ENGINEERING TRANSACTIONS • *Engng. Trans.* • **46**, 2, 165–187, 1998
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.650.1998

ON CYLINDRICAL ELASTIC-PLASTIC SHELLS WITH MODERATELY LARGE ROTATIONS

A. SŁAWIANOWSKA (WARSZAWA)

An application of a moderate rotations theory (MRT) to cylindrical elastic-plastic shells is presented. Geometrical and kinematical relations for cylindrical shells with arbitrary cross-section are derived. Following the general procedure formulated in [4] we derive also equilibrium equations for cylindrical shells. Some special cases of loading and cross-section shapes of shells are discussed in more details. Orthotropic elastic-plastic constitutive relations are assumed and expressed in terms of a cylindrical reference frame. In a forthcoming paper we are going to apply these results as a basis for numerical solving of geometrically nonlinear problems for cylindrical shells.

1. Introduction

The aim of this paper is a geometrically nonlinear analysis of elastic-plastic cylindrical shells subject to quasi-static loads. Our approach is based on the refined theory of shells with moderate rotations, presented in [4].

What concerns the constitutive relations, an orthotropic elastic-plastic material with kinematical hardening is assumed. We attempt to get rid of any infinitesimal assumptions as far as possible. Thus, the shells need not be thin. Geometrical nonlinearity is also admitted, therefore, one uses the Lagrangian description and finite deformation measure, i.e., the full Green tensor, involving terms quadratic in the displacement vector gradient. Later on, some approximation is assumed, namely, one of small strains but moderate rotations of material elements. This assumption cancels a part of the Green tensor, nevertheless there are terms quadratic in derivatives which survive and give rise to kinematical nonlinearity. A dimensional reduction to two independent variables is performed, and then, on the basis of an appropriate variational procedure, a general nonlinear rate theory of shells is derived.

In the recent literature on geometrically nonlinear shells, new papers appear which are more and more general and mathematically advanced, cf., e.g. [5, 8]. The approach to nonlinear shells presented in [8] contains two theoretical schemes: degenerate solid theory (DS) and stress resultant based theory (SRB).

Various constitutive models are considered, but, as yet, there is no numerical realization of this approach. In a series of papers [5], belonging to the class of SRB-theories, some numerical results are obtained, however, they concern only elastic shells.

The moderate rotations theory (1986), that has included the first order shear deformations, provides a natural, general and consistent treatment of the nonlinear shell problems. It is reasonable and convenient both from the analytical and numerical point of view. This theory may also be classified as one of the SRB-theories.

In the sequel, this general scheme is applied to cylindrical shells with arbitrary shape of the cross-section. An explicit form of moderate rotation equations is derived; it is thought on as a basis for numerical treatment in a forthcoming paper. We begin with deriving geometrical and kinematical relations for cylindrical shells with moderate rotations. In particular, the nonlinear Green strain tensor is obtained in an appropriate approximation. Further, following the general pattern outlined in [4], we derive the rate equilibrium equations for shells in a quasi-static state of external loading. The special stress is laid on circular cylindrical shells. In particular, presented is the geometrically nonlinear theory of such shells working in the conditions of rotationally symmetric loading. Obtained are the formulas for the finite Green tensor and the rate equilibrium equations for elastic-plastic shells. In the special case of thin shells, there is a good correspondence with the results obtained in [2]. Such a comparison of two different geometrically nonlinear theories of shells is reasonable and instructive, since, as it was shown in [6], the theory of shells with moderately large deflections [2] is equivalent to the theory of shells with moderately large rotations (used in the case when the shell thickness is small, as [2] concerns such shells only).

What concerns the constitutive assumptions for the anisotropic elastic-plastic material of the shell, we follow [4]. It was necessary to express the corresponding tensor equations in terms of cylindrical coordinates.

This work will be continued on in a numerical way. Equations derived here provide the necessary, theoretical basis for that practical step.

