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A Multi-Layered Ring Under Parabolic Pressure

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The stress- and displacement-fields developed in a circular ring consisting of a finite number of linearly elastic homogeneous and isotropic concentric layers are determined. The composite ring is subjected to a distribution of radial stresses (acting along two finite arcs of its periphery) varying according to a parabolic law. The problem is solved analytically adopting Savin's approach for an infinite plate with a hole strengthened by rings. Taking advantage of the analytic solution, a numerical model is properly calibrated and validated by considering the case of a three-layered ring. It is concluded that the constructed model simulates reality in an excellent manner and therefore it can be safely used for a thorough parametric analysis of the numerous factors influencing the stress- and displacement-fields.

Key words: composite ring, multi-layered rings, Brazilian-disc test, ring test, complex potentials.

1. Introduction

The configuration of a circular ring consisting of a finite number of concentric layers is commonly considered in a number of engineering applications, ranging from biomechanics (human aorta) to fluid mechanics (insulated pipes). Such composite rings are usually loaded either by uniformly distributed internal or external pressure or by a combination of them. A more complicated loading mode appears when the ring is compressed between the curved jaws of the device suggested by the International Society for Rock Mechanics (ISRM) [1] for the standardized implementation of the Brazilian-disc test. The ring is then under a cyclic distribution of radial stresses that act along two finite arcs of its periphery (antisymmetric with respect to the geometric centre of the ring) the length of which depends on the stiffness of the ring and the jaws. The specific distribution is accurately enough simulated by a parabolic scheme [2].

In this study, the analytic full-field solutions are introduced for the stresses and displacements developed in a multi-layered ring under a parabolically varying pressure. The analytic solution is achieved by adopting the procedure proposed by Savin [3] for an infinite plate with a hole strengthened by rings. The specific procedure is based on the complex potentials technique introduced by Muskhelishvili [4]. Its main advantage is that it is extendable to rings made up of any number of concentric layers.

As a next step, the problem is reconsidered numerically using the finite element (FE) method. The FE model is validated taking advantage of the analytic results for a ring made up of three concentric layers. The validated model is used for an in-depth parametric analysis of the role of the numerous factors influencing the solution [5].

2. Analytic solution

Consider a multi-layered ring consisting of n concentric constituent rings made of different linearly elastic, homogeneous and isotropic materials, firmly bonded together along their common interfaces. The ring is subjected to in-plane parabolic pressure along two antisymmetric finite arcs of its outer periphery while its inner periphery is free from stresses. The length of the loaded arcs is either determined approximately (from the contact problem of a solid disc smoothly compressed against the ISRM jaws [1]) or it is arbitrarily prescribed. Bonding of adjacent rings is ensured by considering equal displacements and normal and shear stresses of facing material points along their interfaces. This is a first fundamental problem of plane linear elasticity. Assuming the ring's length w comparable to its radius, plane strain conditions are considered. Stresses and displacements are to be determined at any point of the ring. Its cross-section lies in the $z = r\mathrm{e}^{i\theta}$ plane and its centre is the origin of the Cartesian reference system xOy (Fig. 1).

The innermost layer is denoted by 1, while the outermost one by n. L_1 and L_{n+1} denote the inner and outer boundaries of the ring as a whole. The j-ring ($1 \leqslant j \leqslant n$) is bounded by the L_j and L_{j+1} boundaries, for $r = R_j$ and $r = R_{j+1}$. Rings can have different thicknesses. L_1 is stress-free. L_{n+1} is loaded by a parabolic pressure (acting along the two arcs, each one of a length of $2\omega_o R_{n+1}$):

$$\sigma_r^{(n+1)} = -P(\theta) = -P_c\left[1 - \frac{(\sin\phi_o - \theta)^2}{(\sin\omega_o)^2}\right], \qquad P_c \equiv P(\theta)_{\max}. \tag{2.1}$$

