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FREE VIBRATIONS OF THIN PERIODIC PLATES

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The aim of this contribution is to apply a new modelling approach to the thin linear-elastic Kirchhoff plates with a periodic structure proposed in [2] to investigate free vibrations. Using this approach we can describe the length-scale effect on the dynamic plate behaviour. This approach leading to the structural model is compared with the asymptotic homogenization approach, which neglects the length-scale effect and leads to the local model. It is shown under what conditions the local model is sufficient to analyse free vibrations of microperiodic plates and why we have to take into account the length-scale effect using the structural model. The physical correctness of this model is also discussed.

1. Introduction

The main aim of this paper is to apply general equations formulated in [2] to determine the effect of the microstructure size on the dynamic plate behaviour in the case of free vibrations. In the aforementioned paper the new modelling approach to thin elastic Kirchhoff periodic plates was presented. This approach leads to what is called the "refined" or "structural" plate theory, which makes it possible to investigate non-stationary processes in periodic plates, where the size of the periodicity unit cell plays a crucial role and cannot be neglected. The effect of the microstructure size on the macrobehaviour of a body is called the length-scale effect. In order to estimate this effect, we will investigate both *the macromodel with the length-scale effect of the microperiodic plate and the local macromodel* in which this effect is neglected.

The results of micromodelling applied in this paper are called *the refined theory* or *the structural macrodynamics of periodic plates*. The word "*structural*" is related to the fact, that the obtained equations take into consideration the length-scale effect on the plate behaviour. We will investigate linear-elastic plates with a microperiodic structure and satisfying the assumptions of the Kirchhoff plate theory. The presented theory can describe plates, the thickness and/or material and inertia properties of which are periodic functions of coordinates parametrizing the plate midplane.

The thesis of this research is that in dynamic problems for periodic plates such as the free vibrations, the length-scale effects play the important role and cannot be neglected. In this paper we will analyse the problem of free vibrations for the cylindrical bending of the plate strip simply supported on the opposite edges and made of an isotropic homogeneous material, and having periodic thickness. In the framework of structural models we will obtain equations involving the terms depending on the size of the periodicity unit cell and investigate the effect of the plate microstructure parameter on the plate behaviour. Moreover, from the proposed theory, by scaling down the microstructure parameter, we shall derive an "effective stiffness" theory, which will be called the local model. Using both theories to the analysis of free vibrations plate problem we will show the condition under which the microperiodic aspect of this problem can or cannot be neglected. In the first case we can use local models to investigate free vibrations, and in the second case we have to investigate this problem in the framework of the structural theory.

Our considerations will be based on the micromodelling procedure, which was presented in [1, 2]. This procedure leads to the governing equations of the refined theory of thin microperiodic plates, which in [2] was called *the structural theory.*

2. Preliminaries

Using the governing equations derived in [2] we will investigate free vibrations of microperiodic plates. To make the analysis more clear let us introduce some notations, following the paper [2].

By $\mathbf{x} \equiv (x_1, x_2)$ we denote the Cartesian coordinates of a point on the plate midplane Π and by z – the Cartesian coordinate in the direction normal to the midplane. By L_Π we denote the smallest characteristic length dimension of Π, and by $\Delta := (0, l_1) \times (0, l_2)$ – the periodicity cell on this plane. The size of the cell is described by *the microstructure length parameter* $l := \sqrt{l_1^2 + l_2^2}$, $l \ll L_\Pi$. By $\varrho = \varrho(\mathbf{x}, z)$ and $c_{ijkl} = c_{ijkl}(\mathbf{x}, z)$ we denote mass density and elastic moduli tensor of the plate material, and by $h = h(\mathbf{x})$ the plate thickness at a point $\mathbf{x} \in \Pi$. We assume that $h(\cdot)$ is a Δ-periodic function of $\mathbf{x}$ and $\varrho(\cdot)$ and $c_{ijkl}(\cdot)$ are Δ-periodic functions of $\mathbf{x}$ and even functions of z. Moreover, let p denote tractions on the upper and lower boundaries normal to Π, and b – the constant body force acting in the x_3-axis direction. Throughout the paper subscripts $a, b, ...$ run over 1, 2, and superscripts $A, B, ...$ run over $1, ..., N$. Summation convention holds for all aforementioned indices.

We shall introduce the following notations:

$$(2.1)\qquad \mu := \int_{-h/2}^{h/2} \varrho\, dz, \qquad j := \int_{-h/2}^{h/2} z^2 \varrho\, dz, \qquad d_{\alpha\beta\gamma\delta} := \int_{-h/2}^{h/2} z^2 c_{\alpha\beta\gamma\delta}\, dz,$$

$$D_{\alpha\beta\gamma\delta} \equiv \langle d_{\alpha\beta\gamma\delta}\rangle, \qquad D^{A}_{\alpha\beta\gamma\delta} \equiv \langle d_{\alpha\beta\gamma\delta} g^{A}_{,\gamma\delta}\rangle, \qquad D^{AB} \equiv \langle d_{\alpha\beta\gamma\delta} g^{A}_{,\gamma\delta} g^{B}_{,\alpha\beta}\rangle.$$

Denoting by E^{AB} elements of the matrix inverse to D^{AB} (it can be shown that the matrix D^{AB} is non-singular [2]), we denote by

$$(2.2)\qquad B_{\alpha\beta\gamma\delta} \equiv D_{\alpha\beta\gamma\delta} - D^{A}_{\alpha\beta} E^{AB} D^{B}_{\gamma\delta}\,,$$

what was called the "effective stiffnesses" tensor of the microperiodic plate in the framework of local models under consideration [2].

The micromodelling procedure based on the assumptions formulated in [2], yields the system of equations for macrodeflections W and inhomogeneity correctors V^A. In [2] the following structural models were obtained:

- *the general structural model,*
- *the structural model without rotational inertia terms,*
- *the model without terms of an order $\mathcal{O}(l^4)$.*

Moreover, neglecting in the governing equations of the structural model terms involving the microstructure parameter l, we have obtained in [2] the following models neglecting the length-scale effect:

- *the general local model,*
- *the local model without rotational inertia terms.*

The governing equations of the above structural models of thin microperiodic plates and the equations of local models were presented in the paper [2] (cf. Eqs. (4.2), (4.5), (4.6) and (5.3), (5.4) in [2]) and will be used here.

3. Governing equations

In this section we will specify the governing equations of structural and local models which were derived in the paper [2] for the plate strip simply supported on the opposite edges, made of an isotropic homogeneous material and having the l-periodic thickness $h(x)$ in the x-axis direction, $x \equiv x_1$, cf. Fig. 1. In this case all material properties are constant values; by ϱ, E, ν we denote the mass density, Young's modulus and Poisson's ratio, respectively. We will investigate the cylindrical bending of the plate. Hence, the unit cell is assumed in the form $\Delta_1 \equiv (0, l)$. The microstructure parameter l, which defines the size of the unit cell, satisfies the condition $l \ll L$, where L is the minimum characteristic length

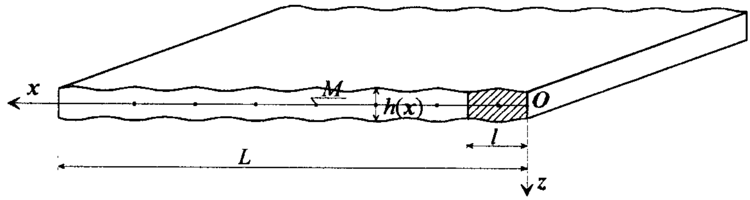


FIG. 1. Example of the plate with the l-periodic thickness.

dimension of the midplane Π. Moreover, we assume that $x = l/2$ is a symmetry plane of the unit cell. Formula for deflections $w(x,t)$ of the midplane Π, which was given in [2], has the form

$$w(x,t) = W(x,t) + g^A(x)V^A(x,t),$$

where W, V^A are called macrodeflections and inhomogeneity correctors, respectively, and g^A are microshape functions, $A = 1, ..., N$. For the sake of simplicity and for a symmetrical form of the unit cell, we introduce one microshape function

$$g(x) \equiv l^2[\cos(2\pi x/l) + c], \qquad g(x) \equiv g^1(x), \tag{3.1}$$

where the constant c is defined by the condition $\langle \mu g \rangle = 0$. Following the line of this approximation we introduce only one inhomogeneity corrector $V(x,t) = V^1(x,t)$. It follows that the basic kinematic unknowns are $W \equiv W(x,t)$ and $V \equiv V(x,t)$ defined for every $x \in \Pi$, $t \in (t_0, t_f)$, and deflections of the midplane can be written in the form

$$w(x,t) = W(x,t) + g(x)V(x,t).$$

In this way we restrict the function describing disturbances in the plate deflections caused by the microperiodic structure of the plate to the first term of a certain Fourier series.

