tensors is fully described by three invariants and three Euler angles. The Euler angles characterize orientation of each tensor with respect to any conceivable reference frame. While these angles change with change of reference frame the *relative orientation* of specific stress tensor with respect to specific non-coaxial strain tensor does not change. Upon their interaction − e.g. taking their scalar product, only their relative orientation is important, what manifests itself in possibility of generating six invariants as indicated by formula 1. So, anchoring of the tensor means that orientation of the principal axes of first or the second tensor takes over the role of reference frame − and any other reference frame is not needed − does not play any role.

Rychlewski's energy orthogonal decomposition resulting from eigenvalues problem delivers yet another very inspiring and prolific hint for research works. It indicates that when with initiation of some physical phenomenon e.g. plastic yield flow there can be associated some fourth order tensor such as e.g. Hooke's tensor in the case of elastic energy plastic flow yield criterion, then loadings inducing the phenomenon can be divided, in the most general case, into 6 *classes of loadings*, depending on specific symmetry of the Hooke's tensor. This suggests, for example, that when safety of a structure is to be assured to preclude plastic yield flow then for each identified class of loadings different value of safety coefficient can be expected to be appropriate. Discussed in this section issues can only be signaled due to space limitations. More detailed treatment of interaction of second order tensor with environment − other tensors, and consequences thereof deserve and require separate research efforts and works.

# 8. Some remarks on conditions and applicability of biaxial (planar) loadings for experimental examination of the influence of skewness (Lode) angle on materials behavior

*8.1 Simple shear versus Planar shear experimental testing layouts*

Considerable attention has been devoted in the present work to the theoretical issues connected with pure shear mode. Let us discuss at present two major experimental layouts leading to actual physical realization of the pure shear mode as defined by , i.e. so called "simple shear" and "planar shear" experimental testing layouts. In experimental mechanics, they are usually considered in terms of strain rather and not stress tensor. When tested material is isotropic then this distinction bears no conceptual difference because in such a case simple equivalence of states exists between stress and strain tensors due to their coaxiality. A lot of misunderstandings exist in the literature regarding difference between simple shear versus planar shear testing layouts. Both these testing layouts belong to the class of biaxial tests and make practical realization of pure shear state. In order to clarify the issues, it is pointed out that no difference between planar shear and simple shear exists in terms of definition , as in both cases trace and determinant of strain tensor during experimental testing are with very good accuracy equal to zero . In the case of testing materials for which one or both of these conditions are not fulfilled variation of specimen thickness is additionally measured in order to introduce relevant corrections. The difference exists between the kinematics of two testing layouts, i.e. motion of material points. While different kinematics results in different deformation gradients, the principal values of stretch tensor   are exactly the same in both layouts, though differently situated in laboratory frame, cf. Figure 5. Here, only the most important characteristics of simple and planar shear are succinctly and explicitly formulated in order to possibly facilitate taking decision on the selection of one or the other experimental layout for attaining specific experimental research tasks. More detailed discussion of planar shear and simple shear interested reader can find for example in Ogden [20], Ziółkowski [33].

The *major* *matching feature* of simple shear and planar shear testing layouts is *identical* *strain pattern* shared by both layouts, finding reflection in identical values of principal stretches. In the case of simple shear they are: , where  and  denotes so called *shear parameter*. In the case of planar shear principal stretches take the form: , where , When  is equated to  one to one correspondence can be immediately found between  and . In both testing schemes volume is preserved , where  denotes elementary volume in actual configuration and  is elementary volume in initial configuration.

The *major* *distinctive feature* differing simple shear from planar shear is that *principal axes constantly rotate* with advancement of shear loading in simple shear layout but in planar shear layout *principal axes remain constant (fixed)* relative to the laboratory frame at all time.

The simple shear testing layout is very popular (standard) in experimental testing of behavior and/or properties of metallic materials. The planar shear testing layout is very often used (standard) in examination of polymeric materials. Many additional factors, besides strain pattern, may have influence on choosing one layout or the other. For example stiffness of metallic samples prevents early warping of the sample during simple shear testing. On the other hand testing metallic sheets in planar shear scheme might require considerably larger forces in comparison to simple shear scheme of testing. It is also worth to indicate that loadings used in testing of metallic samples as a standard *does not induce change of symmetry of the material*.