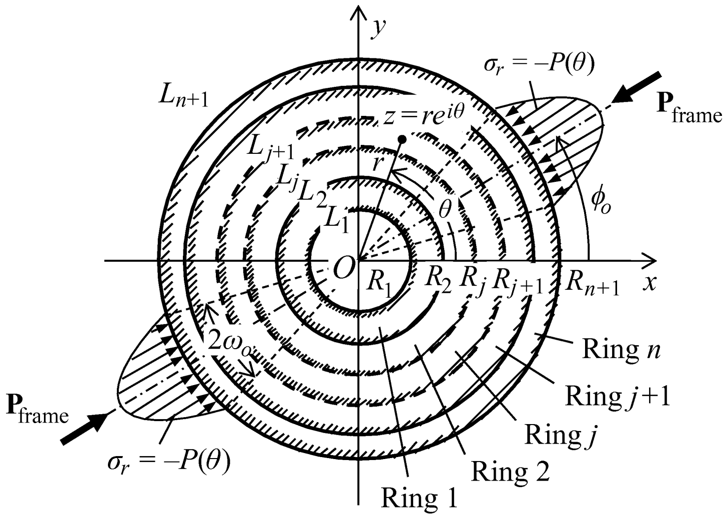

FIG. 1. Configuration of the problem and definition of symbols.

In the case when ω_o is obtained from the respective contact problem [2], the following will hold:

$$\omega_o = \arcsin\sqrt{\frac{6K_n P_{\text{frame}}}{\pi R_{n+1} w}},$$

(2.2)
$$P_c = \sqrt{\frac{3\pi P_{\text{frame}}}{32 K_n R_{n+1} w}},$$

$$K_n = \frac{\kappa_n + 1}{4\mu_n} + \frac{\kappa_J + 1}{4\mu_J}.$$

P_{frame} is the resultant force, and κ_n, κ_J, μ_n, μ_J are Muskhelishvili's constants and shear moduli of the outer ring ($j = n$) and the jaw, respectively. For arbitrarily prescribed ω_o:

(2.3)
$$P_c = \frac{2P_{\text{frame}}(\sin\omega_o)^2}{R_{n+1}(\sin 2\omega_o - 2\omega_o\cos 2\omega_o)\, w}.$$

Each ring j is in elastic equilibrium. The Muskhelishvili's complex potentials read as:

(2.4)
$$\varphi^{(j)}(z) = \sum_{-\infty}^{+\infty} a_k^{(j)} z^k, \qquad \psi^{(j)}(z) = \sum_{-\infty}^{+\infty} b_k^{(j)} z^k, \quad j = 1, 2, \ldots, n.$$

The stress- and displacement-components are written in terms of these functions as follows [4]:

$$\sigma_r^{(j)} - \mathrm{i}\tau_\theta^{(j)} = \varphi^{(j)\prime}(z) + \overline{\varphi^{(j)\prime}(z)} - \mathrm{e}^{\mathrm{i}2\theta}\left[\overline{z}\varphi^{(j)\prime\prime}(z) + \psi^{(j)\prime}(z)\right], \tag{2.5}$$

$$\sigma_\theta^{(j)} + \sigma_r^{(j)} = 4\Re\left[\varphi^{(j)\prime}(z)\right], \tag{2.6}$$

$$u_r^{(j)} - \mathrm{i}u_\theta^{(j)} = \frac{\mathrm{e}^{\mathrm{i}\theta}}{2\mu_j}\left[\kappa_j\overline{\varphi^{(j)}(z)} - \overline{z}\varphi^{(j)\prime}(z) - \psi^{(j)}(z)\right]. \tag{2.7}$$

Prime denotes first derivative and over-bar complex conjugate value, while κ_j and μ_j are Muskhelishvili's constant and shear modulus of the ring j, respectively. Combining Eqs. (2.4)–(2.7) with $z = r\mathrm{e}^{\mathrm{i}\theta}$ ($R_j \leqslant r \leqslant R_{j+1}$), stresses and displacements on the arbitrary j-ring are obtained as:

$$\begin{aligned}\sigma_r^{(j)} - \mathrm{i}\tau_{r\theta}^{(j)} = \sum_{k=0}^{\infty}\Big[&k(2-k)a_k^{(j)}r^{k-1}\mathrm{e}^{\mathrm{i}(k-1)\theta} \\ &- k(2+k)a_{-k}^{(j)}r^{-(k+1)}\mathrm{e}^{-\mathrm{i}(k+1)\theta} + k\left(\overline{a}_k^{(j)}r^{k-1} + b_{-k}^{(j)}r^{-(k+1)}\right)\mathrm{e}^{-\mathrm{i}(k-1)\theta} \\ &-k\left(\overline{a}_{-k}^{(j)}r^{-(k+1)} + b_k^{(j)}r^{k-1}\right)\mathrm{e}^{\mathrm{i}(k+1)\theta}\Big],\end{aligned} \tag{2.8}$$

