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Research Paper

Material Spin and Finite-Strain Hypo-Elasticity for Two-Dimensional Orthotropic Media

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A constitutive material spin tensor in the case of purely elastic finite-strain deformation is introduced for a two-dimensional orthotropic media using the minimizing principle applied to obtain the reloaded configuration of the material volume. This material spin explains the rotation of the orthonormal vector frame which coincides with the material symmetry axes in the initial configuration of the material volume and uniquely corresponds to a set of these axes in the current configuration although it does not coincide with the latter. The given definition is followed by the exact expression which includes the deformation gradient tensor, unit vectors of the initial material anisotropy axes and their axial parameters. This definition allows obtaining a new variant of decomposing any elastic finite-strain motion onto rigid and deformational parts and introducing the material corotational rate. The latter is used for the formulation of the anisotropic rate-type elastic law in the current configuration based on the strain measure which does not belong to the Seth-Hill family. For isotropic as well as for tetragonal media, the introduced material rotation tensor coincides with the rotation tensor from the polar decomposition of a deformation gradient.

Key words: material spin; material corotational rate; elastic anisotropy; finite strains; hypo-elasticity; material strain tensor.

1. Introduction

The question of obtaining a material spin tensor for the imposed finite-strain deformation comes from the problem of motion decomposition onto rigid and deformable parts, which is well known in the nonlinear continuum mechanics of solids. This problem partially corresponds to defining the objective corotational rate in the study of large elastic-plastic deformations of the anisotropic media in the Euler description with the additive decomposition of strain rate into elastic and plastic parts. Several variants of lattice rotation models under elastic-plastic finite strain deformation are known and based on the introduction of an intermediate configuration [1–5] and finding the corresponding plastic spin. Dafalias [4] emphasised that the plastic spin which is conventionally used for the lattice rotation modelling appears as a conjugate notion to the constitutive spin tensor, so the latter is used in the corotational rate equation of evolution of an internal variable and has to be defined for each variable according to its physical meaning. Mandel [2] suggested the existence of a rotating frame of director vectors with respect to which the rate equations of evolution for all internal variables were written. The corresponding spin of the directors is naturally involved in the above mentioned corotational rates. The present paper deals with a different view of introducing such a constitutive or substructural spin considered with the example of the orthotropic medium. Apart from the above aspects, the possibility of recognizing the rigid motion of material within large deformations is the only way to observe the finite-strain history and write the anisotropic elastic law with the unchanged elastic moduli. The often exploited measures of a "material" rotation, such as the proper orthogonal tensor $\mathbf{R}$ from the polar decomposition $\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$ of deformation gradient $\mathbf{F}$ and its spin tensor $\boldsymbol{\Omega}_{\mathbf{R}} = \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}}$ (the upper dot denotes the material time derivative), or the vorticity tensor $\mathbf{W} = (\mathbf{L} - \mathbf{L}^{\mathrm{T}})/2$, where $\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}$, or the logarithmic spin tensor $\boldsymbol{\Omega}_{\log}$ [6, 7] generally have no relation with the real material substructure rotation. Hereinafter, the dot symbol $\cdot$ denotes the single scalar product which may be written in the component form for examples of vectors $\mathbf{a} \cdot \mathbf{b} = a^i b^j g_{ij} = a^i b_i$ or tensors of order two $\mathbf{A} \cdot \mathbf{B} = A^{ik} B^{lj} g_{kl} \mathbf{e}_i \otimes \mathbf{e}_j = A^i_l B^{lj} \mathbf{e}_i \otimes \mathbf{e}_j = A^{ik} B^j_k \mathbf{e}_i \otimes \mathbf{e}_j$, where $\mathbf{e}_i$ are arbitrary basis vectors, the symbol $\otimes$ denotes the tensor product, $g_{kl} = \mathbf{e}_k \cdot \mathbf{e}_l$ is the component matrix of a metric tensor, which is used for raising or lowing the indices, and the summation is assumed over the repeated upper and lower indices according to the Einstein convention. Here the material symmetry frame means a rigid set of directors (mutually orthogonal for the instance of orthotropic materials) which are uniquely related to the deformed material by a certain rule, even if the mutual orientation of the anisotropy axes changes upon deformation. The material symmetry frame rotation is supposed to be associated only with the motion of the material symmetry (anisotropy) frame and has to be defined in a special way. The approach considered here to recognise the material symmetry rotation requires preserving the type of the material anisotropy under deformation. All materials with a regular structure being deformed elastically without phase transitions like, in many cases, metal single crystals, semi-crystalline polymers and some fibre-reinforced composites are appropriate instances. Although the material fibres directed along the anisotropy axes do not co-rotate rigidly without changing the symmetry under deformation, it is supposed that the orthonormal vector frame corresponding to the anisotropy axes may be uniquely introduced and it will coincide with the material frame at the reloaded configuration obtained from the current state. For the sake of simplicity and clearance of defining the material symmetry rotation, a plane orthotropic media and two-dimensional deformations are further considered. Besides some other examples, the class of such materials contains two-dimensional metamaterials with a pantographic structure [8] and in some cases graphene sheets [9] forming orthotropic walls of carbon nanotubes.

2. Rate form of elastic law

Let the properly orthogonal tensor $\mathbf{Q}^{\#}(t)$ be an unknown tensor of the material symmetry frame rotation, let the unit vectors $\mathbf{e}_i^0$ define the initial axes of the material symmetry (anisotropy axes), so vectors $\mathbf{e}_i^{\#} = \mathbf{Q}^{\#} \cdot \mathbf{e}_i^0$ define the current unit axes of the material symmetry, $\dot{\mathbf{e}}_i^{\#} = \mathbf{\Omega}^{\#} \cdot \mathbf{e}_i^{\#}$, so the tensor of material symmetry spin is defined as $\mathbf{\Omega}^{\#} = \dot{\mathbf{Q}}^{\#} \cdot \mathbf{Q}^{\#\mathrm{T}}$. Let the linear elastic law be written in the current configuration as $\mathbf{s} = \mathbf{C} : \mathbf{x}$ (double dot symbol ":" denotes the double scalar product, so $s^{ij} = C^{ijkl} x_{lk}$ is accounting for the Einstein convention), where $\mathbf{s}$ and $\mathbf{x}$ is a certain pair of Eulerian work-conjugated stress and strain measures, and the fourth-rank elastic moduli tensor $\mathbf{C}$ has the same components in the initial $\mathbf{C}^0 = C^{\#\,ijkl} \mathbf{e}_i^0 \otimes \mathbf{e}_j^0 \otimes \mathbf{e}_k^0 \otimes \mathbf{e}_l^0$ and in the current configurations $\mathbf{C} = C^{\#\,ijkl} \mathbf{e}_i^{\#} \otimes \mathbf{e}_j^{\#} \otimes \mathbf{e}_k^{\#} \otimes \mathbf{e}_l^{\#}$ in respect with the axes of material symmetry. This form leads to the rate form of the elastic law $\dot{\mathbf{s}} = \dot{\mathbf{C}} : \mathbf{x} + \mathbf{C} : \dot{\mathbf{x}}$ and then

$$\mathbf{s}^{\mathbf{\Omega}^{\#}} = \mathbf{C} : \mathbf{x}^{\mathbf{\Omega}^{\#}}, \tag{2.1}$$