2. Geometrical preliminaries

Let $\mathbf{R}(\Theta^\alpha, \Theta^3)$ denote the position vector of a point $(\Theta^\alpha, \Theta^3)$, $\alpha = 1, 2$, in a shell body, and let $\mathbf{n}$ denote the unit vector normal to the undeformed midsurface $\mathcal{M}$ of the shell. Then,

$$\mathbf{R}(\Theta^\alpha, \Theta^3) = \mathbf{r}(\Theta^\alpha) + \Theta^3\mathbf{n} = [X(\Theta^\alpha, \Theta^3),\ Y(\Theta^\alpha, \Theta^3),\ Z(\Theta^\alpha, \Theta^3)], \tag{2.1}$$

where $\mathbf{r}(\Theta^\alpha)$ is the position vector of a point on the midsurface $\mathcal{M}$, Fig. 1.

The parametric description of $\mathcal{M}$ is assumed in the form suitable to its cylindrical shape:

$$x = \Theta^1, \quad y = y(\Theta^2), \quad z = z(\Theta^2). \tag{2.2}$$

A point (Θ^1, Θ^2) of a cylindrical midsurface is represented by the vector

$$\mathbf{r} = \mathbf{r}(\Theta^1, \Theta^2) = (x, y, z). \tag{2.3}$$

Vectors tangent to the midsurface are

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \Theta^\alpha}, \qquad \alpha = 1, 2, \tag{2.4}$$

i.e. for the particular coordinate lines we have

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \Theta^1} = \left[\frac{\partial x}{\partial \Theta^1}, \frac{\partial y}{\partial \Theta^1}, \frac{\partial z}{\partial \Theta^1}\right] = [1,\ 0,\ 0],$$
$$\mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \Theta^2} = \left[\frac{\partial x}{\partial \Theta^2}, \frac{\partial y}{\partial \Theta^2}, \frac{\partial z}{\partial \Theta^2}\right] = [0,\ y',\ z'],$$

where

$$(\,)' = \frac{d(\,)}{d\Theta^2}. \tag{2.5}$$

Now we can determine a vector which is normal to $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\begin{aligned} \mathbf{a}_1 \times \mathbf{a}_2 &= \mathbf{e}_1 \times (y'\mathbf{e}_2 + z'\mathbf{e}_3) \\ &= y'\mathbf{e}_1 \times \mathbf{e}_2 + z'\mathbf{e}_1 \times \mathbf{e}_3 = y'\mathbf{e}_3 - z'\mathbf{e}_2 = [0,\ -z',\ y'], \end{aligned} \tag{2.6}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are basis versors along the coordinate axes x, y and z.

The length of the vector (2.6) is

$$\|\mathbf{a}_1 \times \mathbf{a}_2\| = \sqrt{(y')^2 + (z')^2}. \tag{2.7}$$

So, $\mathbf{n}$ – the versor normal to $\mathcal{M}$ can be determined as

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} = \left[0,\ -\frac{z'}{\sqrt{(y')^2 + (z')^2}},\ \frac{y'}{\sqrt{(y')^2 + (z')^2}}\right]. \tag{2.8}$$

REMARK. If $\mathbf{n}$ is to be an external normal, we should take it with the minus sign. Coming back to (2.1) we obtain

$$\begin{aligned} \mathbf{R}(\Theta^1, \Theta^2, \Theta^3) &= \mathbf{r}(\Theta^1, \Theta^2) + \Theta^3\mathbf{n} \\ &= \left[\Theta^1, y - \frac{z'\Theta^3}{\sqrt{(y')^2 + (z')^2}},\ z + \frac{y'\Theta^3}{\sqrt{(y')^2 + (z')^2}}\right]. \end{aligned} \tag{2.9}$$

Now, we are ready to calculate the first quadratic form of the cylindrical midsurface of the shell (of any shape of the cross-section)

$$(2.10)\qquad \begin{gathered} a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta\,, \\ a_{11} = \mathbf{a}_1 \cdot \mathbf{a}_1 = 1, \qquad a_{22} = \mathbf{a}_2 \cdot \mathbf{a}_2 = (y')^2 + (z')^2, \qquad a_{12} = a_{21} = 0. \end{gathered}$$