Let us introduce the following notations

$$B \equiv \frac{Eh^3}{12(1-\nu^2)}, \tag{3.2}$$

$$D \equiv D^{11}, \qquad D_{11} \equiv D^1_{11}, \qquad D_{1111} = \langle B \rangle, \tag{3.3}$$

where D_{1111}, D_{11}, D^1_{11} are defined by $(2.1)_{4,5,6}$.

For the case under consideration the general equations, which were derived in [2] for structural and local models (cf. Eqs. (4.2), (4.5), (4.6) and (5.3), (5.4) in [2]), will be written down. The underlined terms in the equations given below depend on the size of the microstructure.

3.1. The general structural model

Using the notations for the coefficients introduced above, Eqs. (4.2) of [2] can be written in the form

$$
(3.4)\qquad
\begin{aligned}
\langle B\rangle W_{,1111} + D_{11}V_{,11} + \langle\mu\rangle\ddot{W} - \langle j\rangle\ddot{W}_{,11} - \underline{\langle jg_{,1}\rangle}\ddot{V}_{,1} &= p + b\langle\mu\rangle,\\
D_{11}W_{,11} + DV + \underline{\langle\mu gg\rangle}\ddot{V} + \underline{\langle jg_{,1}\rangle}\ddot{W}_{,1} + \underline{\langle jg_{,1}g_{,1}\rangle}\ddot{V} &= 0,
\end{aligned}
$$

where terms with j describe the effect of the rotational inertia on the dynamic plate behaviour.

3.2. The structural model without rotational inertia terms

Equations (4.5) of [2] for a special case of a cylindrical bending of the microperiodic plate are

$$
(3.5)\qquad
\begin{aligned}
\langle B\rangle W_{,1111} + D_{11}V_{,11} + \langle\mu\rangle\ddot{W} &= p + b\langle\mu\rangle,\\
D_{11}W_{,11} + DV + \underline{\langle\mu gg\rangle}\ddot{V} &= 0.
\end{aligned}
$$

3.3. The model without terms of order $O(l^4)$

Equations (4.6) of [2] in a case of a cylindrical bending of the microperiodic plate are

$$
(3.6)\qquad
\begin{aligned}
\langle B\rangle W_{,1111} + D_{11}V_{,11} + \langle\mu\rangle\ddot{W} - \langle j\rangle\ddot{W}_{,11} - \underline{\langle jg_{,1}\rangle}\ddot{V}_{,1} &= p + b\langle\mu\rangle,\\
D_{11}W_{,11} + DV + \underline{\langle jg_{,1}\rangle}\ddot{W}_{,1} + \underline{\langle jg_{,1}g_{,1}\rangle}\ddot{V} &= 0,
\end{aligned}
$$

where the terms involving j describe the effect of the plate rotational inertia.

3.4. The general local model

Equation (5.3) of [2] for a cylindrical bending of the plate has the form

$$
(3.7)\qquad B_{1111}W_{,1111} + \langle\mu\rangle\ddot{W} - \langle j\rangle\ddot{W}_{,11} = p + b\langle\mu\rangle,
$$

where B_{1111} is the coefficient (the "effective stiffness") defined by Eq. (2.2). In this case for the isotropic homogeneous plate having the l-periodic thickness, we arrive at

$$
(3.8)\qquad B_{1111} \equiv \langle B\rangle - D^{-1}D_{11}D_{11}\,,
$$

where D_{11}, D are defined by (3.3).

3.5. The local model without rotational inertia terms

Equation (5.4) of [2] for a case of a cylindrical bending of the plate has the form

$$B_{1111}W_{,1111} + \langle\mu\rangle\ddot{W} = p + b\langle\mu\rangle, \tag{3.9}$$

where B_{1111} is the coefficient defined by (3.8).

4. Applications: free vibrations

Let us show under what condition the length-scale effect can or cannot be neglected. To this end we will compare the results of the structural macromodelling and these of the local macromodelling. To this end we shall investigate free cylindrical vibrations of the plate strip simply supported on the opposite edges ($x_1 = 0$ and $x_1 = L$) made of an isotropic homogeneous material and having the l-periodic thickness, Fig. 1, in the x_1-axis direction. The thickness h of the plate is assumed for $x \in \Delta_1 = (0, l)$ in the form (Fig. 2)

$$h(x) = \begin{cases} h_1 & \text{if } x \in ((1-\lambda)l/2, (1+\lambda)l/2), \\ h_2 & \text{if } x \in [0, (1-\lambda)l/2] \cup [(1+\lambda)l/2], \end{cases} \tag{4.1}$$

where l is the microstructure parameter, λ is a real number from $[0, 1]$, which defines the relation between the length size of the cell part having the thickness h_1 and the length size of the cell l, which is the microstructure parameter (Fig. 2).

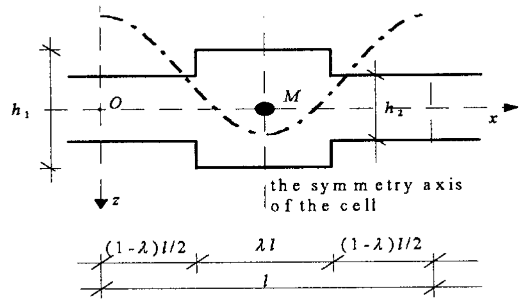


FIG. 2. The unit cell Δ_1 and the microshape function g.

Moreover, we will assume that the concentrated mass M is applied at the center of the unit cell $\Delta_1 = (0, l)$. The averaged density per unit area of the midplane is denoted by $m = \varrho H$, where H is the l-periodic function.

In order to investigate free vibrations we will assume that tractions p on the upper and lower boundaries of the plate and the body force b are equal to zero.

Moreover, assuming the microshape function in the form (3.1), satisfying the condition $\langle \mu g \rangle = 0$, we can show that $\langle j g_{,1} \rangle = 0$.

Under these conditions we will arrive below at the formulae for free vibration frequencies of the microperiodic plates in the framework of models presented by equations (3.4), (3.5), (3.6) and (3.7), (3.9).

4.1. *The general structural model*

Solutions to Eqs. (3.4) satisfying boundary conditions for the simply supported plate will be assumed in the form

$$W(x,t) = A_W \sin(kx)\cos(\omega t), \qquad V(x,t) = A_V \sin(kx)\cos(\omega t), \tag{4.2}$$

where A_W, A_V are vibrations amplitudes, $A_W A_V \neq 0$ and $k = 2\pi/L$ is the wave number. Substituting the right-hand sides of Eqs. (4.2) into Eqs. (3.4) we obtain the system of two linear algebraic equations for A_W, A_V

$$\begin{bmatrix} \langle B\rangle k^4 - \langle \mu\rangle\omega^2 - \langle j\rangle\omega^2 k^2 & -D_{11}k^2 \\ -D_{11}k^2 & -D - \langle \mu gg\rangle\omega^2 - \langle j g_{,1} g_{,1}\rangle\omega^2 \end{bmatrix} \begin{Bmatrix} A_W \\ A_V \end{Bmatrix} = \{0\}. \tag{4.3}$$

This system of equations has nontrivial solutions provided its determinant is equal to zero. In this way we obtain the characteristic equation for free vibrations frequencies.

Using the notations

$$\alpha \equiv \frac{\langle B\rangle}{\langle \mu\rangle}, \qquad \beta \equiv \langle \mu (g)^2\rangle, \qquad \chi \equiv \langle j(g_{,1})^2\rangle, \qquad \gamma \equiv \frac{(D_{11})^2}{\langle B\rangle D}, \qquad \varepsilon^2 \equiv \frac{\langle j\rangle}{\langle \mu\rangle}, \tag{4.4}$$

where μ, j are l-periodic functions defined by (2.1), B is defined by (3.2), and g is the microshape function defined by (3.1), and calculating the constant value c from the condition $\langle \mu g\rangle = 0$, we can write this equation in the form

$$\begin{aligned} &\omega^4 \langle \mu\rangle(\varepsilon^2 k^2 + 1)(\underline{\beta} + \underline{\chi}) \\ &\quad - \omega^2 \left[k^4 \langle B\rangle(\underline{\beta} + \underline{\chi}) + D\langle \mu\rangle(\varepsilon^2 k^2 + 1)\right] + k^4\left[\langle B\rangle D - (D_{11})^2\right] = 0. \end{aligned} \tag{4.5}$$

From the above condition, the values of free vibration frequencies ω can be calculated. We arrive at the following formulae for lower ω_1 and higher ω_2 free vibration frequencies:

$$\begin{aligned} \omega_1^2 &= [2(1+\varepsilon^2k^2)(\beta+\chi)]^{-1}\Big\{\alpha k^4(\beta+\chi) + D(1+\varepsilon^2k^2) \\ &\quad - \sqrt{\alpha(\beta+\chi)k^4[\alpha(\beta+\chi)k^4 - 2D(1+\varepsilon^2k^2)(2\gamma-1)] + D^2(1+\varepsilon^2k^2)^2}\Big\}, \\ \omega_2^2 &= \{2(1+\varepsilon^2k^2)(\beta+\chi)\}^{-1}\Big\{\alpha k^4(\beta+\chi) + D(1+\varepsilon^2k^2) \\ &\quad + \sqrt{\alpha(\beta+\chi)k^4[\alpha(\beta+\chi)k^4 - 2D(1+\varepsilon^2k^2)(2\gamma-1)] + D^2(1+\varepsilon^2k^2)^2}\Big\}. \end{aligned} \tag{4.6}$$

The above formulae were obtained in the framework of the general structural model, which is described by Eqs. (3.4), for the plates under consideration.

