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Equilibrium and Stability of Nonlinearly Elastic Cylinder Made of Blatz-Ko Material

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The Blatz-Ko model describing the behavior of nonlinearly elastic compressible material was used to model the process of stretching and inflation of hollow cylinder, both thick- and thin-walled. By means of semi-inverse method, the three-dimensional problem was reduced to the analysis of boundary-value problem for ordinary differential equation of second order. The presented numerically constructed loading diagrams show in this the model that even the regions of positive stresses can contain zones of instability. Within the framework of bifurcation analysis, size of these zones and their dependence on material parameters were determined.

Key words: Blatz-Ko material, hollow cylinder, inflation, stretching, buckling.

1. Introduction

Equilibrium and stability problems of deformed solids are of a great importance for both practice and theory. Especially, this concerns the behavior of thin-walled and thick-walled solids in the form of rods, shells and plates due to the non-linear deformation effects. A number of methods and approaches were developed for solving the buckling problems during the last fifty years, which are based on three-dimensional elasticity, and a common approach of stability analysis in the elasticity theory is the bifurcation method. This technique allows reducing the problem of the homogeneous boundary-value to the initial-value problem for ordinary differential equation of second-order, linearized in the small neighborhood of the equilibrium state.

One of the most common and important equilibrium and stability problems concerns a hollow cylinder. Although this problem has been investigated for a long time, the solutions were most often developed for incompressible materials [2] or compressible material, but under simple type of loading [1, 4, 5]. In this paper, the stability problem for a hollow finite circular cylinder made of a compressible Blatz-Ko material is considered based on/in the context of three-dimensional elasticity theory the stability problem

2. Constitutive relations in prestressed state

We consider the process of stretching and inflation of a hollow cylinder using the semi-inverse method [1, 2, 4, 5]:

$$R = P(r), \qquad \Phi = \varphi, \qquad Z = \eta z, \tag{2.1}$$

where R, Φ, Z denote the coordinates of material particle after deformation and r, φ, z denote the coordinates of the same material particle in reference configuration, h, r_0 and r_1 denote height, inner radius and outer radii, respectively, $r_0 \leqslant r \leqslant r_1$, $0 \leqslant z \leqslant h$, and η is a constant of extension along the direction of the axis of the cylinder.

The strain energy function W for the Blatz-Ko material is [3]

$$W = \frac{1}{2}\mu(1-\beta)\left(\frac{I_2}{I_3} + \frac{I_3^{\alpha} - 1}{\alpha} - 3\right) + \frac{1}{2}\mu\beta\left(I_1 + \frac{I_3^{-\alpha} - 1}{\alpha} - 3\right),$$

where μ is Lame's coefficient and $\alpha = \nu/(1-2\nu)$, where ν is the Poisson's ratio. $I_k = I_k(\mathbf{G})$, $k = 1, 2, 3$ are the principal invariants of the left Cauchy-Green deformation tensor $\mathbf{G} = \mathbf{C} \cdot \mathbf{C}^{\mathrm{T}}$. $\mathbf{C}$ is the gradient of the deformation in the cylindrical polar coordinates and is computed as

$$\mathbf{C} = P' e_r e_R + \frac{P}{r} e_\varphi e_\Phi + \eta e_z e_Z, \tag{2.2}$$

where e_r, e_φ, e_z and e_R, e_Φ, e_Z are the unit vectors for the cylindrical coordinates in reference and after deformation configurations, respectively. The accent denotes a derivative with respect to r.

Piola's stress tensor $\mathbf{D}$ is used to describe the stress configuration assuming that each point in the stress references configuration is unloaded, where $\mathbf{D} = \partial W/\partial \mathbf{C}$ is a constituting relation. The equilibrium equations are given in reference configuration by

$$\mathrm{div}\mathbf{D} = 0. \tag{2.3}$$

Using (2.1), (2.2) the equations (2.3) reduce to the nonlinear second-order differential equation of $P(r)$ function.