where the material corotational derivative of a second rank $\mathbf{T}$ tensor is defined by the expression $\mathbf{T}^{\mathbf{\Omega}^{\#}} \equiv \dot{\mathbf{T}} + \mathbf{T} \cdot \mathbf{\Omega}^{\#} - \mathbf{\Omega}^{\#} \cdot \mathbf{T} = \dot{T}^{ij} \mathbf{e}_i^{\#} \otimes \mathbf{e}_j^{\#}$. The rate form (2.1) is found using the equalities $\dot{C}^{\#ijkl} = 0$ suitable only for the case of the material symmetry frame $\mathbf{e}_i^{\#}$ and followed by the identity $\mathbf{C}^{\mathbf{\Omega}^{\#}} = \mathbf{0}$. For any arbitrarily chosen vector frame $\mathbf{e}_i$, the time derivatives $\dot{C}^{ijkl} \neq 0$ do not vanish for the case of anisotropy due to the $\mathbf{e}_i$ frame rotation with respect to the material symmetry axes. The very close arguments for the inequalities $\dot{C}^{ijkl} \neq 0$ were introduced by Palmov [10], but with the use of the rotation tensor $\mathbf{B}$ which corresponds to the vorticity tensor $\mathbf{W}$ via the equation $\dot{\mathbf{B}} = -\mathbf{W} \cdot \mathbf{B}$, $\mathbf{B}(t = 0) = \mathbf{I}$. So the $\mathbf{W}$ tensor is a spin tensor for the $\mathbf{B}$ rotation, which explains the rotation of a certain small material volume. Variables with the excluded rotation $\mathbf{B}$ are called corotational. In spite of the general similarity with $\mathbf{Q}^{\#}$, tensor $\mathbf{B}$ is not related to the rotation of the material symmetry axes because the small material volume under consideration not only rotates but also deforms simultaneously, according to the Cauchy-Helmholtz theorem. It is worth emphasizing that for any isotropic medium the mentioned rates $\dot{C}^{ijkl} = 0$ vanish for all the chosen vector frames.

The definition of the work-conjugated Lagrangian stress $\mathbf{S}$ and strain $\mathbf{E}$ tensors in the initial configuration was given by Hill [11, 12] and may be written as a condition of fulfilling the equality $\boldsymbol{\kappa} : \mathbf{D} = \mathbf{S} : \dot{\mathbf{E}}$, where $\boldsymbol{\kappa}$ is the weighted Kirchhoff stress $\boldsymbol{\kappa} = J\,\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$ is Cauchy stress, $J = \det \mathbf{F}$) and $\mathbf{D}$ is the stretching tensor. For the current configuration several definitions of the Eulerian symmetric work-conjugated stress $\mathbf{s}$ and strain $\mathbf{x}$ measures are known:

1) based on a corotational rate, $\boldsymbol{\kappa} : \mathbf{D} = \mathbf{s} : \mathbf{x}^{\boldsymbol{\Omega}}$ ($\boldsymbol{\Omega}$ is a spin tensor from the family discussed in [13]),
2) based on the convective rates [14], $\boldsymbol{\kappa} : \mathbf{D} = \mathbf{s} : \mathbf{x}^{\mathrm{Conv}}$,
3) based on the material time derivative, $\boldsymbol{\kappa} : \mathbf{D} = \mathbf{s} : \dot{\mathbf{x}}$, analogous to the Hill definition for Lagrangian measures, where Eulerian strain measure $\mathbf{x}$ is called the material strain tensor [15].

The latter definition together with the objectivity requirement of the tensors $\boldsymbol{\kappa}$, $\mathbf{D}$, $\mathbf{s}$, $\mathbf{x}$ and the properties of their pair-wise contractions $\boldsymbol{\kappa} : \mathbf{D}$ and $\mathbf{s} : \dot{\mathbf{x}}$ leads to the relation

$$\mathbf{s} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{s}. \tag{2.2}$$

Indeed, under the superimposed rigid rotation $\mathbf{O}(t)$ on the current configuration, the next transformations will follow $\boldsymbol{\kappa}^* = \mathbf{O} \cdot \boldsymbol{\kappa} \cdot \mathbf{O}^{\mathrm{T}}$, $\mathbf{D}^* = \mathbf{O} \cdot \mathbf{D} \cdot \mathbf{O}^{\mathrm{T}}$, $\mathbf{s}^* = \mathbf{O} \cdot \mathbf{s} \cdot \mathbf{O}^{\mathrm{T}}$, $\mathbf{x}^* = \mathbf{O} \cdot \mathbf{x} \cdot \mathbf{O}^{\mathrm{T}}$, $\boldsymbol{\kappa}^* : \mathbf{D}^* = \boldsymbol{\kappa} : \mathbf{D}$, $\mathbf{s}^* : \mathrm{d}\mathbf{x}^*/\mathrm{d}t = \mathbf{s} : \dot{\mathbf{x}}$. So for arbitrary rigid rotation $\mathbf{O}(t)$,the latter double dot scalar product takes the form $\mathbf{s} : (\dot{\mathbf{x}} + \boldsymbol{\Omega} \cdot \mathbf{x} - \mathbf{x} \cdot \boldsymbol{\Omega}) = \mathbf{s} : \dot{\mathbf{x}}$ with the spin $\boldsymbol{\Omega} = \dot{\mathbf{O}} \cdot \mathbf{O}^{\mathrm{T}}$. This leads to the equality $\mathbf{s} : (\boldsymbol{\Omega} \cdot \mathbf{x} - \mathbf{x} \cdot \boldsymbol{\Omega}) = \mathbf{0}$ or $\boldsymbol{\Omega} : (\mathbf{x} \cdot \mathbf{s} - \mathbf{s} \cdot \mathbf{x}) = \mathbf{0}$. The above bracketed expression vanishes due to the arbitrariness of spin tensor $\boldsymbol{\Omega}$. Let $\boldsymbol{\xi}_i$ be the eigenvectors and x_i be the eigenvalues of the $\mathbf{x}$ tensor, so the product $\boldsymbol{\xi}_i \cdot (\mathbf{s} \cdot \mathbf{x}) \cdot \boldsymbol{\xi}_j = \boldsymbol{\xi}_i \cdot (\mathbf{x} \cdot \mathbf{s}) \cdot \boldsymbol{\xi}_j$ gives $x_j \boldsymbol{\xi}_i \cdot \mathbf{s} \cdot \boldsymbol{\xi}_j = x_i \boldsymbol{\xi}_i \cdot \mathbf{s} \cdot \boldsymbol{\xi}_j$ or $(x_i - x_j) s_{ij} = 0$, $\forall i, j$. In a general case $x_i \neq x_j$ for any $i \neq j$, and the non-diagonal components of tensor $\mathbf{s}$ are vanishing $s_{i \neq j} = 0$ in the basis of the $\mathbf{x}$ tensor's eigenvectors, so the $\mathbf{x}$ and $\mathbf{s}$ tensors are coaxial [15]. For the isotropic elasticity when $\mathbf{s} = \mathbf{C} : \mathbf{x}$ and $\mathbf{C}$ is the isotropic tensor of order four, the coaxiality of the $\mathbf{x}$ and $\mathbf{s}$ tensors is unconditionally true.

The coaxiality condition (2.2) with the use of any arbitrary spin tensor $\mathbf{A}$ leads to the following transformations $\mathbf{s} : \dot{\mathbf{x}} = \mathbf{s} : \dot{\mathbf{x}} + \mathbf{A} : (\mathbf{s} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{s}) = \mathbf{s} : (\dot{\mathbf{x}} + \mathbf{x} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{x}) = \mathbf{s} : \mathbf{x}^{\mathbf{A}}$ and so $\boldsymbol{\kappa} : \mathbf{D} = \mathbf{s} : \mathbf{x}^{\mathbf{A}}$. The bracketed expression $\dot{\mathbf{x}} + \mathbf{x} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{x}$ is a corotational rate $\mathbf{x}^{\mathbf{A}}$ of $\mathbf{x}$ tensor with a certain spin $\mathbf{A}$. Therefore, the third definition of the work conjugated stress and strain tensors is similar to the first one, but in contradiction to this the third definition supposes the coaxiality of $\mathbf{x}$ and $\mathbf{s}$ tensors. Different conventional stress and strain tensors may be chosen as coaxial $\mathbf{x}$ and $\mathbf{s}$ tensors in the isotropic elasticity. If the stress tensor $\mathbf{s}$ is equal to the weighted Kirchhoff stress $\boldsymbol{\kappa}$ which is conventionally used for the elastic law formulation in the current configuration, then the relation $\boldsymbol{\kappa} : \mathbf{D} = \boldsymbol{\kappa} : \dot{\mathbf{x}} = \boldsymbol{\kappa} : \mathbf{x}^{\mathbf{A}}$ is followed by the equation with respect to the unknown $\mathbf{x}$ tensor and a certain skew-symmetric spin tensor $\mathbf{A}$

$$\mathbf{D} = \dot{\mathbf{x}} + \mathbf{x} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{x}. \tag{2.3}$$