Fig. 5 Difference between experimental testing layouts of *simple shear* and *planar shear* is shown graphically. The  denote Lagrangian principal axes,  denotes orientation angle of Lagrangian principal axes with respect to fixed laboratory frame,  denotes logarithmic Lagrangian strain measure,  are principal stretches.

In the case of polymeric materials loadings used in their testing as a standard *do induce change of their symmetry* – due to entropic origin of polymeric elasticity, e.g. initially isotropic polymeric material changes its symmetry to transversely isotropic under testing load. From the above discussion it can be concluded that execution of simple shear and planar shear tests on the same material allows to evaluate the influence of principal axes rotation on the behavior of the material.

*8.1.1 Efficient test for finding out whether linear elastic material is isotropic.*

A very interesting experimental application of pure shear modes is that results of 5 tests in which linear elastic material is submitted to a set of 5 linearly independent pure shear loadings enable to uniquely determine experimentally whether the material is elastically isotropic. The above suggestion is a direct consequence resulting from Theorem 4.1 in Blinowski, cf. [2]. It gives at the same time information what is the minimum number of tests necessary for finding out whether the material is isotropic. Indeed, when linear elastic material is submitted to five tests with pure shear loadings, for example, the ones listed by Blinowski in proof of Theorem 4.1 with the following representations in the fixed laboratory frame,



and in response to these loadings, the shear moduli determined from elaborated experimental data in charts  will show to have the same value . Then, this will prove that the tested material is isotropic, linear elastic. Technical realization of such testing program cannot be further discussed here in detail due to limited space.

*8.2 Deficiency of biaxial (planar) tests for finding out experimentally material behavior sensitivity to skewness (Lode) angle.*

In 1959 Davies and Connelly introduced so called *triaxiality factor*, defined as quotient of stress first principal invariant divided by effective stress  cf. formula (35) in [5]. They were motivated in this proposal by supposition, correct in view of their own and later research, that spherical tension  called by them rather exotically *triaxial tension* has strong influence on the loss of ductility of metals, and the need to have some parameter to describe this effect. The name triaxial tension for spherical tension is rather unfortunate because it gives false impression that not (negative) pressure but general 3D multiaxial stress states are subject of description with this parameter. The triaxiality factor gained considerable attention and use when Wierzbicki and his collaborators pointed out that not only spherical tension (negative pressure) but also Lode angle can considerably influence ductility and other properties of metals. The important issue in this was that in 2005 Wierzbicki and Xue found that in the class of biaxial tests  unique relation exists between Lode angle − normalized principal third invariant of deviator, and triaxiality factor in the form  − cf. formula (8) in Bai and Wierzbicki [1]. Wierzbicki and collaborators adopted slightly modified definition of triaxiality factor than the original one . It seems to be worth introducing the concept of *isomorphic triaxiality factor*, naturally corresponding to *isomorphic cylindrical coordinates*, defined as follows − cf. formulas ,



The class of *biaxial tests* is defined by the condition that always one of the principal values of stress tensor is equal to zero. According to the ordering convention of principal values this could be smallest, middle or the largest principal value  but usually, conventionally it is written that the third principal value is zero, regardless of the standard ordering convention. During whatever kind of biaxial tests, in view of  two control parameters, e.g. two principal values of stress , uniquely determine any set of *three principal stress invariants* fully characterizing properties of stress tensor treated as sovereign object, e.g. . Some other convenient pair of control parameters can be selected, for example . Simple transformations lead to the following relations valid for biaxial stress states, cf. ,



Taking advantage of relation 6, it is easy to show that following inequalities are always valid for any planar (2D) stress state – during whatever biaxial test,



The observation can be formulated in the form of the following ***Property I***:

The *modulus of deviatoric (shearing) part* of any non-zero (non-trivial) *planar stress state* is *always greater* than the *modulus of its spherical part*.

Direct conclusion from the Property I is that *no purely spherical (isotropic) planar tensor exists* or equivalently *the only purely spherical planar tensor is zero tensor*. Thus, it can be stated that in the case of any not trivial planar stress tensor – biaxial tests domain, its shearing part dominates over its spherical part.

Wierzbicki and Xue constraint relation valid for biaxial tests can be expressed in the equivalent form of classical third power polynomial equation,



where Lode angle was replaced by the skewness angle (). This equation can be solved with the same method as the one used for finding stress principal values from characteristic equation, cf. . The solution can be written in the following form,





In the above, standard denotation convention of principal stresses was employed  and the following identities: , , .