In a matrix form we can write down $[a_{\alpha\beta}]$ and $[a^{\alpha\beta}]$ as:

$$(2.11)\qquad [a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & (y')^2 + (z')^2 \end{bmatrix},$$

$$(2.12)\qquad [a^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & [(y')^2 + (z')^2]^{-1} \end{bmatrix},$$

and the determinant a is expressed by

$$(2.13)\qquad a = \det[a_{\alpha\beta}] = (y')^2 + (z')^2.$$

Next, the second fundamental form of the midsurface, $b_{\alpha\beta}$, can be derived

$$(2.14)\qquad b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n} = \mathbf{r}_{,\alpha\beta} \cdot \mathbf{n} = \frac{\partial^2 \mathbf{r}}{\partial \Theta^\alpha \partial \Theta^\beta} \cdot \mathbf{n}.$$

It is easy to see that

$$(2.15)\qquad \begin{aligned} \mathbf{a}_{1,1} &= [0,\,0,\,0], \qquad & \mathbf{a}_{1,2} &= [0,\,0,\,0], \\ \mathbf{a}_{2,1} &= [0,\,0,\,0], \qquad & \mathbf{a}_{2,2} &= [0,\,y'',\,z'']. \end{aligned}$$

So, it yields

$$\begin{aligned} b_{11} &= 0, \qquad b_{12} = 0, \qquad b_{21} = 0, \\ b_{22} &= \mathbf{a}_{2,2} \cdot \mathbf{n} = \frac{y''z' - y'z''}{\sqrt{(y')^2 + (z')^2}}\,. \end{aligned}$$

So we have obtained the second quadratic form of the midsurface

$$(2.16)\qquad [b_{\alpha\beta}] = \frac{y''z' - y'z''}{\sqrt{(y')^2 + (z')^2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Making use of the metric tensor we can also determine

$$(2.17)\qquad [b^\alpha{}_\beta] = [a^{\alpha\lambda} b_{\lambda\beta}] = \frac{y''z' - y'z''}{((y')^2 + (z')^2)^{3/2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, we get

$$[b^{\alpha\beta}] = [b^{\alpha}{}_{\lambda}a^{\lambda\beta}] = \frac{y''z' - y'z''}{((y')^2 + (z')^2)^{3/2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.18}$$

As an example, let us write down all the above forms for circular cylindrical shells

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & (r_0)^2 \end{bmatrix}, \qquad [a^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & (r_0)^{-2} \end{bmatrix}, \tag{2.19}$$

$$[b_{\alpha\beta}] = -\begin{bmatrix} 1 & 0 \\ 0 & (r_0) \end{bmatrix}, \qquad [b^{\alpha\beta}] = -\begin{bmatrix} 0 & 0 \\ 0 & (r_0)^{-3} \end{bmatrix}, \tag{2.20}$$

$$[b^{\alpha}{}_{\beta}] = -\begin{bmatrix} 0 & 0 \\ 0 & (r_0)^{-1} \end{bmatrix},$$

where r_0 is the radius of the circular cross-section of the midsurface $\mathcal{M}$.

The parametric description of the midsurface is very simple in this case:

$$y = r_0 \cos\Theta^2, \quad y' = -r_0 \sin\Theta^2 = -z, \quad y'' = -r_0 \cos\Theta^2 = -y, \tag{2.21}$$

$$z = r_0 \sin\Theta^2, \quad z' = r_0 \cos\Theta^2 = y, \quad z'' = -r_0 \sin\Theta^2 = -z, \tag{2.22}$$

and it follows that

$$(y')^2 + (z')^2 = (r_0)^2, \qquad y''z' - y'z'' = -(r_0)^2. \tag{2.23}$$

Let us consider the first and the second quadratic forms of the shell midsurface for the cross-section of an elliptic shape, described by a parametrization

$$y = a\cos\Theta^2, \qquad z = b\sin\Theta^2, \tag{2.24}$$

where a, b – semi-axes of the ellipse; then we get

$$(y')^2 + (z')^2 = (a^2 - b^2)\sin^2\Theta^2 + b^2, \qquad y''z' - y'z'' = -ab. \tag{2.25}$$

So, the first and the second forms of the midsurface for the elliptic cylindrical shell are

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix}, \qquad [a^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & (a_{22})^{-1} \end{bmatrix}, \tag{2.26}$$

where

$$a_{22} := a^2 \sin^2\Theta^2 + b^2 \cos^2\Theta^2, \tag{2.27}$$

and

$$[b_{\alpha\beta}] = -\frac{ab}{\sqrt{a_{22}}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [b^{\alpha\beta}] = -\frac{ab}{\sqrt{a_{22}}}\frac{1}{(a_{22})^2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.28}$$

$$[b^{\alpha}{}_{\beta}] = -\frac{ab}{\sqrt{a_{22}}}\frac{1}{(a_{22})}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.29}$$