If the outer lateral surface is free and the inner lateral surface is loaded, then taking Eq. (2.3) into account the boundary conditions have the form:

$$e_r \cdot \mathbf{D} = 0, \qquad e_r \cdot \mathbf{D} = -pJ\mathbf{C}^{-1} \cdot e_r, \tag{2.4}$$

where p is the coefficient of internal pressure applied at the inner lateral surface, and $J = \det \mathbf{C}$. The conditions of the cylinder ends are given by the relations

$$D_{zR} = 0, \qquad D_{z\Phi} = 0. \tag{2.5}$$

3. Perturbed equilibrium

To obtain the equilibrium equations, the small perturbation of prestressed equilibrium state defined in the previous section is considered [1, 2]. The linearized equilibrium compressible solid equation has the form

$$\operatorname{div}\dot{\mathbf{D}} = 0, \qquad \dot{\mathbf{D}} = \left[\frac{\partial}{\partial \varepsilon}\mathbf{D}\left(\mathbf{r} + \varepsilon \mathbf{u}\right)\right]_{\varepsilon=0}, \qquad \mathbf{u} = u e_r + v e_\varphi + w e_z. \tag{3.1}$$

In this formula, $\dot{\mathbf{D}}$ is linearized Piola's stressed tensor, $\mathbf{r}$ is the radius vector in the unloaded state and $\mathbf{u}$ is the perturbation vector. The linearized boundary conditions at the lateral surfaces of the hollow cylinder have the form

$$e_r \cdot \dot{\mathbf{D}} = 0, \qquad e_r \cdot \dot{\mathbf{D}} = -p\left(\dot{J}\mathbf{C}^{-1} + J\dot{\mathbf{C}}^{-1}\right) \cdot e_r. \tag{3.2}$$

We will seek the perturbation vector components u, v, w in the form

$$\begin{gathered} u = U(r)\cos\left(\frac{z\pi m}{h}\right)\cos\left(n\varphi\right), \qquad v = V(r)\cos\left(\frac{z\pi m}{h}\right)\sin\left(n\varphi\right), \\ w = W(r)\sin\left(\frac{z\pi m}{h}\right)\cos\left(n\varphi\right). \end{gathered} \tag{3.3}$$

m and n are the integers in (3.3). This representation permits to separate the variables in the linearized Eq. (3.1) and conditions (3.2) and reduce the stability analysis to the solution for a set of ordinary differential equations. At the same time, the linearized conditions (2.5) are performed automatically.

4. Numerical results

It is important to mention that the function $P(r)$ cannot be found in an explicit form and there is a need to define the function for the previously solved nonlinear boundary value problem at each step, and in addition to that the dependence of p parameter on the tension process is quite difficult to define. One of the ways to simplify the solving process is using another defined parameter, for example, loaded cylinder radius.

Now, we present the obtained results. They show that the boundary of any stability region consists of four bifurcation curves with the corresponding mode sets $(m, n) = (1, 0)$, $(1, 1)$, $(1, 2)$ and (2,2). The accuracy of used methods has been confirmed by the results obtained in [1, 4].

Using varying parameters β and α and the previously saved geometric data in reference configuration, the stability region forms are shown in Table 2.

Table 1. Forms of cylinder pre-bucling state.

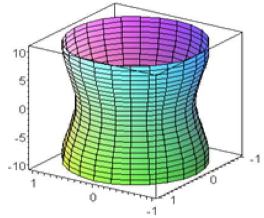
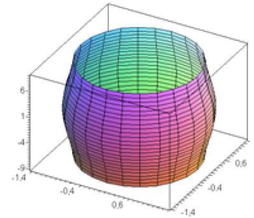
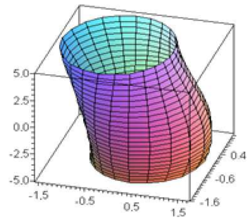
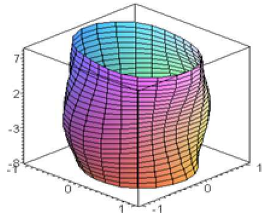
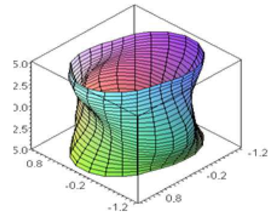
Geometrical and physical cylinder data	$m=1$, $n=0$, $\eta=2.1$, $\xi=0.92$ (left side from local maximum point)	$m=1$, $n=0$, $\eta=1.8$, $\xi=1.28$ (right side from local maximum point)
$\beta=0$ $\alpha=0.5$ $\dfrac{h}{r_1}=10$ $\dfrac{r_0}{r_1}=0.9$		
$m=1$, $n=1$, $\eta=0.98$, $\xi=1.195$	$m=1$, $n=2$, $\eta=1.6$, $\xi=0.88$	$m=2$, $n=2$, $\eta=0.975$, $\xi=1.004$
		