$$\begin{aligned}u_r^{(j)} - \mathrm{i}u_\theta^{(j)} = \frac{1}{2\mu_j}\sum_{k=0}^{\infty}\Big[&-k\,a_k^{(j)}r^k\mathrm{e}^{\mathrm{i}(k-1)\theta} + k\,a_{-k}^{(j)}r^{-k}\mathrm{e}^{-\mathrm{i}(k+1)\theta} \\ &+ \left(\kappa_j\,\overline{a}_k^{(j)}r^k - b_{-k}^{(j)}r^{-k}\right)\mathrm{e}^{-\mathrm{i}(k-1)\theta} + \left(\kappa_j\,\overline{a}_{-k}^{(j)}r^{-k} - b_k^{(j)}r^k\right)\mathrm{e}^{\mathrm{i}(k+1)\theta}\Big].\end{aligned} \tag{2.9}$$

Constants $a_k^{(j)}$ and $b_k^{(j)}$ are determined by fulfilling the boundary conditions for the composite ring and the conditions along the interfaces between the constituent j-rings: zero stresses on L_1 imply that Eq. (2.8) should be set equal to zero for $j = 1$ and $r = R_1$. In addition, parabolic pressure on L_{n+1} implies that Eq. (2.8), for $j = n$ and $r = R_{n+1}$, should be set equal to the Fourier series expansion of the parabolic pressure of Eq. (2.1). Finally, on each one of the $n-1$ interfaces L_{j+1}, Newton's third law and bonding of adjacent rings along L_{j+1} imply that $\sigma_r^{(j)} - i\tau_{r\theta}^{(j)} = \sigma_r^{(j+1)} - i\tau_{r\theta}^{(j+1)}$, $u_r^{(j)} - \mathrm{i}u_\theta^{(j)} = u_r^{(j+1)} - \mathrm{i}u_\theta^{(j+1)}$ for $r = R_{j+1}$ with $j = 1, \ldots, n-1$. Comparing coefficients of $\mathrm{e}^{\mathrm{i}\theta}$ terms of the same order yields the following four systems of equations providing the sought coefficients $a_k^{(j)}$ and $b_k^{(j)}$:

$$2\alpha_1^{(1)}+R_1^{-2}\overline{b}_{-1}^{(1)}=0,\qquad 2\alpha_1^{(n)}+R_{n+1}^{-2}\overline{b}_{-1}^{(n)}=-\tfrac{P_c}{\pi}\left(2\omega_o-\tfrac{2\omega_o-\sin 2\omega_o}{2\sin^2\omega_o}\right),$$

$$\left.\begin{aligned}&2\alpha_1^{(j)}+R_{j+1}^{-2}\overline{b}_{-1}^{(j)}-2\alpha_1^{(j+1)}-R_{j+1}^{-2}\overline{b}_{-1}^{(j+1)}=0,\\ &\tfrac{\mu_{j+1}}{\mu_j}\left[(\kappa_j-1)R_{j+1}\alpha_1^{(j)}-R_{j+1}^{-1}\overline{b}_{-1}^{(j)}\right]\\ &\qquad-(\kappa_{j+1}-1)\,R_{j+1}\alpha_1^{(j+1)}+R_{j+1}^{-1}\overline{b}_{-1}^{(j+1)}=0\end{aligned}\right\},\quad j=1,2,...,n-1, \tag{2.10}$$

$$\begin{aligned}&R_1a_2^{(1)}+R_1^{-3}\overline{b}_{-2}^{(1)}=0,\qquad R_{n+1}a_2^{(n)}+R_{n+1}^{-3}\overline{b}_{-2}^{(n)}=0,\\ &2\tfrac{\mu_2}{\mu_1}R_2^2a_2^{(1)}-b_0^{(2)}-2R_2^2a_2^{(2)}=0,\\ &\tfrac{\mu_{j+1}}{\mu_j}\left(b_0^{(j)}+2R_{j+1}^2a_2^{(j)}\right)-b_0^{(j+1)}-2R_{j+1}^2a_2^{(j+1)}=0,\quad j=2,3,...,n-1,\\ &\left.\begin{aligned}&R_{j+1}a_2^{(j)}+R_{j+1}^{-3}\overline{b}_{-2}^{(j)}-R_{j+1}a_2^{(j+1)}-R_{j+1}^{-3}\overline{b}_{-2}^{(j+1)}=0,\\ &\tfrac{\mu_{j+1}}{\mu_j}\left(\kappa_jR_{j+1}^2a_2^{(j)}-R_{j+1}^{-2}\overline{b}_{-2}^{(j)}\right)\\ &\qquad-\kappa_{j+1}R_{j+1}^2a_2^{(j+1)}+R_{j+1}^{-2}\overline{b}_{-2}^{(j+1)}=0\end{aligned}\right\},\quad j=1,2,...,n-1,\end{aligned} \tag{2.11}$$