4.2. The structural model without rotational inertia terms

Solutions to Eqs. (3.5) satisfying boundary conditions for the simply supported plate can be assumed in the form (4.2). Substituting the right-hand sides of Eqs. (4.2) into Eqs. (3.5) we obtain the system of two linear algebraic equations for A_W, A_V

$$\begin{bmatrix} \langle B\rangle k^4 - \langle\mu\rangle\omega'^2 & -D_{11}k^2 \\ -D_{11}k^2 & D - \langle\mu gg\rangle\omega'^2 \end{bmatrix} \begin{Bmatrix} A_W \\ A_V \end{Bmatrix} = \{0\}. \tag{4.7}$$

This system has nontrivial solutions if its determinant is equal to zero. Hence we obtain the characteristic equation for the free vibration frequencies. Using notations (4.4), we can write this equation as

$$\omega_1'^4\langle\mu\rangle\underline{\beta} - \omega'^2(k^4\langle B\rangle\underline{\beta} + D\langle\mu\rangle) + k^4[\langle B\rangle D - (D_{11})^2] = 0. \tag{4.8}$$

From the above condition, after some manipulations, we arrive at the following formulae for lower ω_1' and higher ω_2', the free vibration frequencies

$$\begin{aligned} \omega_1'^2 &= \frac{1}{2}\alpha k^4 + (2\beta)^{-1}\left[D - \sqrt{(\alpha\beta k^4 - D)^2 + 4\gamma\alpha\beta D k^4}\right], \\ \omega_2'^2 &= \frac{1}{2}\alpha k^4 + (2\beta)^{-1}\left[D + \sqrt{(\alpha\beta k^4 - D)^2 + 4\gamma\alpha\beta D k^4}\right]. \end{aligned} \tag{4.9}$$

The above formulae were obtained in the framework of the structural model without rotational inertia terms described by Eqs. (3.5).

4.3. The model without terms of order $\mathcal{O}(l^4)$

Solutions to Eqs. (3.6) satisfying boundary conditions for the simply supported plate can be assumed in the form (4.2). Substituting the right-hand sides of Eqs. (4.2) into Eqs. (3.6), we obtain the system of two linear algebraic equations for A_W, A_V

$$\begin{bmatrix} \langle B\rangle k^4 - \langle\mu\rangle\omega''^2 - \langle j\rangle\omega''^2k^2 & -D_{11}k^2 \\ -D_{11}k^2 & D - \langle jg_{,1}g_{,1}\rangle\omega''^2 \end{bmatrix} \begin{Bmatrix} A_W \\ A_V \end{Bmatrix} = \{0\}. \tag{4.10}$$

This system of equations has nontrivial solutions if its determinant is equal to zero; hence we arrive at the characteristic equation for frequencies. Using notations (4.4) we can write the aforementioned equation in the form

$$\begin{aligned}(4.11)\qquad &\omega''^4\langle\mu\rangle(1+\varepsilon^2k^2)\underline{\chi}\\ &\quad-\omega''^2\left[k^4\langle B\rangle\underline{\chi}+D\langle\mu\rangle(1+\varepsilon^2k^2)\right]+k^4[\langle B\rangle D-(D_{11})^2]=0.\end{aligned}$$

From the above condition the values of the free vibration frequencies ω'' can be calculated. We arrive at the following formulae for lower ω''_1 and higher ω''_2 resonance frequencies

$$\begin{aligned}&\omega_1''^2=[2\chi(1+\varepsilon^2k^2)]^{-1}\Big\{\alpha\chi k^4+D(1+\varepsilon^2k^2)\\ &\qquad-\sqrt{\alpha\chi k^4[\alpha\chi k^4+2D(1+\varepsilon^2k^2)(2\gamma-1)]+D^2(1+\varepsilon^2k^2)^2}\Big\},\\ (4.12)\qquad&\omega_2''^2=[2\chi(1+\varepsilon^2k^2)]^{-1}\Big\{\alpha\chi k^4+D(1+\varepsilon^2k^2)\\ &\qquad+\sqrt{\alpha\chi k^4[\alpha\chi k^4+2D(1+\varepsilon^2k^2)(2\gamma-1)]+D^2(1+\varepsilon^2k^2)^2}\Big\}.\end{aligned}$$

The above results were obtained in the framework of the model without terms of order $\mathcal{O}(l^4)$, which is described by Eqs. (3.6).

4.4. The general local model

Solutions to Eqs. (3.7), satisfying boundary conditions for the simply supported plate, can be assumed in the form

$$(4.13)\qquad W(x,t)=A_W\sin(kx)\cos(\omega t),$$

where A_W is a vibration amplitude, $A_W\neq 0$ and $k=2\pi/L$ is the wave number. Substituting the right-hand side of Eq. (4.13) into Eqs. (3.7) and using $(4.4)_5$ we obtain the linear algebraic equation for A_W in the form

$$(4.14)\qquad [B_{1111}k^4-\tilde{\omega}^2\langle\mu\rangle(1+\varepsilon^2k^2)]A_W=0,$$

which leads to the characteristic equation for frequencies

$$(4.15)\qquad -\tilde{\omega}^2\langle\mu\rangle(1+\varepsilon^2k^2)+B_{1111}k^4=0.$$

From the above condition we can calculate the value of the free vibration frequency $\tilde{\omega}$. Using notations (3.8) and (4.4), we arrive at the following formula for the lower free vibration frequency

$$(4.16)\qquad \tilde{\omega}^2=\left(\alpha-\frac{\gamma}{D}\right)\frac{k^4}{1+\varepsilon^2k^2}\cdot$$

The above result was obtained in the framework of the general local model, which is described by Eqs. (3.7).

4.5. The local model without rotational inertia terms

Solutions to Eqs. (3.9) satisfying boundary conditions for the simply supported plate can be assumed in the form (4.13). Substituting the right-hand side of Eq. (4.13) into Eqs. (3.9) and using the relation $(4.4)_5$, we obtain the linear algebraic equations for A_W in the form

$$(B_{1111}k^4 - \langle\mu\rangle\widetilde{\omega}'^2)A_W = 0, \tag{4.17}$$

which leads to the characteristic equation for free vibration frequencies

$$-\langle\mu\rangle\widetilde{\omega}'^2 + B_{1111}k^4 = 0. \tag{4.18}$$

From the above condition we can calculate the value of the free vibration frequency $\widetilde{\omega}'$. Using notations (3.8) and (4.4), we arrive at the following formula for the lower free vibration frequency

$$\widetilde{\omega}'^2 = \left(\alpha - \frac{\gamma}{D}\right)k^4. \tag{4.19}$$

This formula was obtained in the framework of the local model without rotational inertia terms described by Eqs. (3.9).

It can be seen that the structural models, which are described by Eqs. (3.4), (3.5) and (3.6) lead to formulae for lower and higher free vibration frequencies. However, using the local models (Eqs. (3.7) and (3.9)), we can obtain only lower frequencies of the microperiodic plates under consideration.

5. Diagrams of free vibration frequencies

In this section we will present diagrams of free vibration frequencies for the isotropic homogeneous plate simply supported on the opposite edges and having the l-periodic thickness. We will make these diagrams using the obtained formulae for frequencies, which will be shown below in the dimensionless form. To this end we will introduce the notation

$$\kappa \equiv \frac{E}{12(1-\nu^2)}, \tag{5.1}$$

where E – is the Young modulus, ν – is the Poisson ratio, and some dimensionless coefficients $\overline{\varepsilon}^2$, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\chi}$, $\overline{D}$, satisfying relations

$$\varepsilon^2 = \overline{\varepsilon}^2 l^2, \qquad \alpha = \overline{\alpha}\frac{\kappa l^2}{\varrho}, \qquad \beta = \overline{\beta}\varrho l^5, \qquad \chi = \overline{\chi}\varrho l^5, \qquad D = \overline{D}\kappa l^3, \tag{5.2}$$

where D is defined by $(3.3)_1$, ε^2, α, β, χ are defined by $(4.4)_{5,1,2,3}$, ϱ is the constant value of the mass density and l is the microstructure parameter.