The rate form of the elastic law (2.1) in an isotropic case is suitable for any spin tensor $\boldsymbol{\Omega}^{\#} = \mathbf{A}$, so the formulae (2.1) and (2.3) result in the rate-type elastic law in the following form $\boldsymbol{\kappa}^{\mathbf{A}} = \mathbf{C} : \mathbf{D}$. The hypo-elasticity is usually introduced as $\boldsymbol{\kappa}^{\mathrm{Cor}} = \mathbf{C} : \mathbf{D}$, which is valid for the isotropic media only [16, 17]. Among the widespread corotational rates $\boldsymbol{\kappa}^{\mathrm{Cor}}$ Zaremba-Jaumann and Green-Naghdi derivatives are the most popular. However, using these rates leads to the energy dissipation under pure elastic deformation [18, 19], and Zaremba-Jaumann derivative predicts the stress oscillation with a monotonically increasing shear strain [3, 20–22]. The works [6, 7, 23] state that in addition to the vanishing energy dissipation, the objective rate has to assure zero stresses after the deformation along any closed path starting from (and ending with) the natural stress-free state. The logarithmic corotational rate [6, 7, 23] is free from the above disadvantages, but it is also suitable for isotropic materials only.

For the non-dissipative isothermal elastic deformation, the internal energy equation takes the form of $\widehat{\rho}\dot{u} = \boldsymbol{\sigma} : \mathbf{D}$, where $\widehat{\rho}$ is the current mass density and ρ_0 is the initial one. The internal energy increment $\Delta u(t) = u(t) - u(t_1)$ at any instance of time t under deformation along an arbitrary path starting at the instance of time t_1 without any assumptions about the spin $\mathbf{A}$ is found as

$$\Delta u(t) = \int_{t_1}^{t} \dot{u}\,\mathrm{d}t = \rho_0^{-1} \int_{t_1}^{t} \rho_0 \widehat{\rho}^{\,-1} \boldsymbol{\sigma} : \mathbf{D}\,\mathrm{d}t = \rho_0^{-1} \int_{t_1}^{t} \boldsymbol{\kappa} : \mathbf{D}\,\mathrm{d}t$$
$$= \rho_0^{-1} \int_{t_1}^{t} \mathbf{s} : \mathbf{x}^{\mathbf{A}}\,\mathrm{d}t = \rho_0^{-1} \int_{t_1}^{t} \mathbf{x} : \mathbf{C} : \mathbf{x}^{\mathbf{A}}\,\mathrm{d}t.$$

The elasticity tensor spectral decomposition [24–28] $\mathbf{C} = \sum\limits_K \lambda_K \boldsymbol{\omega}_K^{\#} \otimes \boldsymbol{\omega}_K^{\#}$ ($\boldsymbol{\omega}_K^{\#}$ are the second-rank eigenstates defined with respect to the current material symmetry frame, for which $\boldsymbol{\omega}_K^{\#} : \boldsymbol{\omega}_L^{\#} = \delta_{KL}$, and the sum $\sum\limits_K \boldsymbol{\omega}_K^{\#} \otimes \boldsymbol{\omega}_K^{\#} = \mathbf{I}^{(4)}$ is the fourth-rank identity tensor, $\dot{\lambda}_K = 0$, $\dot{\boldsymbol{\omega}}_K^{\#} = \boldsymbol{\Omega}^{\#} \cdot \boldsymbol{\omega}_K^{\#} - \boldsymbol{\omega}_K^{\#} \cdot \boldsymbol{\Omega}^{\#}$, $\mathbf{x} = \sum\limits_K x_K^{\#} \boldsymbol{\omega}_K^{\#}$, where $x_K^{\#} = \mathbf{x} : \boldsymbol{\omega}_K^{\#}$) allows fulfilling the following transformations:

$$\int_{t_1}^{t} \mathbf{x} : \mathbf{C} : (\dot{\mathbf{x}} + \mathbf{x}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{x})\,\mathrm{d}t = \sum_K \int_{t_1}^{t} \mathbf{x} : \lambda_K \boldsymbol{\omega}_K^{\#} \otimes \boldsymbol{\omega}_K^{\#} : (\dot{\mathbf{x}} + \mathbf{x}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{x})\,\mathrm{d}t$$

$$= \sum_K \lambda_K \int_{t_1}^{t} x_K^{\#} \boldsymbol{\omega}_K^{\#} : \left\{ \sum_J \dot{x}_J^{\#} \boldsymbol{\omega}_J^{\#} + \boldsymbol{\Omega}^{\#}\cdot\mathbf{x} - \mathbf{x}\cdot\boldsymbol{\Omega}^{\#} + \mathbf{x}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{x} \right\} \mathrm{d}t$$

$$= \sum_K \lambda_K \left\{ \int_{t_1}^{t} x_K^{\#} \dot{x}_K^{\#}\,\mathrm{d}t + \int_{t_1}^{t} x_K^{\#} \boldsymbol{\omega}_K^{\#} : \Big((\boldsymbol{\Omega}^{\#} - \mathbf{A})\cdot\mathbf{x} - \mathbf{x}\cdot(\boldsymbol{\Omega}^{\#} - \mathbf{A}) \Big)\,\mathrm{d}t \right\}$$

$$= \sum_K \left\{ \lambda_K \int_{x_K^{\#1}}^{x_K^{\#}} x_K^{\#}\,\mathrm{d}x_K^{\#} + \lambda_K \int_{t_1}^{t} x_K^{\#} (\boldsymbol{\Omega}^{\#} - \mathbf{A}) : (\mathbf{x}\cdot\boldsymbol{\omega}_K^{\#} - \boldsymbol{\omega}_K^{\#}\cdot\mathbf{x})\,\mathrm{d}t \right\}$$

$$= \sum_K \lambda_K \left. \frac{x_K^{\#} x_K^{\#}}{2} \right|_{x_K^{\#1}}^{x_K^{\#}} + \sum_K \lambda_K \int_{t_1}^{t} x_K^{\#} (\boldsymbol{\Omega}^{\#} - \mathbf{A}) : (\mathbf{x}\cdot\boldsymbol{\omega}_K^{\#} - \boldsymbol{\omega}_K^{\#}\cdot\mathbf{x})\,\mathrm{d}t$$

$$= \sum_K \lambda_K \left. \frac{(\mathbf{x} : \boldsymbol{\omega}_K^{\#})(\boldsymbol{\omega}_K^{\#} : \mathbf{x})}{2} \right|_{x_K^{\#1}}^{x_K^{\#}} + \sum_K \lambda_K \int_{t_1}^{t} x_K^{\#} (\boldsymbol{\Omega}^{\#} - \mathbf{A}) : (\mathbf{x}\cdot\boldsymbol{\omega}_K^{\#} - \boldsymbol{\omega}_K^{\#}\cdot\mathbf{x})\,\mathrm{d}t$$

$$= \frac{1}{2}(\mathbf{x} : \mathbf{C} : \mathbf{x} - \mathbf{x}^1 : \mathbf{C} : \mathbf{x}^1) + \sum_K \lambda_K \int_{t_1}^{t} x_K^{\#} (\boldsymbol{\Omega}^{\#} - \mathbf{A}) : (\mathbf{x}\cdot\boldsymbol{\omega}_K^{\#} - \boldsymbol{\omega}_K^{\#}\cdot\mathbf{x})\,\mathrm{d}t.$$