Fig. 7 Graphical illustration of biaxial tests domain parameterization in terms of parameters . The  marks some hypothetical convex, critical surface (e.g. plastic yield), for which critical values of effective stress depend on skewness angle. The second quarter of the chart  makes domain of states attainable in standard tests on tubular specimen (with no internal pressure).

Explicit relations linking triaxiality factor and skewness angle , cf. Fig. 6, are *three bijections* (one to one relations) in three sharing edges separate subdomains, which altogether make entire *two parameter* domain (half-plane) of biaxial tests stress states, see also Figure 7.

From relation it can be easily concluded that the constant value of triaxiality factor  corresponds to the constant value of skewness (Lode) angle . However, what are the curves of constancy paths of these parameters in biaxial tests domain (plane) is not obvious. The following Theorem I is helpful in this issue.

***Theorem I***

The *radial lines (rays) coming out from the origin*  of coordinates frame of biaxial tests domain, i.e. half plane – , are lines of constant values of triaxiality factor and at the same time lines of constant values of skewness (Lode) angle .

*Proof*

The radial lines running from the origin can be described as follows,



In the case ,  can take any value, and it is  *q.e.d*.

***Theorem II***

The relations , , resulting from relations , valid for plane stress states, are bijections (one to one relations) in three sharing edges but otherwise separate subdomains of the whole domain of biaxial tests stress states, except on the line , on which  for any value of .

*Proof*

It is straightforward to find that – *skewness angle*  maintains constant value on *radial lines* running from the origin  of biaxial tests domain coordinates frame, – *mean value of stress* maintains constant value on *45 degrees slanted lines*, – *effective stress* maintains constant value on *ellipsoids with centers in the origin* of biaxial tests domain, cf. also Figure 7.

In view of the above, at any specific point of three complementary subdomains of biaxial tests domain, the value of any variable chosen from triple set  can be uniquely determined by the values of two remaining ones. On line  it is  . *q.e.d*.

An important open problem of *experimental mechanics of materials* comprise determination of *critical stress states surfaces* conditioning initiation of some physical processes in materials, for example plastic yield flow, damage, cracking or start of phase transition. The following observation can be formulated taking advantage of Theorems I and II.

***Corollary I***

In the case of *convex critical surface* with the aid of *whatever type of biaxial test,* for any *fixed value of mean stress (pressure)* , *critical effective stress* can be determined for *only single value of skewness (Lode) angle* .

In the case of *convex critical surface* with the aid of *whatever type of biaxial test,* for any *fixed value of skewness (Lode) angle* , *critical effective stresses* can be determined for *only three values of mean stress (pressure)* .

The direct conclusion from the **Corollary I** is that planar (biaxial) tests, among them very common tension (compression)-torsion tests on tubular samples – also the ones with internal pressure, are *not sufficient for executing methodologically correct experimental examination of the influence of skewness (Lode) angle on materials behavior*. This is so because upon executing only biaxial tests no adequate experimental data results can be collected to reliably separate the influence of mean stress and/or skewness angle on the possible variations of critical effective stresses. One value for any fixed pressure and/or three values for any fixed skewness angle are surely rather insufficient for such purpose. This observation delivers a clear incentive for development and use of experimental techniques in which all three parameters characterizing stress state can be independently controlled to induce in the specimen not only planar stress state (2D) but fully three dimensional (3D) stress state loadings. They should make possible determination of critical effective stress, or other critical parameters, for example effective fracture strains in the whole range of skewness angle values at freely prescribed, fixed mean stress.

Relations valid for *biaxial* (*plane) tests* show that in such a case the values of triaxiality factor must always remain in the range , while in general case of *three dimensional multiaxial tests* the triaxiality factor can take any value from the range. In many *experimental mechanics* publications, in which results from *biaxial tests* are presented, there can be noticed values of triaxiality factor exceeding the two third value , which may seem to be *incorrect*. However, experimental observation of *triaxiality factor greater than* rather indicates that *biaxiality condition of test was lost*, and in the sample general (3D) stress state started to exist. This delivers a hint to develop experimental methodologies, in which *triaxiality factor* is used as an effective *indicator of passing from plane state of stress to three dimensional state of stress*.