Multiplying both sides of relations (4.6), (4.9), (4.12), (4.16), (4.19) by $l^2\varrho\kappa^{-1}$, we obtain the following dimensionless formulae for frequencies:

- free vibration frequencies for the general structural model:

$$
\begin{aligned}
\Omega_1 &= \left[2(\overline{\beta}+\overline{\chi})(1+\overline{\varepsilon}^2q^2)\right]^{-1/2}\Big\{\overline{\alpha}q^4(\overline{\beta}+\overline{\chi})+\overline{D}(1+\overline{\varepsilon}^2q^2)\\
&\quad-\sqrt{\overline{\alpha}(\overline{\beta}+\overline{\chi})q^4\left[\overline{\alpha}(\overline{\beta}+\overline{\chi})q^4+2\overline{D}(1+\overline{\varepsilon}^2q^2)(2\gamma-1)\right]+\overline{D}^2(1+\overline{\varepsilon}^2q^2)^2}\Big\}^{1/2},\\
\Omega_2 &= \left[2(\overline{\beta}+\overline{\chi})(1+\overline{\varepsilon}^2q^2)\right]^{-1/2}\Big\{\overline{\alpha}q^4(\overline{\beta}+\overline{\chi})+\overline{D}(1+\overline{\varepsilon}^2q^2)\\
&\quad+\sqrt{\overline{\alpha}(\overline{\beta}+\overline{\chi})q^4\left[\overline{\alpha}(\overline{\beta}+\overline{\chi})q^4+2\overline{D}(1+\overline{\varepsilon}^2q^2)(2\gamma-1)\right]+\overline{D}^2(1+\overline{\varepsilon}^2q^2)^2}\Big\}^{1/2},
\end{aligned}
\tag{5.3}
$$

- free vibration frequencies for the structural model without rotational inertia terms:

$$
\begin{aligned}
\Omega_1' &= (2\overline{\beta})^{-1/2}\sqrt{q^4\overline{\alpha}\overline{\beta}+\overline{D}-\sqrt{(\overline{\alpha}\overline{\beta}q^4-\overline{D})^2+4\gamma\overline{\alpha}\overline{\beta}\overline{D}q^4}}\,,\\
\Omega_2' &= (2\overline{\beta})^{-1/2}\sqrt{q^4\overline{\alpha}\overline{\beta}+\overline{D}+\sqrt{(\overline{\alpha}\overline{\beta}q^4-\overline{D})^2+4\gamma\overline{\alpha}\overline{\beta}\overline{D}q^4}}\,,
\end{aligned}
\tag{5.4}
$$

- free vibration frequencies for the model without terms of order $\mathcal{O}(l^4)$:

$$
\begin{aligned}
\Omega_1'' &= \left[2\overline{\chi}(1+\overline{\varepsilon}^2q^2)\right]^{-1/2}\Big\{\overline{\alpha}\overline{\chi}q^4+\overline{D}(1+\overline{\varepsilon}^2q^2)\\
&\quad-\sqrt{\overline{\alpha}\overline{\chi}q^4\left[\overline{\alpha}\overline{\chi}q^4+2\overline{D}(1+\overline{\varepsilon}^2q^2)(2\gamma-1)\right]+\overline{D}^2(1+\overline{\varepsilon}^2q^2)^2}\Big\}^{1/2},\\
\Omega_2'' &= \left[2\overline{\chi}(1+\overline{\varepsilon}^2q^2)\right]^{-1/2}\Big\{\overline{\alpha}\overline{\chi}q^4+\overline{D}(1+\overline{\varepsilon}^2q^2)\\
&\quad+\sqrt{\overline{\alpha}\overline{\chi}q^4\left[\overline{\alpha}\overline{\chi}q^4+2\overline{D}(1+\overline{\varepsilon}^2q^2)(2\gamma-1)\right]+\overline{D}^2(1+\overline{\varepsilon}^2q^2)^2}\Big\}^{1/2},
\end{aligned}
\tag{5.5}
$$

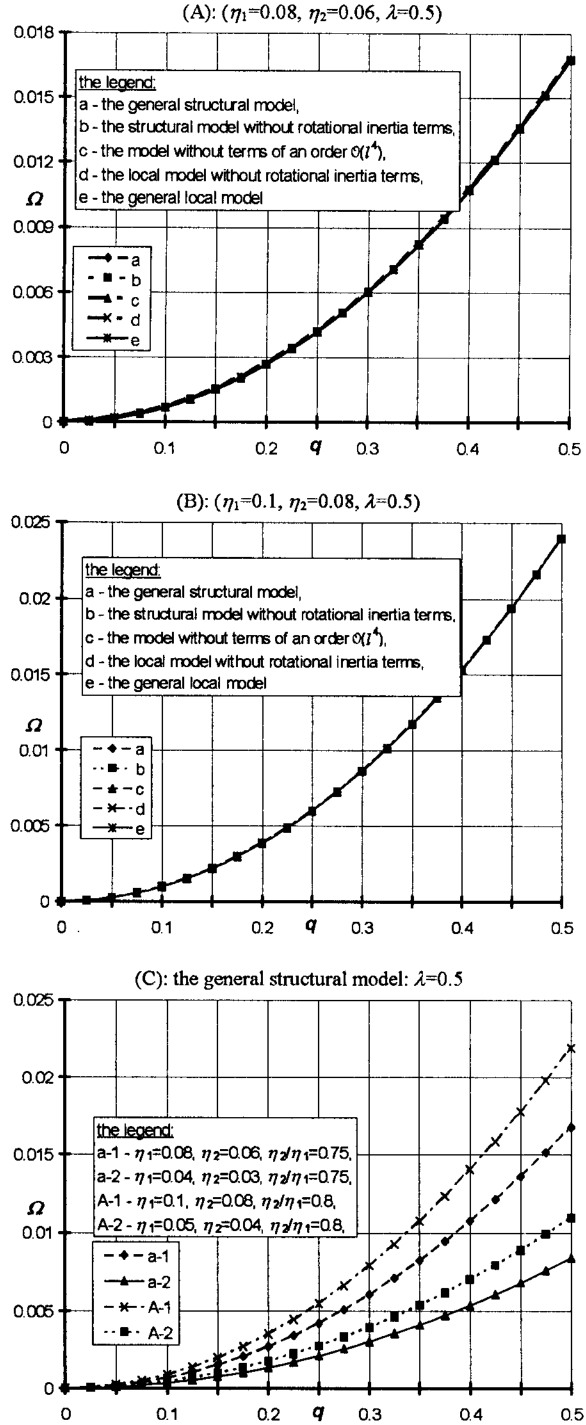
- free vibration frequency for the general local model:

$$
\tilde{\Omega}=\sqrt{\overline{\alpha}(1-\gamma)}\,\frac{q^2}{\sqrt{1+\overline{\varepsilon}^2q^2}}\,,
\tag{5.6}
$$

- free vibration frequency for the local model without rotational inertia terms

$$
\tilde{\Omega}'=\sqrt{\overline{\alpha}(1-\gamma)}\,q^2.
\tag{5.7}
$$

In the above formulae q is the dimensionless wave number ($q \equiv kl$, where $k = 2\pi/L$ is the wave number, l is the microstructure parameter, L is the span of the plate along the x-axis).


(A): (η_1=0.08, η_2=0.06, λ=0.5)
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order $O(l^4)$,
d - the local model without rotational inertia terms,
e - the general local model
a
b
c
d
e
Ω
q
0.018
0.015
0.012
0.009
0.006
0.003
0
0
0.1
0.2
0.3
0.4
0.5
(B): (η_1=0.1, η_2=0.08, λ=0.5)
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order $O(l^4)$,
d - the local model without rotational inertia terms,
e - the general local model
a
b
c
d
e
Ω
q
0.025
0.02
0.015
0.01
0.005
0
0
0.1
0.2
0.3
0.4
0.5
(C): the general structural model: λ=0.5
the legend:
a-1 - η_1=0.08, η_2=0.06, η_2/η_1=0.75,
a-2 - η_1=0.04, η_2=0.03, η_2/η_1=0.75,
A-1 - η_1=0.1, η_2=0.08, η_2/η_1=0.8,
A-2 - η_1=0.05, η_2=0.04, η_2/η_1=0.8,
a-1
a-2
A-1
A-2
Ω
q
0.025
0.02
0.015
0.01
0.005
0
0
0.1
0.2
0.3
0.4
0.5

[FIG. 3]