For the deformation along any closed path from the instance of time t_1 until the second instance t_2, for which $\mathbf{x}|_{t_1} = \mathbf{x}|_{t_2} = \mathbf{x}_0$ the internal energy increment has to be zero

$$(2.4) \qquad \Delta u(t_2) = u(t_2) - u(t_1) = \rho_0^{-1} \sum_K \lambda_K \int_{t_1}^{t_2} x_K^{\#} (\boldsymbol{\Omega}^{\#} - \mathbf{A}) : (\mathbf{x}\cdot\boldsymbol{\omega}_K^{\#} - \boldsymbol{\omega}_K^{\#}\cdot\mathbf{x})\,\mathrm{d}t = 0.$$

The condition (2.4) of the non-dissipative elastic deformation along any closed path holds absolutely $\forall \mathbf{x}(t)$ with $\mathbf{x}|_{t_1} = \mathbf{x}|_{t_2} = \mathbf{x}_0$ only when the equality $\mathbf{A} = \boldsymbol{\Omega}^{\#}$ is true. Consequently the first and the third definitions of the work-conjugated measures are followed by the equation $\mathbf{x}^{\boldsymbol{\Omega}^{\#}} = \mathbf{D}$ and then the rate-type elastic law takes the next form $\boldsymbol{\kappa}^{\boldsymbol{\Omega}^{\#}} = \mathbf{C} : \mathbf{D} = \mathbf{C} : \mathbf{x}^{\boldsymbol{\Omega}^{\#}}$, so the internal energy rate due to the identity $\mathbf{C}^{\boldsymbol{\Omega}^{\#}} = \mathbf{0}$ has the next form $\rho_0 \dot{u} = \boldsymbol{\kappa} : \mathbf{D} = \boldsymbol{\kappa} : \mathbf{x}^{\boldsymbol{\Omega}^{\#}} = \mathbf{x} : \mathbf{C} : \mathbf{x}^{\boldsymbol{\Omega}^{\#}} = \mathbf{x} : (\mathbf{C} : \mathbf{x})^{\boldsymbol{\Omega}^{\#}} = \boldsymbol{\kappa}^{\boldsymbol{\Omega}^{\#}} : \mathbf{x}$. In addition, the third definition [15] requires the coaxiality of the stress $\boldsymbol{\kappa}$ and strain $\mathbf{x}$ measures.

For any isotropic medium, the equation $\mathbf{x}^{\boldsymbol{\Omega}} = \mathbf{D}$ is fulfilled when $\mathbf{x}$ is equal to the left Hencky strain tensor $\mathbf{x} = \ln \mathbf{V} = \widehat{\mathbf{H}}$ [29, 30], and $\boldsymbol{\Omega}$ is the logarithmic spin tensor $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\log}$ [6, 7, 23]. For that solution and the elastic law $\boldsymbol{\kappa} = \mathbf{C} : \widehat{\mathbf{H}}$, the internal energy increment is written as

$$\Delta u(t) = \rho_0^{-1} \int_{t_1}^{t} \boldsymbol{\kappa} : \mathbf{D}\, \mathrm{d}t = \rho_0^{-1} \int_{t_1}^{t} \boldsymbol{\kappa} : \widehat{\mathbf{H}}^{\boldsymbol{\Omega}^{\log}}\, \mathrm{d}t = \rho_0^{-1} \int_{t_1}^{t} \widehat{\mathbf{H}} : \mathbf{C} : \widehat{\mathbf{H}}^{\boldsymbol{\Omega}^{\log}}\, \mathrm{d}t,$$

where $\mathbf{C} = \lambda \mathbf{C}_{\mathrm{I}} + \mu(\mathbf{C}_{\mathrm{II}} + \mathbf{C}_{\mathrm{III}})$, $\mathbf{C}_{\mathrm{I}} = \mathbf{I} \otimes \mathbf{I}$, $\mathbf{C}_{\mathrm{II}} = \mathbf{e}_i \otimes \mathbf{e}^k \otimes \mathbf{e}^i \otimes \mathbf{e}_k$, $\mathbf{C}_{\mathrm{III}} = \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}^i$ are isotropic tensors of order four, λ and μ are Lamé's moduli [29, 30], and $\mathbf{I} = \mathbf{e}_k \otimes \mathbf{e}^k$ is the identity tensor of order two. The contraction $\widehat{\mathbf{H}} : \mathbf{C} : \widehat{\mathbf{H}}^{\boldsymbol{\Omega}^{\log}}$ contains two parts

$$\widehat{\mathbf{H}} : \mathbf{C} : \widehat{\mathbf{H}}^{\boldsymbol{\Omega}^{\log}} = \widehat{\mathbf{H}} : \mathbf{C} : \dot{\widehat{\mathbf{H}}} + \widehat{\mathbf{H}} : \mathbf{C} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}).$$

The first term gives $\int_{t_1}^{t} \widehat{\mathbf{H}} : \mathbf{C} : \dot{\widehat{\mathbf{H}}}\, \mathrm{d}t = \frac{1}{2}\, \widehat{\mathbf{H}} : \mathbf{C} : \widehat{\mathbf{H}} \Big|_{t_1}^{t}$, and the second term vanishes:

$$\begin{aligned}
\widehat{\mathbf{H}} : \mathbf{C} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) &= \lambda \widehat{\mathbf{H}} : \mathbf{C}_{\mathrm{I}} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) \\
&\quad + \mu \widehat{\mathbf{H}} : (\mathbf{C}_{\mathrm{II}} + \mathbf{C}_{\mathrm{III}}) : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) \\
&= \lambda \widehat{\mathbf{H}} : \mathbf{I} \otimes \mathbf{I} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) + 2\mu \widehat{\mathbf{H}} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) \\
&= \lambda \widehat{\mathbf{H}} : \mathbf{I}(\widehat{\mathbf{H}} : \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} : \widehat{\mathbf{H}}) + 2\mu \boldsymbol{\Omega}^{\log} : (\widehat{\mathbf{H}} \cdot \widehat{\mathbf{H}} - \widehat{\mathbf{H}} \cdot \widehat{\mathbf{H}}) = \mathbf{0}.
\end{aligned}$$

So for any isotropic medium, there is no elastic energy dissipation under deformation along any closed path. However, if the $\mathbf{C}$ tensor is anisotropic, then using the left Hencky strain and the logarithmic spin lead to the opposite result, indeed $\forall \widehat{\mathbf{H}} \neq \alpha \mathbf{I}$, $\alpha \in \mathbb{R}$:

$$\begin{aligned}
\widehat{\mathbf{H}} : \mathbf{C} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) &= \widehat{\mathbf{H}} : \sum_K \lambda_K \boldsymbol{\omega}_K^{\#} \otimes \boldsymbol{\omega}_K^{\#} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) \\
&= \sum_K \widehat{\mathbf{H}}_K \lambda_K \boldsymbol{\omega}_K^{\#} : (\widehat{\mathbf{H}} \cdot \boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \cdot \widehat{\mathbf{H}}) \\
&= \boldsymbol{\Omega}^{\log} : \sum_K \widehat{\mathbf{H}}_K \lambda_K (\boldsymbol{\omega}_K^{\#} \cdot \widehat{\mathbf{H}} - \widehat{\mathbf{H}} \cdot \boldsymbol{\omega}_K^{\#}) \neq \mathbf{0}.
\end{aligned}$$

So for the anisotropic medium in the considered case, the elastic energy dissipation arises under deformation along a closed path for which $\widehat{\mathbf{H}}$ is not a ball tensor.

3. Material rotation and material spin

The material spin $\mathbf{\Omega}^{\#}$ in the case of the finite-strain elastic deformation is defined by involving an adopted intermediate configuration which corresponds to the motion decomposition for any material volume onto deformation and rigid motion parts. This configuration could not be reached by the medium in any real motion and represents an imaginary configuration, according to which all parameters of the anisotropic material take clear physical meanings. The paper supposes an introduction of this state by reloading the small material volume from the current elastically deformed configuration and gives the following definition of the material rotation tensor.