$$\begin{aligned}&3R_1^2a_3^{(1)}+R_1^{-2}\overline{a}_{-1}^{(1)}+b_1^{(1)}=0,\qquad R_1^2a_3^{(1)}-R_1^{-2}\overline{a}_{-1}^{(1)}+R_1^{-4}\overline{b}_{-3}^{(1)}=0,\\ &3R_{n+1}^2a_3^{(n)}+R_{n+1}^{-2}\overline{a}_{-1}^{(n)}+b_1^{(n)}\\ &\qquad=\tfrac{P_c}{\pi}\left[\sin 2\omega_o-\tfrac{1}{2\sin^2\omega_o}\left(\sin 2\omega_o-\omega_o-\tfrac{\sin 2\omega_o\cos 2\omega_o}{2}\right)\right]\mathrm{e}^{-\mathrm{i}2\phi_o},\\ &R_{n+1}^2a_3^{(n)}-R_{n+1}^{-2}\overline{a}_{-1}^{(n)}+R_{n+1}^{-4}\overline{b}_{-3}^{(n)}\\ &\qquad=-\tfrac{P_c}{3\pi}\left[\sin 2\omega_o-\tfrac{1}{2\sin^2\omega_o}\left(\sin 2\omega_o-\omega_o-\tfrac{\sin 2\omega_o\cos 2\omega_o}{2}\right)\right]\mathrm{e}^{-\mathrm{i}2\phi_o},\\ &R_{j+1}^2a_3^{(j)}-R_{j+1}^{-2}\overline{a}_{-1}^{(j)}+R_{j+1}^{-4}\overline{b}_{-3}^{(j)}-R_{j+1}^2a_3^{(j+1)}+R_{j+1}^{-2}\overline{a}_{-1}^{(j+1)}-R_{j+1}^{-4}\overline{b}_{-3}^{(j+1)}=0,\\ &3R_{j+1}^2a_3^{(j)}+R_{j+1}^{-2}\overline{a}_{-1}^{(j)}+b_1^{(j)}-3R_{j+1}^2a_3^{(j+1)}-R_{j+1}^{-2}\overline{a}_{-1}^{(j+1)}-b_1^{(j+1)}=0,\\ &\tfrac{\mu_{j+1}}{\mu_j}\left(\kappa_jR_{j+1}^3a_3^{(j)}+R_{j+1}^{-1}\overline{a}_{-1}^{(j)}-R_{j+1}^{-3}\overline{b}_{-3}^{(j)}\right)\\ &\qquad-\kappa_{j+1}R_{j+1}^3a_3^{(j+1)}-R_{j+1}^{-1}\overline{a}_{-1}^{(j+1)}+R_{j+1}^{-3}\overline{b}_{-3}^{(j+1)}=0,\\ &\tfrac{\mu_{j+1}}{\mu_j}\left(3R_{j+1}^3a_3^{(j)}-\kappa_jR_{j+1}^{-1}\overline{a}_{-1}^{(j)}+R_{j+1}b_1^{(j)}\right)\\ &\qquad-3R_{j+1}^3a_3^{(j+1)}+\kappa_{j+1}R_{j+1}^{-1}\overline{a}_{-1}^{(j+1)}-R_{j+1}b_1^{(j+1)}=0\end{aligned} \tag{2.12}$$

(the last four equations of the (2.12) system are valid for $j=1,2,\dots,n-1$).