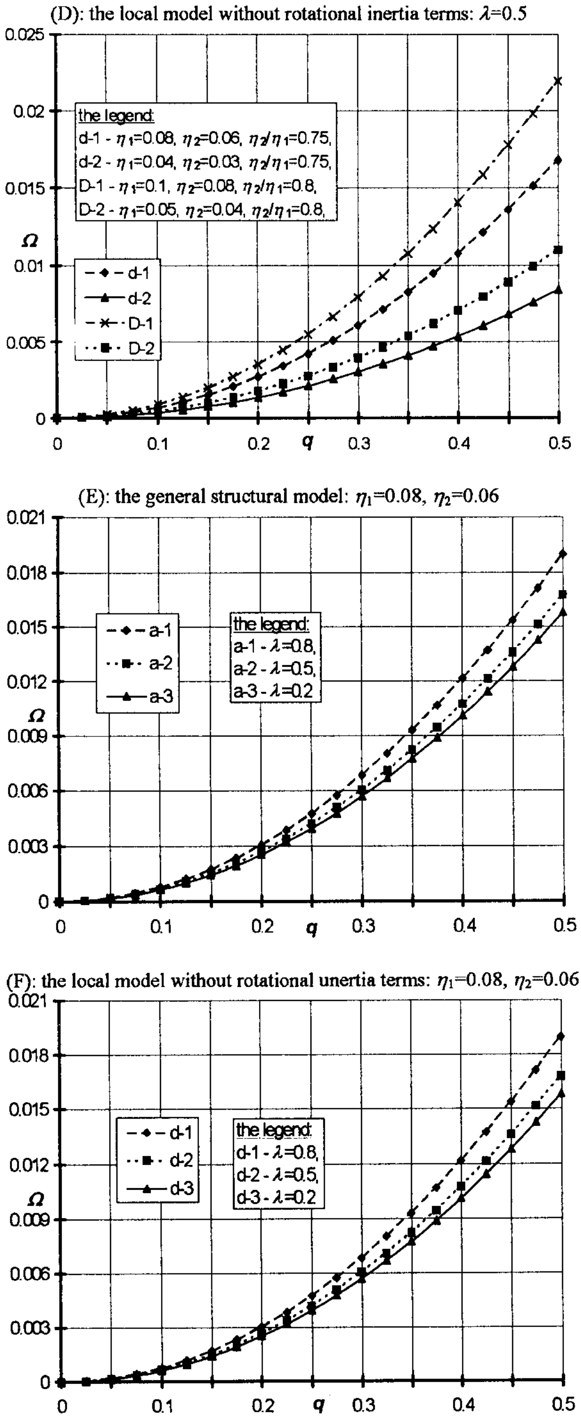
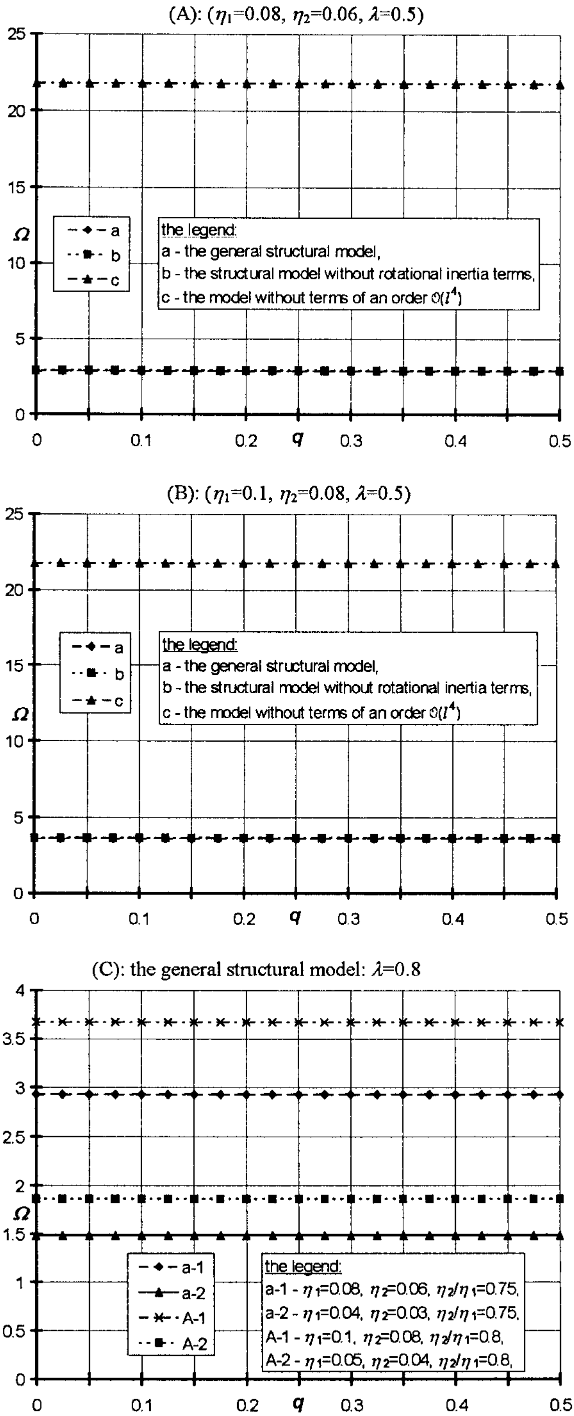


FIG. 3. Lower free vibration frequencies for the plate without concentrated masses ($\eta = 0$).



(A): (η_1=0.08, η_2=0.06, λ=0.5)
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order $O(l^4)$
a
b
c
Ω
q
(B): (η_1=0.1, η_2=0.08, λ=0.5)
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order $O(l^4)$
(C): the general structural model: λ=0.8
a-1
a-2
A-1
A-2
the legend:
a-1 - η_1=0.08, η_2=0.06, η_2/η_1=0.75,
a-2 - η_1=0.04, η_2=0.03, η_2/η_1=0.75,
A-1 - η_1=0.1, η_2=0.08, η_2/η_1=0.8,
A-2 - η_1=0.05, η_2=0.04, η_2/η_1=0.8,

[FIG. 4]

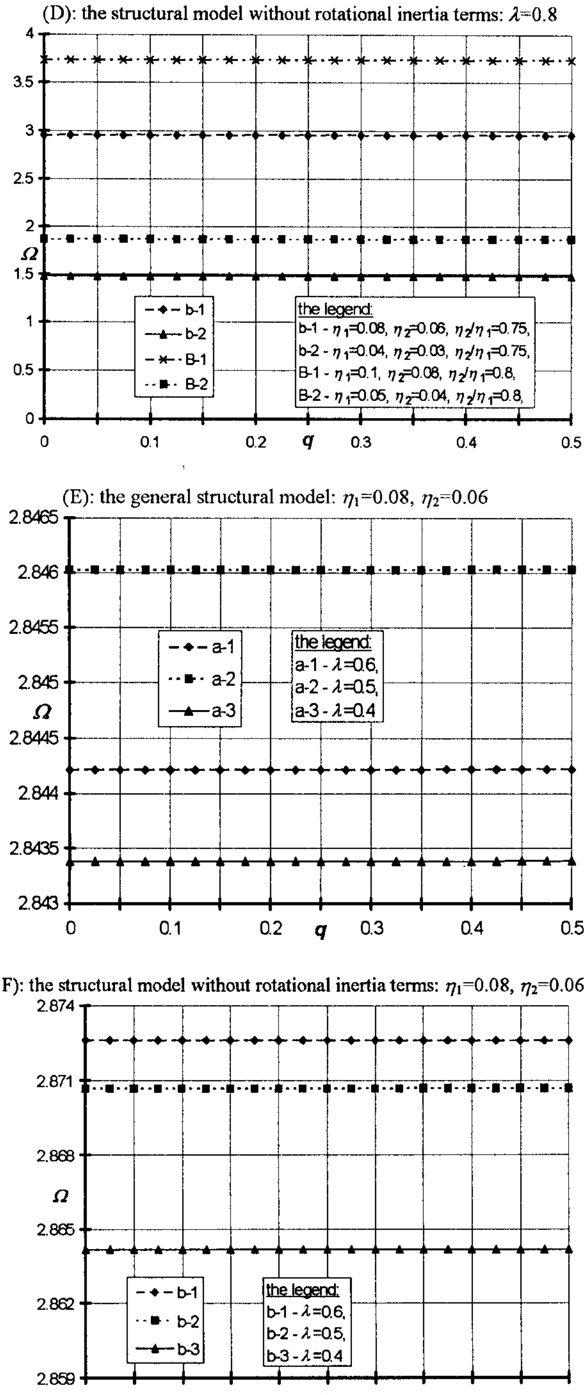
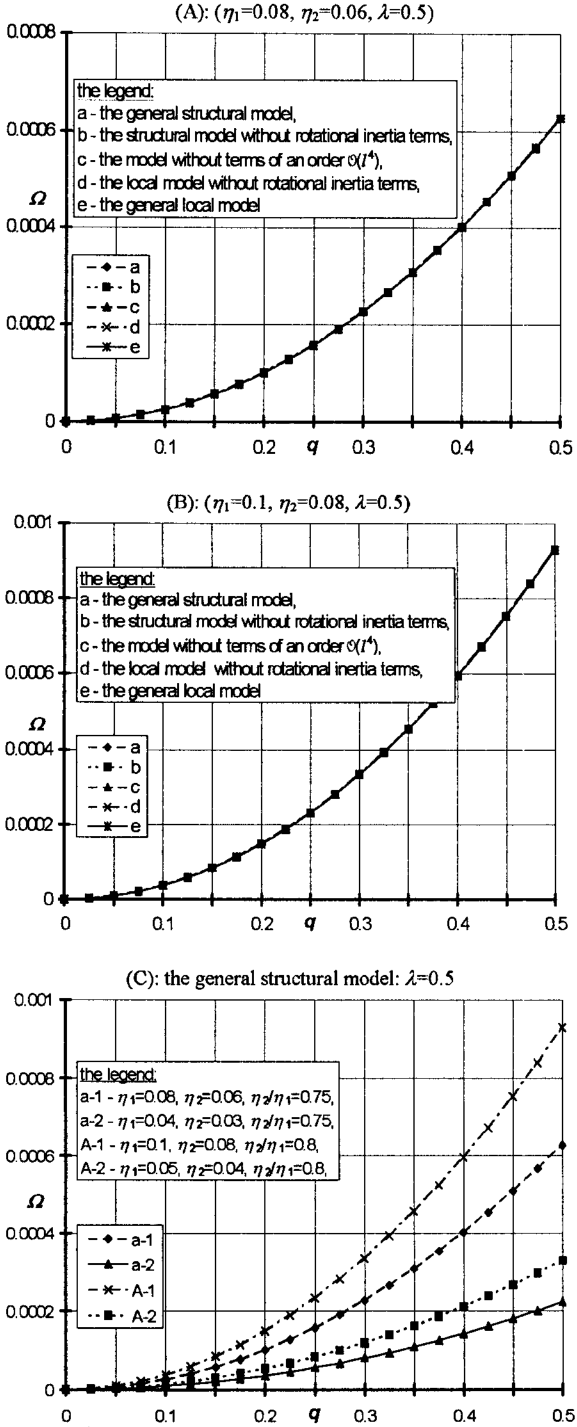