Definition: tensor $\mathbf{Q}^{\#}$ of the material symmetry rotation at any chosen material point P with the radius-vector $\mathbf{r}_P$ in the current configuration is a proper orthogonal second-order tensor connecting the current, but not deformed set of the material symmetry axes $\mathbf{a}_i^{\#}$ (obtained in the current configuration by releasing a small material neighbourhood of the considered point P) with initial set of the material symmetry axes $\mathbf{a}_i$ at the same material point P: $\mathbf{a}_i^{\#} = \mathbf{Q}^{\#} \cdot \mathbf{a}_i$, $\mathbf{Q}^{\#} = \mathbf{a}_i^{\#}\mathbf{a}^i$, $\mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j$.

The deformed material axes in the current configuration are $\widehat{\mathbf{a}}_i = \mathbf{F} \cdot \mathbf{a}_i$. Suppose that after releasing, the chosen material volume reaches its final reloaded state corresponding to the minimal summary displacements of all the material points (across the immovable mass centre $\mathbf{r}_P = \mathbf{F} \cdot \mathbf{R}_P$ of the small current material volume) from the deformed configuration into the one released. Let the small material volume under consideration be homogeneous and have the shape of the elementary material symmetry cell with sizes proportional to an integer number m of the symmetry axes parameters. Then the reloading will be considered as the motion of the chosen volume's apexes. In the case of the plane material, the unit rotation axis $\mathbf{n}$ is fixed and orthogonal to the plane of the material substructure vectors $\mathbf{a}_i$, $\mathbf{a}_1 \perp \mathbf{a}_2$, and the axes parameters are not the same $|\mathbf{a}_1| \neq |\mathbf{a}_2|$. Four initial apexes positions of the volume are given by the $\mathbf{R}_P \pm m\,\mathbf{a}_1 \pm m\,\mathbf{a}_2$ vectors, $m \in \mathbb{N}$. Suppose that these four vectors are transformed into $\mathbf{r}_P \pm m\,\widehat{\mathbf{a}}_1 \pm m\,\widehat{\mathbf{a}}_2$ in the current state, their representations after the release will be $\mathbf{r}_P \pm m\,\mathbf{a}_1^{\#} \pm m\,\mathbf{a}_2^{\#}$, and then the following minimizing condition for the positive convex function

$$\sum_{i,\,j \in \{-1,1\}} \Big\{ (i\,m\,\widehat{\mathbf{a}}_1 + j\,m\,\widehat{\mathbf{a}}_2) - \Big(i\,m\,\mathbf{a}_1^{\#} + j\,m\,\mathbf{a}_2^{\#} \Big) \Big\}^2 \to \min_{\mathbf{Q}_{\#}}$$

comes from the above supposition, which is valid for any potential interaction of the material elements, so

$$\sum_{j\in\{-1,1\}} \left\{(\widehat{\mathbf{a}}_1 + j\,\widehat{\mathbf{a}}_2) - \left(\mathbf{a}_1^{\#} + j\,\mathbf{a}_2^{\#}\right)\right\}^2 \to \min_{\mathbf{Q}_{\#}}, \tag{3.1}$$

where the power two of any vector $\mathbf{a}$ supposes the scalar product $\mathbf{a}^2 \equiv \mathbf{a}\cdot\mathbf{a}$. The left-hand side expression in the problem (3.1) transforms into the following one:

$$\begin{aligned}\{(\widehat{\mathbf{a}}_1 - \widehat{\mathbf{a}}_2) - (\mathbf{a}_1^{\#} - \mathbf{a}_2^{\#})\}^2 + \{(\widehat{\mathbf{a}}_1 + \widehat{\mathbf{a}}_2) - (\mathbf{a}_1^{\#} + \mathbf{a}_2^{\#})\}^2 \\ = \{\mathbf{F}\cdot(\mathbf{a}_1 - \mathbf{a}_2) - \mathbf{Q}^{\#}\cdot(\mathbf{a}_1 - \mathbf{a}_2)\}^2 + \{\mathbf{F}\cdot(\mathbf{a}_1 + \mathbf{a}_2) - \mathbf{Q}^{\#}\cdot(\mathbf{a}_1 + \mathbf{a}_2)\}^2 \\ = \{(\mathbf{a}_1 + \mathbf{a}_2)\otimes(\mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{a}_1 - \mathbf{a}_2)\otimes(\mathbf{a}_1 - \mathbf{a}_2)\} : \{(\mathbf{F} - \mathbf{Q}^{\#})^{\mathrm{T}}\cdot(\mathbf{F} - \mathbf{Q}^{\#})\},\end{aligned}$$

which leads to the equation

$$\frac{\mathrm{d}\{(\mathbf{a}_1\otimes\mathbf{a}_1 + \mathbf{a}_2\otimes\mathbf{a}_2) : (\mathbf{F} - \mathbf{Q}_{\#})^{\mathrm{T}}\cdot(\mathbf{F} - \mathbf{Q}_{\#})\}}{\mathrm{d}s} = 0$$

or

$$(\mathbf{a}_1\otimes\mathbf{a}_1 + \mathbf{a}_2\otimes\mathbf{a}_2) : \left\{-\mathbf{F}^{\mathrm{T}}\cdot\frac{\mathrm{d}\mathbf{Q}_{\#}}{\mathrm{d}s} - \frac{\mathrm{d}\mathbf{Q}_{\#}^{\mathrm{T}}}{\mathrm{d}s}\cdot\mathbf{F}\right\} = 0, \tag{3.2}$$

where $\mathbf{Q}^{\#}(s) = \mathbf{I}\cos(s) + \mathbf{n}\otimes\mathbf{n}(1-\cos(s)) + \mathbf{n}\times\mathbf{I}\sin(s)$, and the single parameter s of the rotation angle is used due to the fact that the axis of rotation $\mathbf{n}$ is permanent, the cross sign "$\times$" denotes the vector product. So, only the rotation angle s has to be found from (3.2). The solution of (3.2) takes the form

$$\tan s = \frac{(\xi\,\mathbf{e}_1^0\otimes\mathbf{e}_2^0 - \xi^{-1}\mathbf{e}_2^0\otimes\mathbf{e}_1^0) : \mathbf{F}}{(\xi\,\mathbf{e}_1^0\otimes\mathbf{e}_1^0 + \xi^{-1}\mathbf{e}_2^0\otimes\mathbf{e}_2^0) : \mathbf{F}}, \tag{3.3}$$

where

$$\mathbf{e}_i^0 \equiv \frac{\mathbf{a}_i}{|\mathbf{a}_i|}, \qquad \xi = \frac{|\mathbf{a}_1|}{|\mathbf{a}_2|}.$$

In the three-dimensional case, the axis of rotation is changing in the course of deformation; thus, the resulting expression becomes much more complicated.