$$kR_1^{k-1}a_k^{(1)} + R_1^{-(k-1)}\overline{a}_{-(k-2)}^{(1)} + R_1^{k-3}b_{k-2}^{(1)} = 0,$$

$$R_1^{k-1}a_k^{(1)} - (k-2)R_1^{-(k-1)}\overline{a}_{-(k-2)}^{(1)} + R_1^{-(k+1)}\overline{b}_{-k}^{(1)} = 0,$$

$$(k-2)\left(kR_{n+1}^{k-1}a_k^{(n)} + R_{n+1}^{-(k-1)}\overline{a}_{-(k-2)}^{(n)} + R_{n+1}^{k-3}b_{k-2}^{(n)}\right) = \frac{P_c}{\pi}\left\{\frac{\sin(k-1)\omega_o}{k-1} - \frac{1}{2\sin^2\omega_o}\left[\frac{\sin(k-1)\omega_o}{k-1} + \frac{(k-1)\cos 2\omega_o \sin(k-1)\omega_o - 2\sin 2\omega_o \cos(k-1)\omega_o}{4-(k-1)^2}\right]\right\}\left(1 - \mathrm{e}^{-\mathrm{i}k\pi}\right)\mathrm{e}^{-\mathrm{i}(k-1)\phi_o},$$

$$k\left[R_{n+1}^{k-1}a_k^{(n)} - (k-2)R_{n+1}^{-(k-1)}\overline{a}_{-(k-2)}^{(n)} + R_{n+1}^{-(k+1)}\overline{b}_{-k}^{(n)}\right] = -\frac{P_c}{\pi}\left\{\frac{\sin(k-1)\omega_o}{k-1} - \frac{1}{2\sin^2\omega_o}\left[\frac{\sin(k-1)\omega_o}{k-1} + \frac{(k-1)\cos 2\omega_o \sin(k-1)\omega_o - 2\sin 2\omega_o \cos(k-1)\omega_o}{4-(k-1)^2}\right]\right\}\left(1 - \mathrm{e}^{-\mathrm{i}k\pi}\right)\mathrm{e}^{-\mathrm{i}(k-1)\phi_o},$$

(2.13)
$$kR_{j+1}^{k-1}a_k^{(j)} + R_{j+1}^{-(k-1)}\overline{a}_{-(k-2)}^{(j)} + R_{j+1}^{k-3}b_{k-2}^{(j)} - kR_{j+1}^{k-1}a_k^{(j+1)} - R_{j+1}^{-(k-1)}\overline{a}_{-(k-2)}^{(j+1)} - R_{j+1}^{k-3}b_{k-2}^{(j+1)} = 0,$$

$$R_{j+1}^{k-1}a_k^{(j)} - (k-2)R_{j+1}^{-(k-1)}\overline{a}_{-(k-2)}^{(j)} + R_{j+1}^{-(k+1)}\overline{b}_{-k}^{(j)} - R_{j+1}^{k-1}a_k^{(j+1)} + (k-2)R_{j+1}^{-(k-1)}\overline{a}_{-(k-2)}^{(j+1)} - R_{j+1}^{-(k+1)}\overline{b}_{-k}^{(j+1)} = 0,$$

$$\frac{\mu_{j+1}}{\mu_j}\left(kR_{j+1}^{k}a_k^{(j)} - \kappa_j R_{j+1}^{-(k-2)}\overline{a}_{-(k-2)}^{(j)} + R_{j+1}^{k-2}b_{k-2}^{(j)}\right) - kR_{j+1}^{k}a_k^{(j+1)} + \kappa_{j+1}R_{j+1}^{-(k-2)}\overline{a}_{-(k-2)}^{(j+1)} - R_{j+1}^{k-2}b_{k-2}^{(j+1)} = 0,$$

$$\frac{\mu_{j+1}}{\mu_j}\left[\kappa_j R_{j+1}^{k}a_k^{(j)} + (k-2)\,R_{j+1}^{-(k-2)}\overline{a}_{-(k-2)}^{(j)} - R_{j+1}^{-k}\overline{b}_{-k}^{(j)}\right] - \kappa_{j+1}R_{j+1}^{k}a_k^{(j+1)} - (k-2)\,R_{j+1}^{-(k-2)}\overline{a}_{-(k-2)}^{(j+1)} + R_{j+1}^{-k}\overline{b}_{-k}^{(j+1)} = 0,$$

$$(k = 5, 7, 9, \ldots).$$

Again, the last four equations of the (2.13) system are valid for $j = 1, 2, \ldots, n-1$. Notice that with zero shear stresses on L_{n+1}, all coefficients $a_k^{(j)}$, $b_k^{(j)}$ found above are real. Eventually, the complex potentials for each j-ring (recalling that $b_0^{(1)} = 0$ and $a_0^{(1)} = a_0^{(2)} = \ldots = a_0^{(n)} = 0$, since these terms are related to rigid body motion) are written as:

$$\varphi^{(j)}(z) = \alpha_1^{(j)}z + a_3^{(j)}z^3 + \sum_{k=5,7,9,\ldots} a_k^{(j)}z^k + a_{-1}^{(j)}z^{-1} + \sum_{k=5,7,9,\ldots} a_{-(k-2)}^{(j)}z^{-(k-2)},$$

$$j = 1, 2, \ldots, n,$$

$$\psi^{(j)}(z) = b_1^{(j)}z + \sum_{k=5,7,9,...} b_{k-2}^{(j)} z^{k-2} + b_{-1}^{(j)} z^{-1} + b_{-3}^{(j)} z^{-3} + \sum_{k=5,7,9,...} b_{-k}^{(j)} z^{-k}, \qquad j = 1, 2, ..., n.$$

3. The numerical model and concluding remarks

The problem is now resolved numerically under plane strain conditions, using the ANSYS software. For optimum simulation of the boundary conditions, both the ring and the jaws were modelled. The model was meshed with PLANE182 element. Three layers were considered, perfectly bonded to each other. In contrast, various friction coefficients f were considered along the ring-jaw interface, which was modelled with CONTA171 and TARGE169 elements. The lower side of the lower jaw was clamped and a uniform displacement was imposed on the upper side of the upper jaw. Convergence analysis indicated that 60 000 elements provided sufficient accuracy. One quarter of the model is shown in Fig. 2. The data used are: $R_j = 15, 31, 35, 50$ mm $(j = 1, 2, 3, 4)$; $E_j = 10, 2.1, 3.2$ GPa, $\nu_j = 0.30, 0.38, 0.36$ $(j = 1, 2, 3)$. For the jaws: $R_{\text{jaw}} = 75$ mm, $E_{\text{jaw}} = 210$ GPa, $\nu_{\text{jaw}} = 0.3$.

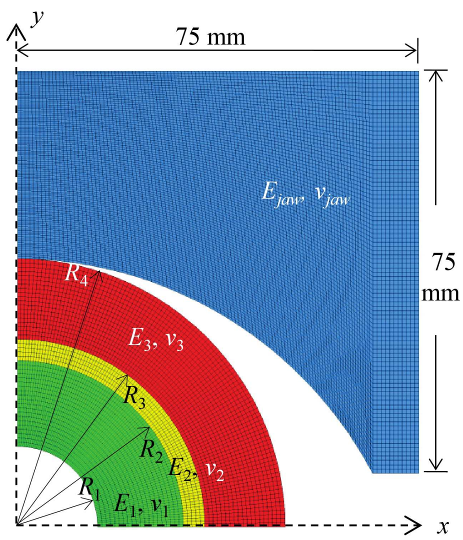

Fig. 2. The numerical model.

In Fig. 3, the transverse stress σ_θ along y-axis is plotted for a series of f-values, together with the respective analytic results. It can be seen that the agreement is satisfactory. Some discrepancies in the vicinity of the ring-jaw contact arc $(y \to R_4)$ for increased values of f are due to the fact that the analytic solution did not take friction into account. Thus, it is concluded that the

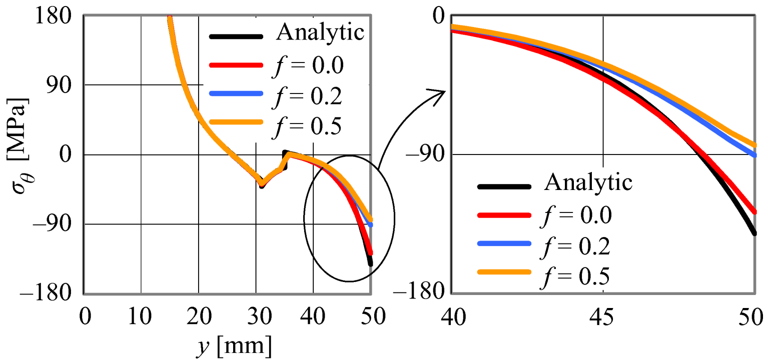

FIG. 3. Analytic and numerical results for the σ_θ stress along y-axis (left) and a detailed view close to the ring-jaw contact arc (right).

present numerical model can be safely used for practical purposes and parametric studies of the numerous factors influencing the stress field. The specific project is already in progress with very encouraging results [5].

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