FIG. 4. Higher free vibration frequencies for the plate without concentrated masses ($\eta = 0$).


(A): (η_1=0.08, η_2=0.06, λ=0.5)
0.0008
0.0006
0.0004
0.0002
0
Ω
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order O(l^4),
d - the local model without rotational inertia terms,
e - the general local model
a
b
c
d
e
0
0.1
0.2
q
0.3
0.4
0.5
(B): (η_1=0.1, η_2=0.08, λ=0.5)
0.001
0.0008
0.0006
0.0004
0.0002
0
Ω
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order O(l^4),
d - the local model without rotational inertia terms,
e - the general local model
a
b
c
d
e
0
0.1
0.2
q
0.3
0.4
0.5
(C): the general structural model: λ=0.5
0.001
0.0008
0.0006
0.0004
0.0002
0
Ω
the legend:
a-1 - η_1=0.08, η_2=0.06, η_2/η_1=0.75,
a-2 - η_1=0.04, η_2=0.03, η_2/η_1=0.75,
A-1 - η_1=0.1, η_2=0.08, η_2/η_1=0.8,
A-2 - η_1=0.05, η_2=0.04, η_2/η_1=0.8,
a-1
a-2
A-1
A-2
0
0.1
0.2
q
0.3
0.4
0.5

[FIG. 5]

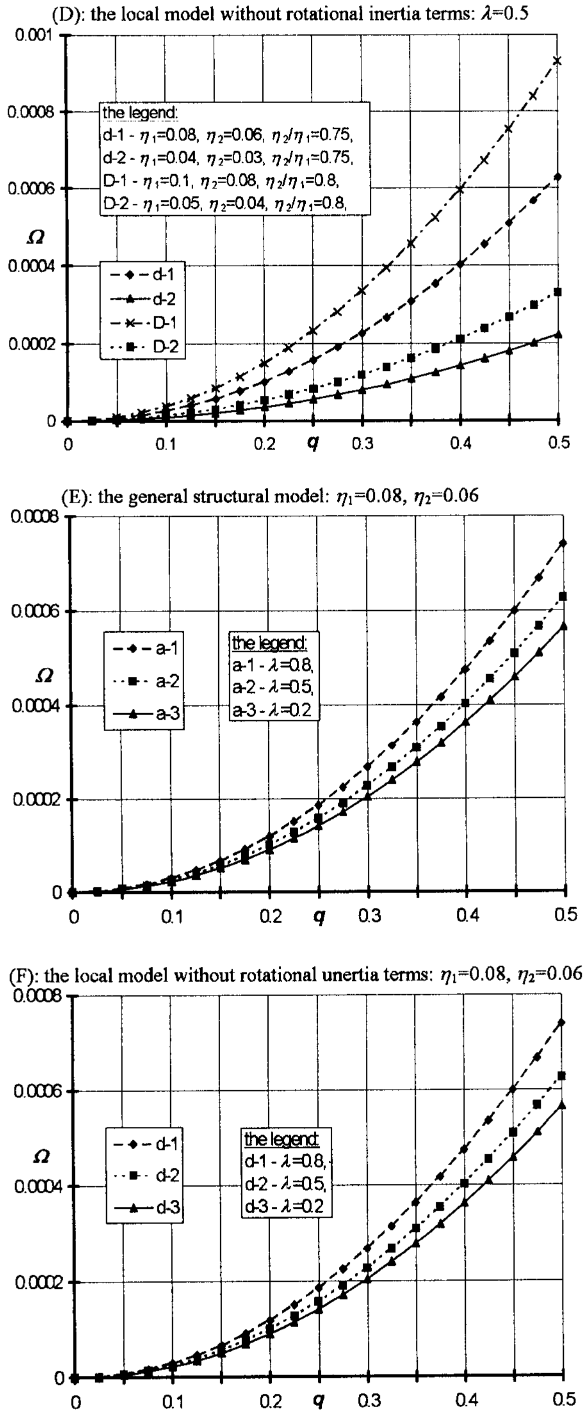
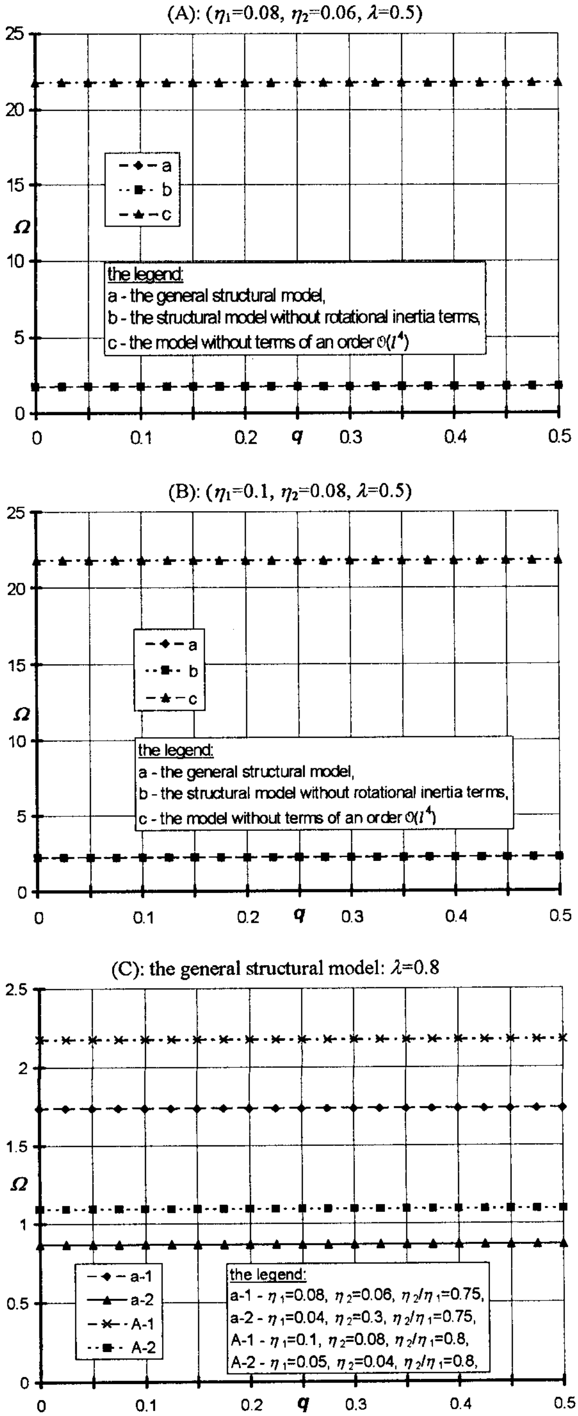


FIG. 5. Lower free vibration frequencies for the plate with concentrated masses ($\eta = 50$).