For the plane orthotropic medium, the elastic eigenstates are written using the initial anisotropy axes $\mathbf{e}_i^0$ in the next invariant form $\boldsymbol{\omega}_{\mathrm{I}} = \cos\varphi\,\mathbf{e}_1^0\otimes\mathbf{e}_1^0 + \sin\varphi\,\mathbf{e}_2^0\otimes\mathbf{e}_2^0$, $\boldsymbol{\omega}_{\mathrm{II}} = \sin\varphi\,\mathbf{e}_1^0\otimes\mathbf{e}_1^0 - \cos\varphi\,\mathbf{e}_2^0\otimes\mathbf{e}_2^0$, $\boldsymbol{\omega}_{\mathrm{III}} = (\mathbf{e}_1^0\otimes\mathbf{e}_2^0 + \mathbf{e}_2^0\otimes\mathbf{e}_1^0)/\sqrt{2}$ [24–26], the angle φ is a stiffness distributor. In the particular case of a plane tetragonal material with $|\mathbf{a}_1| = |\mathbf{a}_2|$, $\varphi = \pi/4$, one obtains the following eigenstates: $\boldsymbol{\omega}_{\mathrm{I}} = (\mathbf{e}_1^0\otimes\mathbf{e}_1^0 + \mathbf{e}_2^0\otimes\mathbf{e}_2^0)/\sqrt{2}$, $\boldsymbol{\omega}_{\mathrm{II}} = (\mathbf{e}_1^0\otimes\mathbf{e}_1^0 - \mathbf{e}_2^0\otimes\mathbf{e}_2^0)/\sqrt{2}$, $\boldsymbol{\omega}_{\mathrm{III}} = (\mathbf{e}_1^0\otimes\mathbf{e}_2^0 + \mathbf{e}_2^0\otimes\mathbf{e}_1^0)/\sqrt{2}$ [28] and then

$$\tan s = \frac{(\mathbf{e}_1^0\otimes\mathbf{e}_2^0 - \mathbf{e}_2^0\otimes\mathbf{e}_1^0) : \mathbf{F}}{(\mathbf{e}_1^0\otimes\mathbf{e}_1^0 + \mathbf{e}_2^0\otimes\mathbf{e}_2^0) : \mathbf{F}}.$$

Let the vector $\mathbf{u}$ describe the displacements of any material particle of interest. Then the deformation gradient tensor is written as $\mathbf{F} = \mathbf{I} + (\nabla_0 \otimes \mathbf{u})^{\mathrm{T}}$ (∇_0 is the initial vector operator of the gradient) and the (3.3) expression is transformed into the form

$$\tan s = \frac{\xi u_{2,1} - \xi^{-1} u_{1,2}}{\xi(u_{1,1} + 1) + \xi^{-1}(u_{2,2} + 1)},$$

where $u_{i,j} = \partial u_i / \partial X^j$, X^j are the Lagrangian (material) coordinates of a particle.

The example of the elastic motion decomposition for the elementary cell of a plane single crystal according to (3.3) is shown in Fig. 1 for randomly chosen components of the deformation gradient tensor $\mathbf{F}$.

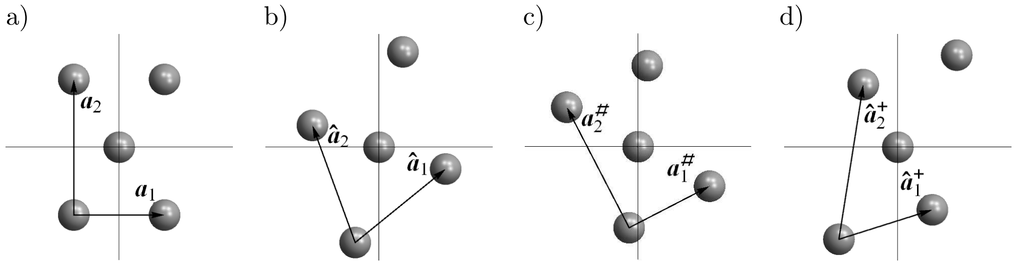

FIG. 1. An example of material rotation: a) initial equilibrium configuration, b) arbitrary deformed configuration, c) released material volume, d) "pure" deformations of the volume in respect to the rotated material frame defined by $\mathbf{F}^{\mathrm{def}} = \mathbf{F} \cdot \mathbf{Q}^{\#\mathrm{T}}$, $\widehat{\mathbf{a}}_i^{+} = \mathbf{F}^{\mathrm{def}} \cdot \mathbf{a}_i$.

For the particular case of the rigid motion $\mathbf{F} = \mathbf{R}$, where $\mathbf{R}$ is an orthogonal tensor describing rotation around the unit normal vector $\mathbf{n}$ by the angle β the (3.3) expression results in $\tan s = \tan \beta$. Under the conditions of small displacements $|\mathbf{u}| \ll 1$ and their gradient $\|\nabla_0 \otimes \mathbf{u}\| \ll 1$ one becomes $\mathbf{F} \approx \mathbf{I} + \boldsymbol{\varepsilon} - \boldsymbol{\omega}$, $\mathbf{R} \approx \mathbf{I} - \boldsymbol{\omega}$ ($\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ are the infinitesimal strain and rotation tensors), so (3.3) takes the form

$$\tan s \approx \omega_{12} + \varepsilon_{12} \frac{\xi - \xi^{-1}}{\xi + \xi^{-1}}, \tag{3.4}$$

where $\varepsilon_{12} = \mathbf{e}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{e}_2$, $\omega_{12} = \mathbf{e}_1 \cdot \boldsymbol{\omega} \cdot \mathbf{e}_2$. For the particular case when $\xi = 1$, $\varphi = \pi/4$ and the material possesses the plane tetragonal symmetry, the expression (3.4) arrives at the classical result $\tan s \approx \omega_{12}$ suitable also for the material isotropy.

Deformations by plane pure tension $\mathbf{F}_{(1)} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2$ and two types of plane simple shear $\mathbf{F}_{(2)} = \mathbf{I} + 2\gamma \mathbf{e}_1 \otimes \mathbf{e}_2$, $\mathbf{F}_{(3)} = \mathbf{I} + 2\gamma \mathbf{e}_2 \otimes \mathbf{e}_1$ lead to the rotations

$$\tan s_{(1)} = 0, \qquad \cos s_{(1)} = 1, \qquad \sin s_{(1)} = 0,$$

$$\tan s_{(2)} = -\frac{2\gamma}{1+\xi^2}, \qquad \cos s_{(2)} = \frac{1+\xi^2}{\sqrt{4\gamma^2+(1+\xi^2)^2}},$$

$$\sin s_{(2)} = -\frac{2\gamma}{\sqrt{4\gamma^2+(1+\xi^2)^2}},$$

$$\tan s_{(3)} = \frac{2\gamma}{1+\xi^{-2}}, \qquad \cos s_{(3)} = \frac{1+\xi^2}{\sqrt{4\gamma^2\xi^4+(1+\xi^2)^2}},$$

$$\sin s_{(3)} = \frac{2\gamma\,\xi^2}{\sqrt{4\gamma^2\xi^4+(1+\xi^2)^2}}.$$

Under the condition $\xi = 1$, these expressions give the rotation tensors which coincide with the tensors $\mathbf{R}$ from the polar decomposition for the same motions, e.g.,

$$\mathbf{Q}^{\#}_{(2)} = \mathbf{R}_{(2)} = (\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2)\frac{1}{\sqrt{1+\gamma^2}} + (\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_1)\,\frac{\gamma}{\sqrt{1+\gamma^2}}.$$

When the rigid motion described by a proper orthogonal tensor $\mathbf{O}^{\phi}_{\mathbf{n}}(t)$ with the angle ϕ of the rotation around $\mathbf{n}$ is superimposed onto the current configuration, the material symmetry axes are rotated together with the material volume. Indeed, $\mathbf{F}^* = \mathbf{O}^{\phi}_{\mathbf{n}}\cdot\mathbf{F}$, so the resulting rotation (3.3) is explained with the expression

$$\frac{(\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_2-\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_1):(\mathbf{O}^{\phi}_{\mathbf{n}}\cdot\mathbf{F})}{(\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_1+\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_2):(\mathbf{O}^{\phi}_{\mathbf{n}}\cdot\mathbf{F})}$$

$$= \frac{(\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_2-\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_1):\mathbf{F}+(\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_1+\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_2):\mathbf{F}\,\tan\phi}{(\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_1+\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_2):\mathbf{F}-(\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_2-\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_1):\mathbf{F}\,\tan\phi} = \tan(s+\phi).$$