Relations delivering explicit connection between triaxiality factor and skewness angle  are *very convenient for numerical computations*, because they enable determination of the value of skewness (Lode) angle  from the value of triaxiality factor  *much* *more efficiently* *numerically* than e.g. from Wierzbicki and Xue formula, not to speak about computing it from definition formula what necessitates computation of third invariant of deviator . Selection of the correct subformula is very easy because it can be decided upon *the value of* falling into specific range of values. For example when  then it belongs to the range  hence . The reverse of formulas are not so convenient because in order to select correct reverse subformula to compute  it must be known combination of signs of two non zero principal values of stress tensor .

Since popularization of *triaxiality factor* *η* by Wierzbicki et. al. it started to be very frequently used in charts as argument (governing parameter) – often together with Lode angle in 3D chart, to present experimental results obtained in *biaxial tests* in order to present influence of Lode angle on various properties of metals and other materials. However, the present study shows that t*riaxiality factor* in general *is not convenient operand* to be used *for presentation of experimental biaxial tests results*. This is so because when taken at its face value it contains tangled together information on two in principle linearly independent parameters characterizing stress tensor (loading), i.e.  and . Such entanglement projects to the presented results making them somehow blurred. In the case of *biaxial tests* one to one relation exists between triaxiality factor and skewness (Lode) angle, i.e. *constant value of triaxiality factor corresponds to constant value of skewness (Lode) angle*. Due to that it is advisable to directly use *skewness angle as governing parameter in charts* presenting experimental results from biaxial tests. Possibly, with information indicating the mode of loading: tensioning , mixed  or compressive . In this manner specific information presented in the chart from biaxial test will be delivered in transparent, methodologically unambiguous manner.

Obtained in the present section results deliver sound methodological grounds for rational and effective designing of experimental programs aiming at determination of mechanical properties of complex materials, as they enable, precise evaluation which material characteristics can be acquired in biaxial test and which one can only be obtained in truly triaxial tests.

# 9. Concluding Remarks

In the paper concise historical survey is presented on tensor notion. The survey gives grounds to the view that the key features, which decided that tensors nowadays became the language of all advanced technological sciences, are linearity and invariance. In view of universal use of tensors to model real phenomena in all kinds of applied sciences profound understanding of what the tensors actually are, what their specific features are, and how they mutually interact is of great importance. The present work addresses these issues on the example of stress tensor, generic instance of second order symmetric tensor. It has been pointed out that tensors can be viewed and/or understood from several perspectives as either: algebraic objects, linear transformations or oriented geometrical objects. This versatility might be another feature, which decided on attractiveness of tensors. Attention is focused here on identification and finding possibly best manner of description of eigenproperties of second order symmetric tensors. The results and conclusions obtained here specifically for stress tensor are of general character and *mutatis mutandis* translate to second order symmetric tensors, which may have other interpretation in modeling of real physical phenomena and/or objects.

The executed analysis showed that it is convenient to introduce new parameterization of second order tensor eigenproperties by introduction of the concept of isotropy angle and skewness angle. The first parameter allows for very quick and transparent, at first sight, evaluation/separation of isotropic and anisotropic parts of the tensor, which actually coincides with decomposition of the tensor into spherical and deviatoric part. It is elucidated the special role played by pure shear modes which explains and justifies using them as comparison reference states in definition of the skewness angle. At the same time it gives strong rationale for replacement of Lode angle with skewness angle in characterization of second order tensors, in view of much more clear and comprehensive physical interpretation of skewness angle. A new very simple formula is presented for anisotropy factor of stress tensor, based on tensor orbit notion, expressed in terms of isotropy angle and skewness angle. An original statistical interpretation of principal invariants of tensor deviator is developed, which allowed explaining why anisotropy factor of the second order tensor diminishes with departure of the tensor from pure shear mode. The reason for that can be attributed to growth of internal entropy of the tensor understood as increasing orientational disorder of elementary pure shears population generating given macro stress state. It is shortly outlined that interaction of second order tensor with other tensorial objects, e.g. fourth order Hooke's tensor representing elastic properties of material results in possibility of constructing not only three but six parameters of second order symmetric tensor remaining invariant upon change of coordinates frame (basis). It is indicated that this gives premises for introduction and development of *weighted effective stress* notion, which takes into account interaction of stress tensor with other tensorial object characterizing material in order to improve the classical effective stress notion.