(A): (η_1=0.08, η_2=0.06, λ=0.5)
Ω
q
a
b
c
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order $O(l^4)$
(B): (η_1=0.1, η_2=0.08, λ=0.5)
Ω
q
a
b
c
the legend:
a - the general structural model,
b - the structural model without rotational inertia terms,
c - the model without terms of an order $O(l^4)$
(C): the general structural model: λ=0.8
Ω
q
a-1
a-2
A-1
A-2
the legend:
a-1 - η_1=0.08, η_2=0.06, η_2/η_1=0.75,
a-2 - η_1=0.04, η_2=0.3, η_2/η_1=0.75,
A-1 - η_1=0.1, η_2=0.08, η_2/η_1=0.8,
A-2 - η_1=0.05, η_2=0.04, η_2/η_1=0.8,

[Fig. 6]

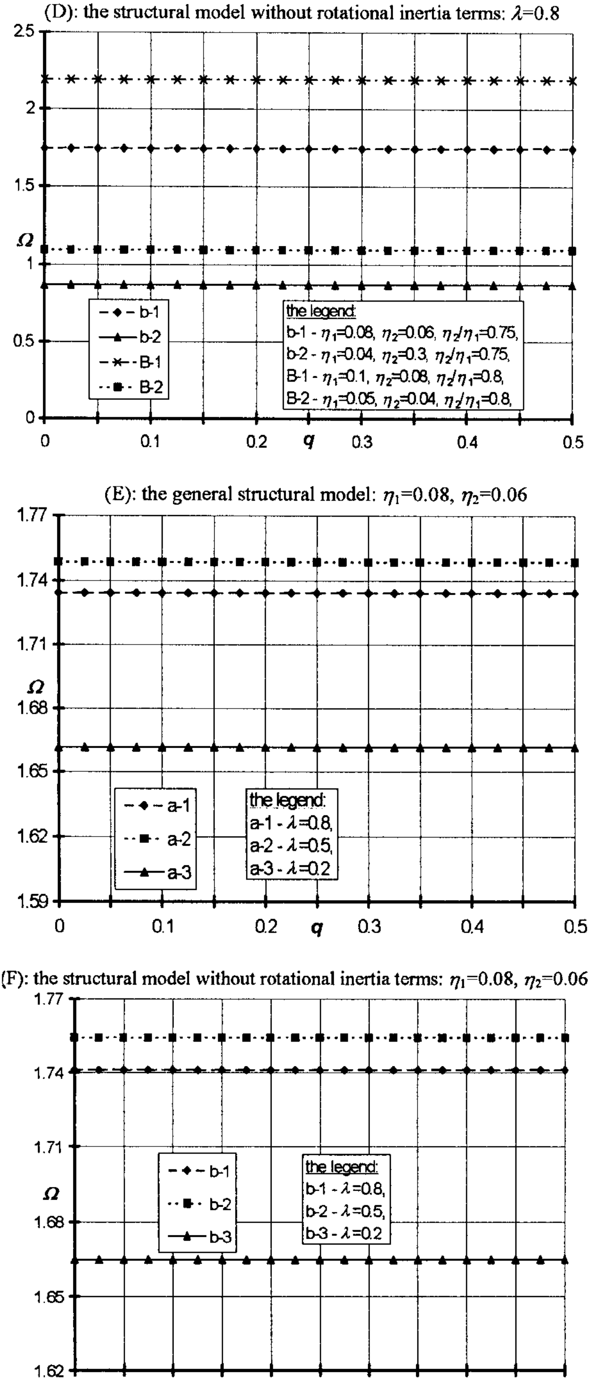


FIG. 6. Higher free vibration frequencies for the plate with concentrated masses ($\eta = 50$).

From formulae (5.3)–(5.7) we obtain below the diagrams shown in Figs. 3–6. These diagrams are made for the plate having the l-periodic thickness h, which is defined by (4.1) and Fig. 2. In order to investigate different cases of the thickness, we will introduce dimensionless coefficients

$$\eta_1 \equiv \frac{h_1}{l}, \qquad \eta_2 \equiv \frac{h_2}{l}, \tag{5.8}$$

where h_1, h_2 are thicknesses of the plate in the cell Δ_1 (Fig. 2) and l is the microstructure parameter (the length size of this cell). Moreover, the concentrated masses M can be put at the center of every cell. Because the averaged density per unit area of this mass can be denote by $m = \varrho H$, where H is the l-periodic function and ϱ is the mass density of the plate, we will introduce the notation

$$\eta \equiv \frac{H}{l}. \tag{5.9}$$

The value of η is equal to zero in the entire region of the cell except the center, and in this point it can be greater than zero (if the mass is located at this point) or equal to zero (if the mass is equal to zero).

In Figs. 3–6 the following diagrams are shown:

• Figures 3, 4 – diagrams of lower and higher free vibration frequencies, respectively, for the plate under consideration without concentrated masses ($\eta = 0$ in the whole region of the cell, where η is defined by (5.9));

• Figures 5, 6 – diagrams of lower and higher free vibration frequencies, respectively, for this plate with the concentrated mass in the middle of every cell (hence, in this point $\eta = 50$).

Below, we will comment the above diagrams.

• Diagrams in Fig. 3 show lower free vibrations frequencies.

∘ Diagrams (A) and (B) in Fig. 3 show the values of lower free vibrations frequencies obtained for different models. These diagrams are made for parameter $\lambda = 0.5$, which describes the length of the part of the cell having the thickness h_1 (Fig. 2). The parameters η_1, η_2 are defined by (5.8) and describe relation between the thickness of the plate and the microstructure parameter l.

∘ In diagrams (C) and (D) in Fig. 3 we can see the values of lower frequencies for the general structural model and the local model without rotational inertia terms, respectively. These diagrams are made for the value $\lambda = 0.5$, for different values of the quotient η_2/η_1 (where parameters η_1, η_2 are defined by (5.8)), and for different values of these parameters describing the plate thickness. It can be observed that the frequencies for all the structural and local models grow together with increasing thickness of the plate (with increasing parameters η_1, η_2).

∘ Diagrams (E) and (F) in Fig. 3 are made for the general structural model and the local model without rotational inertia terms, respectively, for the constant quotient η_2/η_1 and different values of λ. It can be observed that for every model, lower frequencies grow with an increasing $\lambda \in [0, 1]$.

• In diagrams in Fig. 4 we can see higher free vibration frequencies.

∘ Diagrams (A), (B) show higher frequencies for the general structural model (Eq. $(5.3)_2$), for the model without rotational inertia terms (Eq. $(5.4)_2$), and for the model without terms of order $\mathcal{O}(l^4)$ (Eq. $(5.5)_2$). We can observe that the lowest values of the frequencies can be obtained from the general structural model.

∘ Diagrams (C) and (D) demonstrate, that values of higher free vibration frequencies obtained from any structural model, for constant parameter λ and constant relation η_2/η_1, increase with increasing parameters η_1, η_2 (which describe the relation between the plate thickness and the microstructure parameter l).

∘ Diagrams (E) and (F) in Fig. 4 are made for the general structural model and the model without terms of order $\mathcal{O}(l^4)$, respectively, for the constant quotient η_2/η_1 and different values of λ. It can be observed that for every model, higher frequencies grow for the parameter $\lambda \in [0, \lambda']$, then decrease for $\lambda \in [\lambda', \lambda'']$ and grow for $\lambda \in [\lambda'', 1]$, where λ', λ'' depend on parameters η_1, η_2.

• Diagrams in Figs. 5 and 6, which show free vibration frequencies for the plate with concentrated masses (the parameter η defined by (5.9) is equal to $\eta = 50$ in the middle of the cell), can be commented in the following form.

∘ Diagrams (A)–(F) in Fig. 5 and (A)–(F) in Fig. 6 can be described in the same form as in the case of Figs. 3 and 4 (free vibration frequencies of the plate for the parameter $\eta = 0$ – without concentrated masses).

• Comparing diagrams of free vibration frequencies for the plate without and with additional concentrated masses (e.g. Fig. 3(A)–(B) and Fig. 5(A)–(B)), we can see that these masses decrease the values of free vibration frequencies.

6. Physical correctness of the proposed approach

6.1. *The structural model*

In order to verify the presented modelling approach, let us investigate in this section free vibrations of a special case of microperiodic plates. It will be shown that assumption of only one microshape function is sufficient to consider the micromotion of a plate described by this function.

Let us assume the plate discussed in Secs. 3–5 – the plate strip simply supported on the opposite edges ($x = 0$ and $x = L$) and made of an isotropic homogeneous material, having the l-periodic thickness h and with periodically distributed concentrated masses M. An example of this plate is shown in the

Fig. 1. Moreover, we would like to investigate micromotions described by the microshape function in the form

$$g(x) \equiv l^2 \sin(2\pi x/l). \tag{6.1}$$

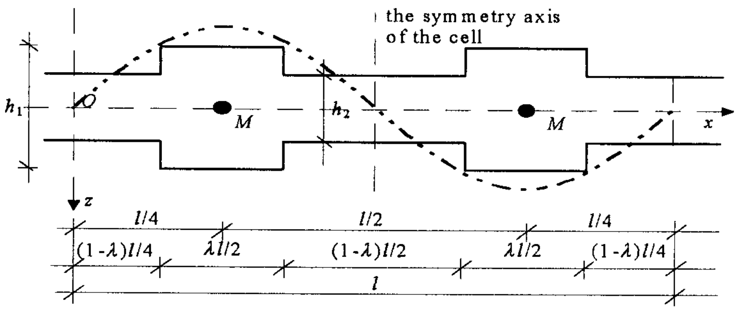


FIG. 7. The unit cell Δ_1 and the microshape function g.