Table 2. Dependence of stability region forms on constants α and β

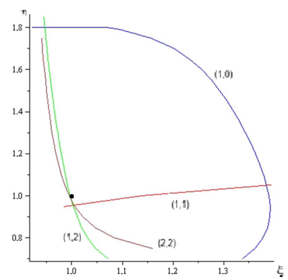
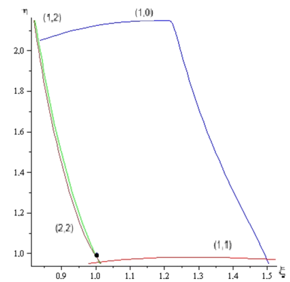
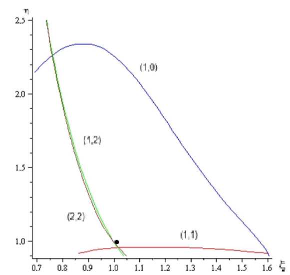
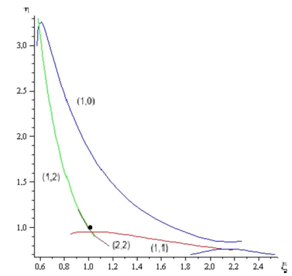
$\dfrac{h}{r_1}=10$, $\dfrac{r_0}{r_1}=0.9$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.97$	$\alpha=4.5$
$\beta=0$				

Table 2. [Cont.]