Several observations are presented concerning multiaxial tests targeted at examination of the influence of skewness angle on material behavior. In particular, it is clearly demonstrated that the only difference between two, very popular in experimental mechanics, testing layouts so called simple shear and planar shear tests is that in the first case principal axes rotate constantly with increasing loading while in the second the orientation of principal axes remains all the time constant with respect to laboratory reference frame. The kinematics, deformation gradient , in the two tests is different but principal stretches (strains) in both cases are exactly the same. New original formulae have been derived, which delivers explicit relation between so called triaxiality factor and skewness angle (normalized third invariant of deviator) valid for biaxial tests. It is shown that executing biaxial tests only does not allow for methodologically correct determination of critical surfaces e.g. of plastic yielding, phase transition start or initiation of fracture (effective fracture strain). This is so, because in the case of biaxial tests for any fixed value of mean stress (pressure) only single critical value can be determined of some indicator (e.g. effective stress, thermodynamic force of phase transition, effective fracture strain, etc.) corresponding to only one value of skewness (Lode) angle of critical stress. This situation calls for development of new experimental testing lay­outs enabling independent control of all three invariant parameters characterizing stress loading of test specimen. Such testing layouts should enable determination of critical values of stress (strain) loadings for the entire range of skewness angles  at fixed value of pressure.

In view of rapid developments in computer technology a very strong need and demand can be noticed for development of efficient methods of visualization of not only vector but also second order (and higher) tensor fields. The classical principal axes ellipsoid can be noted down as first attempt of this kind, which completely fails when some principal values are negative. The present study shows that due to very rich structure of second order tensors the task of reasonable visualization of second order tensor fields grasping simultaneously all the characteristics and flavors of these objects is rather desperate. At the same time it shows that the way out from this dilemma is proper structuralization of the second order tensor to single out, construct parameters proper for description of the specific problem of interest, and only later visualization of the fields of such parameters. Otherwise the visualization results may prove to be obscure, incomprehensible and intricate. Proposed here new parametrization of second order tensors eigenproperties can be very helpful in such tasks.

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**Supplement**

In order to make the work as self-contained as possible essential definitions connected with *external symmetry* of tensors are recalled here, cf. e.g. [21].

*Definition S1.* A set of second order tensors  with properties,

 (S.1)

is a group and is called the *group of orthogonal tensors*.

*Definition S2.*A subset of orthogonal tensors for which 

 (S.2)

is a group and is called the *proper orthogonal group* or *rotation group*.

*Definition S3.* A *group of external symmetry* of tensor ( denotes order of the tensor) we call a subset of all orthogonal tensors , which satisfy the following condition

. (S.3)

Tensors  that satisfy condition (S.3) are called *symmetric* with respect to orthogonal transformations .

*Definition S4.* Tensor is *isotropic* when the group of its external symmetry is the whole set of orthogonal tensors, cf. (S.1).

*Definition S5.* Tensor is *hemitropic* (also called *proper*-*isotropic*)when the group of its external symmetry is the whole set of proper orthogonal tensors, cf. (S.2).

Note: The above definitions plainly show that symmetry property is a characteristic of a tensor as integrated entity of basis and representation (components in the basis) and not the tensor matrix of components (representation).

1. Algebraic structure  composed of  treated as a set of points, associated with this set vector space , and operation  defining adding of vectors to points makes an *affine space*. [↑](#footnote-ref-1)
2. *Coordinates frame* and *reference frame* "physically" are both sets composed of some anchoring point and a set of basis vectors . The difference is rather in their functionality. The coordinates frame makes a reference for determination of vector (tensor) components, while the reference frame makes a reference for examination of e.g. motions (kinematics). Naturally, the same pair  can be adopted to be simultaneously coordinates frame and reference frame or different pairs can be adopted depending on the need and convenience in examination of specific problem. [↑](#footnote-ref-2)
3. Definition of *isotropic tensors* is recalled in the Supplement at the end of the present work. [↑](#footnote-ref-3)
4. Note: In mathematical terms H-W space can also be interpreted (understood) as a set of *numerical markers of stress tensors orbits* (*with respect to proper rotations group – special orthogonal group SO3*); every point of H-W space represents one orbit, more information on this aspect interested reader can find in Rychlewski's book [27]. [↑](#footnote-ref-4)