Then $\mathbf{Q}^{\#*} = \mathbf{O}^{\phi}_{\mathbf{n}}\cdot\mathbf{Q}^{\#}$, $\boldsymbol{\Omega}^{\#*} = \mathbf{O}^{\phi}_{\mathbf{n}}\cdot\boldsymbol{\Omega}^{\#}\cdot\mathbf{O}^{\phi\mathrm{T}}_{\mathbf{n}} + \dot{\mathbf{O}}^{\phi}_{\mathbf{n}}\cdot\mathbf{O}^{\phi\mathrm{T}}_{\mathbf{n}}$, and the material corotational rate of any objective tensor $\mathbf{T}$ (for which $\mathbf{T}^* = \mathbf{O}^{\phi}_{\mathbf{n}}\cdot\mathbf{T}\cdot\mathbf{O}^{\phi\mathrm{T}}_{\mathbf{n}}$) is also objective:

$$(\mathbf{T}^*)^{\boldsymbol{\Omega}^{\#}} = \mathbf{O}^{\phi}_{\mathbf{n}}\cdot(\dot{\mathbf{T}}+\mathbf{T}\cdot\boldsymbol{\Omega}^{\#}-\boldsymbol{\Omega}^{\#}\cdot\mathbf{T})\cdot\mathbf{O}^{\phi\mathrm{T}}_{\mathbf{n}} = \mathbf{O}^{\phi}_{\mathbf{n}}\cdot\mathbf{T}^{\boldsymbol{\Omega}^{\#}}\cdot\mathbf{O}^{\phi\mathrm{T}}_{\mathbf{n}}.$$

The notions $\mathbf{A}\equiv\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_2-\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_1$, $\mathbf{B}\equiv\xi\mathbf{e}^0_1\otimes\mathbf{e}^0_1+\xi^{-1}\mathbf{e}^0_2\otimes\mathbf{e}^0_2$ for the tensors introduced into the right side of (3.3) (note that $\det\mathbf{A} = \det\mathbf{B} = 1$ and $\mathbf{A}:\mathbf{B} = 0$) allow obtaining the compact representation of the material spin tensor:

$$\mathbf{\Omega}^{\#} = \dot{s}\,\mathbf{n}\times\mathbf{I} = \frac{\left(\dot{\mathbf{f}}^{(s)}\times\mathbf{f}^{(s)}\right)\cdot\mathbf{n}}{\mathbf{f}^{(s)}\cdot\mathbf{f}^{(s)}}\left(\mathbf{e}_2^0\otimes\mathbf{e}_1^0-\mathbf{e}_1^0\otimes\mathbf{e}_2^0\right), \tag{3.5}$$

where the $\mathbf{f}^{(s)}$ vector is defined as $\mathbf{f}^{(s)} \equiv (\mathbf{A}:\mathbf{F})\,\mathbf{e}_1^0 + (\mathbf{B}:\mathbf{F})\,\mathbf{e}_2^0$.

If the initial orientation of the material axes is posed by the angle θ of the axis $\mathbf{a}_1$ with respect to the laboratory frame, then (3.3) gives the material axes orientation at any arbitrary instance of time. Its evolution under the two-dimensional deformation along the closed trajectory $x^1 = X^1 + a\,H^{-1}X^2\sin t$, $x^2 = X^2 + b\,H^{-1}X^2(1-\cos t)$ (elliptic path with Eulerian spatial coordinates x^i, material Lagrangian coordinates X^i, elliptic half-axes a and b, the initial height of the specimen H) is shown for different initial angles θ in Fig. 2a. The axes rotation function is smooth and evidently depends on the initial orientation θ in contrast to the orientation β of the eigenvectors of tensor $\mathbf{V}$ from the polar decomposition of the deformation gradient $\mathbf{F}$.

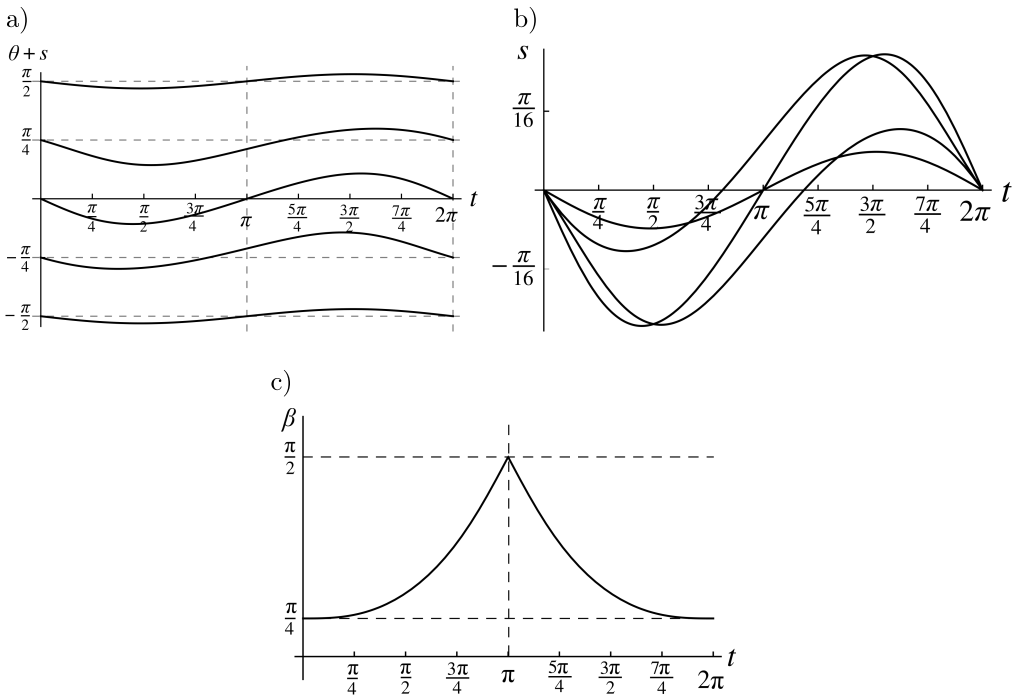

Fig. 2. Time dependences of the material symmetry axes orientations (a), their rotations (b) and orientation of the first eigenvector of tensor $\mathbf{V}$ (c); $H = 1$, $a = 0.5$, $b = 0.25$.

To obtain the strain measure $\mathbf{x}$, the equation (2.1) has to be solved with the found spin involved, and this is discussed further.

4. Material measure of deformations

For the first definition of the work-conjugated stress $\boldsymbol{\kappa}$ and strain $\mathbf{x}$ measures $\boldsymbol{\kappa} : \mathbf{D} = \boldsymbol{\kappa} : \mathbf{x}^{\boldsymbol{\Omega}}$, $\boldsymbol{\kappa} = \mathbf{C} : \mathbf{x}$, and there is no restriction on their coaxiality. Finding the material strain measure $\mathbf{x}$ described in the previous sections requires solving the equation $\dot{\mathbf{x}} + \mathbf{x} \cdot \boldsymbol{\Omega}^{\#} - \boldsymbol{\Omega}^{\#} \cdot \mathbf{x} = \mathbf{D}$ for the given skew-symmetric spin tensor $\boldsymbol{\Omega}^{\#}$. The similar problem with an unknown spin tensor $\boldsymbol{\Omega}$ and a strain measure from the Seth-Hill family [11, 31] results in $\mathbf{x} = \widehat{\mathbf{H}} = \log \mathbf{V}$ solution with the left Hencky strain tensor $\widehat{\mathbf{H}}$ and the logarithmic spin $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\log}$ [6, 7, 23]. The Seth-Hill strain measures are defined in respect to the left triple of the $\mathbf{V}$ tensor eigenvectors. If one considered strain measures within other eigenbasis and an unknown spin $\boldsymbol{\Omega}$ a different result will be obtained. For example, for the case of eigenvectors $\mathbf{q}_i$ of the stretching tensor $\mathbf{D}$, the equation $\dot{\mathbf{x}} + \mathbf{x} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \mathbf{x} = \mathbf{D}$ is fulfilled if the spin tensor coincides with the $\mathbf{W}$ vorticity tensor and the $\mathbf{x}$ strain measure takes the form $\mathbf{x} = \sum\limits_{i=1}^{3} x_i \, \mathbf{q}_i \otimes \mathbf{q}_i$, where eigenvalues x_i are obtained from the equations $\dot{x}_i = d_i$, where d_i are eigenvalues of the $\mathbf{D}$ tensor, i.e., $\mathbf{D} = \sum\limits_{i=1}^{3} d_i \, \mathbf{q}_i \otimes \mathbf{q}_i$. So the components of the $\mathbf{x}$ tensor have the same meaning with the components of the linear deformation tensor but in the rotating basis $\mathbf{q}_i$.