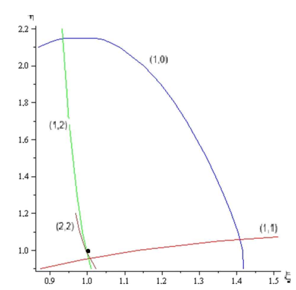
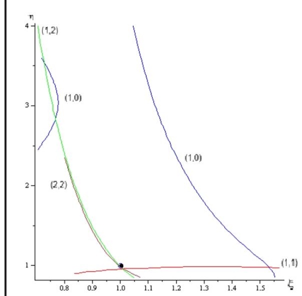
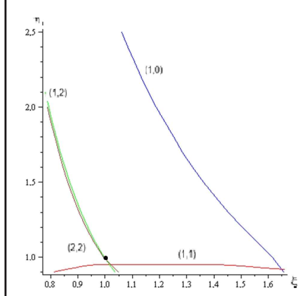
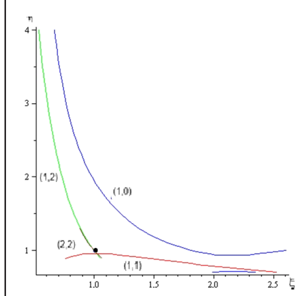
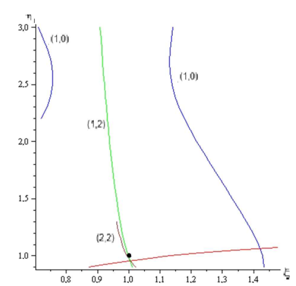
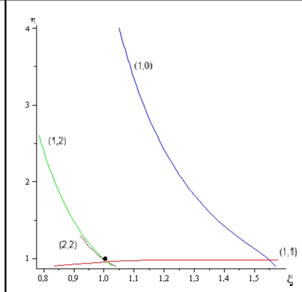
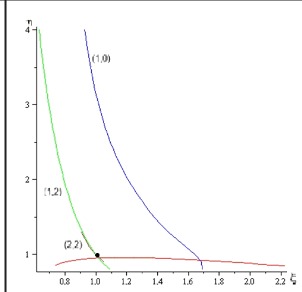
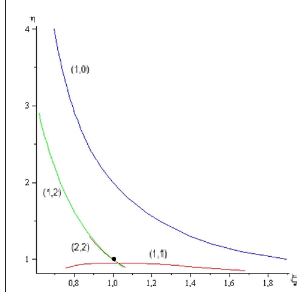
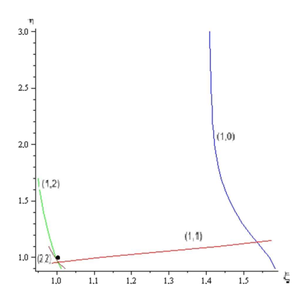
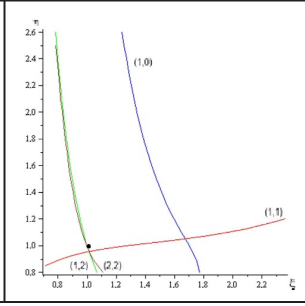
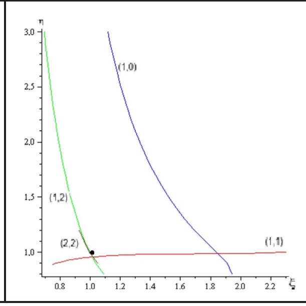
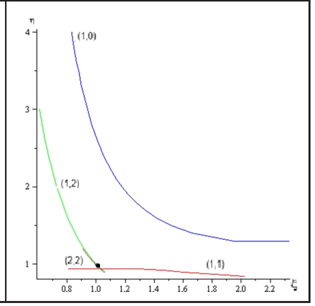
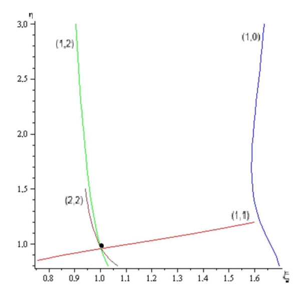
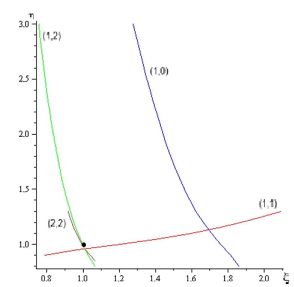
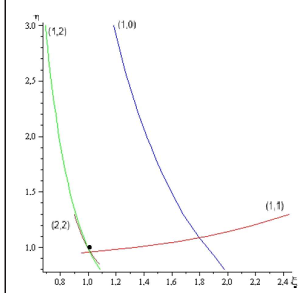
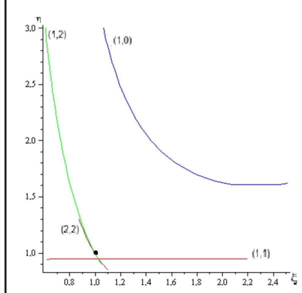
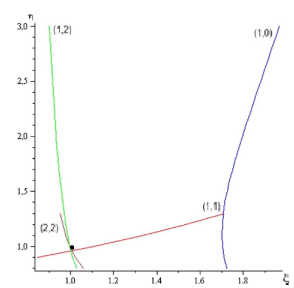
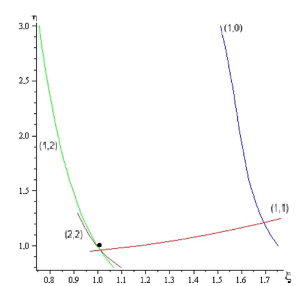
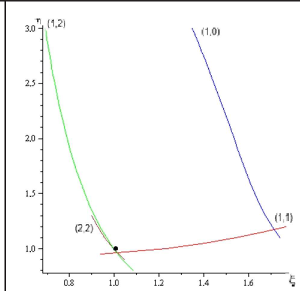
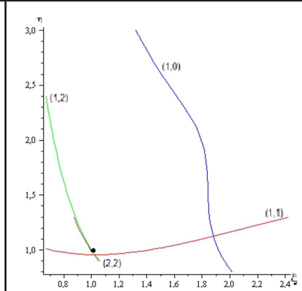
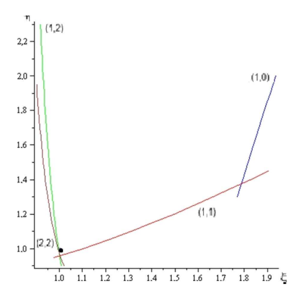
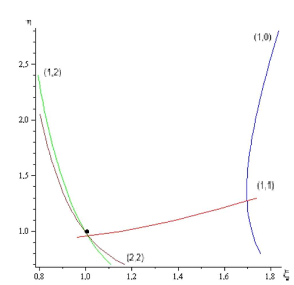
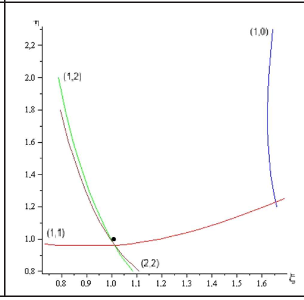
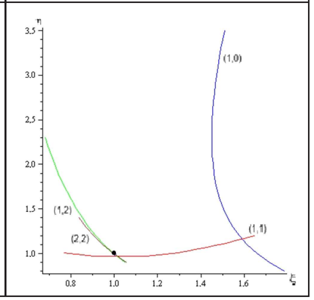
$\frac{h}{r_1}=10,\ \frac{r_0}{r_1}=0.9$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.97$	$\alpha=4.5$
$\beta=0.05$				
$\beta=0.075$				
$\beta=0.3$				

Table 2. [Cont.]

$\dfrac{h}{r_1} = 10,\ \dfrac{r_0}{r_1} = 0.9$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.97$	$\alpha = 4.5$
$\beta = 0.5$				
$\beta = 0.75$				
$\beta = 1$				

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