Hence, we assume the periodicity unit cell in the form shown in the Fig. 7, and the thickness h as

$$h(x) = \begin{cases} h_1 & \text{if } \ x \in ((1-\lambda)l/4, (1+\lambda)l/4) \cup ((3-\lambda)l/4, (3+\lambda)l/4), \\ h_2 & \text{if } \ x \in [(0, (1-\lambda)l/4] \cup ((1+\lambda)l/4), \\ & \qquad (3-\lambda)l/4 \cup [(3+\lambda)l/4), l], \end{cases} \tag{6.2}$$

where l is the microstructure parameter, λ is a real number from the interval $[0,1]$. The concentrated masses M will be applied at two points of the unit cell Δ_1 on the x-axis – at $x = l/4$ and $x = 3l/4$, and the averaged density of each of them is denoted like it was in the Sec. 4.

For the purpose of investigation of free vibrations we will assume that tractions p on upper and lower boundaries of the plate and the body force b are equal to zero. Using the governing equations of the general structural model (3.4), this plate will now be analysed. Describing micromotions by the microshape function in the form (6.1) we obtain that $\langle \mu g \rangle = 0$, $\langle j g_{,1} \rangle = 0$, and moreover, $D_{11} = 0$. In this way Eqs. (3.4) have the form

$$\begin{aligned} \langle B \rangle W_{,1111} + \langle \mu \rangle \ddot{W} - \langle j \rangle \ddot{W}_{,11} &= 0, \\ DV + \underline{\langle \mu g g \rangle} \ddot{V} + \underline{\langle j g_{,1} g_{,1} \rangle} \ddot{V} &= 0. \end{aligned} \tag{6.3}$$

The above equations represent the system of two independent differential equations – the first for the macrodeflections W and the second for the corrector V.

Solutions to Eqs. (6.3) satisfying boundary conditions for the simply supported plate strip will be assumed in the form (4.2). Substituting the right-hand sides of Eqs. (4.2) into Eqs. (6.3), we obtain two independent linear algebraic equations for amplitudes A_W, A_V. After some manipulations we obtain the following formulae for free vibrations frequencies:

- *the lower frequency*

$$(\omega_1)^2 = \frac{\langle B\rangle}{\langle \mu\rangle + \langle j\rangle k^2} k^4, \tag{6.4}$$

- *the higher frequency*

$$(\omega_2)^2 = \frac{D}{\underline{\langle \mu g g\rangle} + \underline{\langle j g_{,1} g_{,1}\rangle}}, \tag{6.5}$$

where $k = 2\pi/L$ is the wave number.

We can observe that in the case of micromotions described by the microshape function (6.1), the lower frequency depends only on the plate macrostructure, and the higher frequency depends only on the microstructure of the plate.

For the sake of simplicity we assume constant thickness h of the plate under consideration. Hence, formulae (6.4)–(6.5) can be written in the following forms:

- *the lower frequency*

$$(\omega_1)^2 = \frac{Eh^3}{\varrho(1-\nu)} \frac{k^4}{12(h+2H) + h^3 k^2}, \qquad k = 2\pi/L, \tag{6.6}$$

- *the higher frequency*

$$(\omega_2)^2 = \frac{Eh^3}{\varrho(1-\nu)} \frac{4\pi^4}{[3l^2(h+4H) + \pi^2 h^3] l^2}, \tag{6.7}$$

where E, ϱ, ν are the Young modulus, the mass density and the Poisson ratio, respectively, h is the constant plate thickness, H is the function describing concentrated masses and l is the microstructure length-parameter.

It can be seen that the lower frequency (6.6) is independent of the size of the plate microstructure, and that the higher frequency depends only on the microstructure parameter l and is independent of the wave number k.

6.2. *The Kirchhoff plate theory*

In the framework of the classical Kirchhoff plate theory, the governing equation for the deflections U of the plate with the constant thickness h can be written in the form

$$BU_{,1111} + \mu \ddot{U} - j\ddot{U}_{,11} = 0. \tag{6.8}$$

Using the above equation we will discuss below two cases.

• The first case – the plate strip simply supported on the opposite edges ($x = 0$ and $x = L$) and made of an isotropic homogeneous material, with constant thickness h and periodically distributed concentrated masses M. The solution to Eq. (6.8) will be assumed in the form

$$U(x,t) = A_U \sin(kx)\cos(\hat{\omega}t), \qquad A_U \neq 0, \quad k = 2\pi/L. \tag{6.9}$$

Substituting (6.9) and coefficients B, μ, j into Eq. (6.8) and after some manipulations, we arrive at the formula for the first frequency

$$\hat{\omega}^2 = \frac{Eh^3}{\varrho(1-\nu^2)} \frac{k^4}{12(h+2H)+h^3k^2}, \qquad k = 2\pi/L, \tag{6.10}$$

which is identical with the lower frequency (6.6) obtained in the framework of the general structural model.

• The second case – the plate strip simply supported on the opposite edges ($x = 0$ and $x = l$) and made of an isotropic homogeneous material, with constant thickness h and two concentrated masses M distributed as in Fig. 7. The solution to Eq. (6.8) will be assumed in the form

$$U(x,t) = \breve{A}_U \sin(2\pi x/l)\cos(\breve{\omega}t), \qquad \breve{A}_U \neq 0. \tag{6.11}$$

Substituting (6.11) and coefficients B, μ, j into Eq. (6.8) and after some manipulations, we arrive at the formula for the second frequency

$$\breve{\omega}^2 = \frac{Eh^3}{\varrho(1-\nu^2)} \frac{4\pi^4}{[3l^2(h+4H)+\pi^2h^3]l^2}, \tag{6.12}$$

which is identical with the higher frequency (6.7) obtained in the framework of the general structural model.

Summing up, we can confirm that the results (6.6) and (6.7) obtained in the framework of the structural model are correct.

7. Conclusions

In order to show that the length-scale effect plays a crucial role in a dynamic macrobehaviour of plates with periodic structure, we have used in this paper governing equations of the structural models which were obtained in [2]. In the Sec. 3 these equations have been written for a special case of a plate – the plate strip simply supported on the opposite edges, made of an isotropic homogeneous material and having the l-periodic thickness along the x-axis. Next, we have investigated free cylindrical vibrations of this plate. The obtained diagrams in Figs. 3–6 of free vibration frequencies for structural and local models have enabled us to formulate below the following essential conclusions concerning the modelling methods for microperiodic plates.

• Lower free vibration frequencies calculated from the general structural model (Eq. $(4.6)_1$ or $(5.3)_1$) can be approximated by similar frequencies derived from this model with additional simplifying assumptions (Eq. $(4.9)_1$ or $(5.4)_1$ and Eq. $(4.12)_1$ or $(5.5)_1$), and also from the local models (Eq. (4.16) or (5.6) and Eq. (4.19) or (5.7)).

• Higher free vibration frequencies can be derived only from the general structural model (the refined theory) and its variant with additional simplifications.

∘ From the general structural model, described by Eqs. (3.4), higher frequencies are determined by Eq. $(4.6)_2$ or $(5.3)_2$.

∘ The structural model without rotational inertia terms given by Eqs. (3.5). In this case, we have higher free vibration frequencies determined by Eq. $(4.9)_2$ or $(5.4)_2$.

∘ The model without terms of an order of $\mathcal{O}(l^4)$ described by Eqs. (3.6) and higher frequencies, derived from this model, are determined by Eq. $(4.12)_2$ or $(5.5)_2$.

Using the structural models, which take into account the length-scale effect and/or the effect of the rotational inertia terms on the dynamic plate behaviour, we can obtain some informations about higher free vibration frequencies.

• The general structural model, governed by Eqs. (3.4), determines the upper bound of higher vibration frequencies, for which the resonance does not take place, because using this model, the lowest from higher free vibration frequencies can be calculated (Eq. $(4.6)_2$ or $(5.3)_2$).

• Comparing diagrams of free vibration frequencies of the plate without and with the concentrated masses, we can confirm that the masses reduce the values of these frequencies.

Hence, we can use the local models, which neglect the effect of the microstructure length parameter l, only to investigate the lower free vibration frequencies of thin microperiodic plates. In this case the structural model can be approximated by the local model. We can observe the advantages of the structural model for periodic plates by investigating higher free vibration frequencies.

Verification of the structural model shown in the Sec. 6, confirms the results obtained in the framework of this model.

References

1. J. Jędrysiak and C. Woźniak, *On the elastodynamics of thin microperiodic plates*, J. Theor. Appl. Mech., **33**, 337–349, 1995.
2. J. Jędrysiak, *On dynamics of thin Kirchhoff plates with a periodic structure*, Engng. Trans., **46**, 1, 73–87, 1998.

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