The fixed material spin tensor $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\#}$ (3.5) and two-dimensional media will be further considered. Using the first definition of the work-conjugated measures one goes to the relation $\mathbf{D} = \mathbf{x}^{\boldsymbol{\Omega}^{\#}} = \sum\limits_{i,j=1}^{2} \dot{x}_{ij} \, \widehat{\mathbf{e}}_i^{\#} \otimes \widehat{\mathbf{e}}_j^{\#}$, $\dot{x}_{ij} = \dot{x}_{ji}$. Hence, components of the $\mathbf{x}$ tensor with respect to the current material frame satisfy the equation $\dot{x}_{ij} = \widehat{\mathbf{e}}_i^{\#} \cdot \mathbf{D} \cdot \widehat{\mathbf{e}}_j^{\#}$. Let the value of α be the angle between the $\widehat{\mathbf{e}}_1^{\#}$, $\mathbf{q}_1$ vectors, $\cos\alpha = \widehat{\mathbf{e}}_1^{\#} \cdot \mathbf{q}_1 = \widehat{\mathbf{e}}_2^{\#} \cdot \mathbf{q}_2$, $\sin\alpha = \widehat{\mathbf{e}}_1^{\#} \cdot \mathbf{q}_2 = -\widehat{\mathbf{e}}_2^{\#} \cdot \mathbf{q}_1$, then the rates of the $\mathbf{x}$ tensor components are found as

$$
\begin{cases}
\dot{x}_{11} = d_1 \cos^2\alpha + d_2 \sin^2\alpha, \\
\dot{x}_{22} = d_1 \sin^2\alpha + d_2 \cos^2\alpha, \\
\dot{x}_{12} = \dot{x}_{21} = (d_1 - d_2) \cos\alpha \ \sin\alpha.
\end{cases}
\tag{4.1}
$$

For some particular cases of deformation, the solution of the Eqs. (4.1) may be written in the explicit form. Consider the simple shear $\mathbf{F} = \mathbf{I} + 2v\,t\,\mathbf{b} \otimes \mathbf{m}$, $\mathbf{b} \perp \mathbf{m}$, $|\mathbf{b}| = |\mathbf{m}| = 1$, $v = \text{const}$, for which $\mathbf{D} = v\,(\mathbf{b} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{b})$, $d_1 = v$, $\mathbf{q}_1 = (\mathbf{b} + \mathbf{m})/\sqrt{2}$, $d_2 = -v$, $\mathbf{q}_2 = (\mathbf{m} - \mathbf{b})/\sqrt{2}$. It is shown above that $\tan s = -2v\,t/(1+\xi^2)$. If the angle θ is the angle of initial symmetry axis orientation $\mathbf{e}_1^0$ with respect to $\mathbf{b}$, $\cos\theta = \mathbf{e}_1^0 \cdot \mathbf{b}$, then $\alpha + \pi/4 = \theta + s$. Thus on the basis of material symmetry axes $\widehat{\mathbf{e}}_1^{\#}$ and $\widehat{\mathbf{e}}_2^{\#}$:

$$
\begin{cases}
x_{11} = -x_{22} = \left(-v\,t + (1+\xi^2) \arctan \dfrac{2v\,t}{1+\xi^2} \right) \sin 2\theta \\
\qquad\qquad + \dfrac{1+\xi^2}{2} \ln \left(1 + \dfrac{4v^2t^2}{(1+\xi^2)^2} \right) \cos 2\theta, \\
x_{12} = x_{21} = \left(-v\,t + (1+\xi^2) \arctan \dfrac{2v\,t}{1+\xi^2} \right) \cos 2\theta \\
\qquad\qquad - \dfrac{1+\xi^2}{2} \ln \left(1 + \dfrac{4v^2t^2}{(1+\xi^2)^2} \right) \sin 2\theta.
\end{cases}
\tag{4.2}
$$

For the small-strain deformation when $|v\,t| \ll 1$ the (4.2) expressions are reduced to

$$
\begin{cases}
x_{11} = -x_{22} = v\,t \sin 2\theta, \\
x_{12} = x_{21} = v\,t \cos 2\theta,
\end{cases}
$$

and do not depend on the material anisotropy parameter ξ. For the tetragonal media $\xi = 1$ and the expressions (4.2) take the form

$$
\begin{cases}
x_{11} = -x_{22} = (-v\,t + 2 \arctan(v\,t)) \sin 2\theta + \ln(1+v^2t^2) \cos 2\theta, \\
x_{12} = x_{21} = (-v\,t + 2 \arctan(v\,t)) \cos 2\theta - \ln(1+v^2t^2) \sin 2\theta.
\end{cases}
$$

The vorticity tensor is equal to $\mathbf{W} = v\,(\mathbf{b} \otimes \mathbf{m} - \mathbf{m} \otimes \mathbf{b})$, so the angular velocity of the material fibres which coincide with $\mathbf{D}$ tensor's eigenvectors at the current instance of time is permanent, $\mathbf{w}_\times \cdot \mathbf{n} = -v$. The material symmetry axes rotation is defined by the angular velocity $\boldsymbol{\omega}^\#$, $\boldsymbol{\omega}^\# \cdot \mathbf{n} = \dot{s} = -2v(1+\xi^2)/\left(4v^2t^2 + (1+\xi^2)^2\right)$, so it approaches zero at the limit $t \to \infty$ in contrast to the $\mathbf{w}_\times$ vector.

The notions $a = -v\,t + (1+\xi^2) \arctan \frac{2v\,t}{1+\xi^2}$, $b = \frac{1+\xi^2}{2} \ln \left(1 + \frac{4v^2t^2}{(1+\xi^2)^2}\right)$ for terms of (4.2) allow to get a compact expression for obtaining the angle ψ of the $\mathbf{x}$ tensor's first eigenvector orientation with respect to the first material symmetry axis $\widehat{\mathbf{e}}_1^\#$:

$$
\sin 2\psi = \frac{a\,\cos 2\theta - b\,\sin 2\theta}{\sqrt{a^2+b^2}}, \qquad \cos 2\psi = \frac{b\,\cos 2\theta + a\,\sin 2\theta}{\sqrt{a^2+b^2}}. \tag{4.3}
$$

Eigenvalues of $\mathbf{x}$ tensor are equal to $x_1 = -\sqrt{a^2+b^2}$, $x_2 = \sqrt{a^2+b^2}$ for the simple shear of interest. In the case of small-strain deformations the (4.3) expressions are reduced to $\sin 2\psi = \operatorname{sign} v\ \cos 2\theta$, $\cos 2\psi = \operatorname{sign} v\ \sin 2\theta$, and the corresponding eigenvalues attain the following expressions $x_1 = -\,|v\,t|$, $x_2 = |v\,t|$.

5. Conclusion

The introduced material rotation for the two-dimensional orthotropic media naturally accounts for the material symmetry and demonstrates the dependence of its rotation angle on the initial orientation of the anisotropy axes under finite-strain deformation. For the particular cases such as infinitesimal deformations or a higher material symmetry (more close to the isotropic one), the introduced material rotation tensor coincides with the conventional measures of rotation introduced in solid mechanics. By using the corresponding material spin tensor for the hypo-elastic law, it becomes possible to take into account the material anisotropy in the finite-strain hypo-elasticity in contrast to all the known variants of the corotational rates.

The obtained results demonstrate that for the introduced anisotropic rate-type elastic law and material symmetry rotation there is no elastic energy dissipation under deformation along any closed paths, and the components of stress tensor are equal to zero at the end of any closed strain path starting from the natural stress-free state. The introduced rate-type elastic law is appropriate for studying the finite-strain deformation of anisotropic materials.

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