Our geometrically nonlinear analysis of the shells requires the application of the Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}a^{\alpha\delta}(a_{\delta\beta,\gamma} + a_{\delta\gamma,\beta} - a_{\beta\gamma,\delta}), \tag{2.30}$$

As $a_{\beta\delta}$ does not depend on Θ^1, $a_{1\beta} = \text{const}$, we obtain $\Gamma^1_{\beta\delta} = 0$, and similarly we arrive at the results: $\Gamma^2_{11} = 0$, $\Gamma^2_{12} = 0$, $\Gamma^2_{21} = 0$.

Then the only non-zero component is Γ^2_{22}:

$$\Gamma^2_{22} = \frac{1}{2}a^{22}a_{22,2} = \frac{(y'^2 + z'^2)'}{2(y'^2 + z'^2)} = \frac{1}{2}\left(\ln(y'^2 + z'^2)\right)'. \tag{2.31}$$

When the coordinate Θ^2 is proportional to s – the length of arc measured along the midsurface cross-section, then also $\Gamma^2_{22} = 0$, because $y'^2 + z'^2 = 1$.

3. The Green strain tensor for cylindrical shells

The displacement vector $\mathbf{V}(\Theta^{\alpha}, \Theta^3)$, ([1, 2], [4, 7]), of a point of the shell can be represented as follows:

$$\mathbf{V} = V^{\alpha}\mathbf{g}_{\alpha} + V^3\mathbf{g}_3 = V_{\alpha}\mathbf{g}^{\alpha} + V_3\mathbf{g}^3, \tag{3.1}$$

$$\mathbf{V} = v^{\alpha}\mathbf{a}_{\alpha} + v^3\mathbf{a}_3 = v_{\alpha}\mathbf{a}^{\alpha} + v_3\mathbf{a}^3, \tag{3.2}$$

where

$$\mathbf{g}_i = \mathbf{R}_{,i}, \quad i = 1,2,3; \quad \mathbf{a}_{\alpha} = \mathbf{r}_{,\alpha}, \quad \alpha = 1,2; \quad \mathbf{n} = \mathbf{g}^3 = \mathbf{g}_3 = \mathbf{a}^3 = \mathbf{a}_3, \tag{3.3}$$

and where $(\cdot)_{,k}$ denotes partial differentiation with respect to Θ^k.

As usual, we have

$$\mathbf{g}_{\alpha} = \mu^{\lambda}_{\alpha}\mathbf{a}_{\lambda}, \qquad \mu^{\lambda}_{\alpha} \text{ – components of a shifter tensor,} \tag{3.4}$$

where $[\mu^\alpha_\beta]$ for cylindrical shells of an arbitrary cross-section, can be written in the form

$$[\mu^\alpha_\beta] = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} - \Theta^3 H \begin{bmatrix} 0, & 0 \\ 0, & 1 \end{bmatrix} = \begin{bmatrix} 1, & 0 \\ 0, & 1 - H\Theta^3 \end{bmatrix}, \tag{3.5}$$

with

$$H = b^\alpha_\beta = \frac{y''z' - y'z''}{(y'^2 + z'^2)^{3/2}}. \tag{3.6}$$

The finite deformation tensor is given by the familiar formula [1, 2, 4, 7]

$$E_{ij} = \frac{1}{2}\left(V_{i;j} + V_{j;i} + V^k_{;i} V_{k;j}\right), \tag{3.7}$$

where $(\cdot)_{;i}$ denotes the covariant differentiation with respect to the metric of the undeformed shell space.

Assumptions of the shell theory with small strains and moderate rotations are:

$$\begin{aligned} E_{ij} &= O\left((\vartheta)^2\right), & (\vartheta)^2 &\ll 1, \\ \Omega_{\alpha\beta} &= O\left((\vartheta)^2\right), & \Omega_{\alpha 3} &= O(\vartheta), \\ \eta_{ij} &= O\left((\vartheta)^2\right), \end{aligned} \tag{3.8}$$

where the linearized strains η_{ij} and the linearized rotations Ω_{ij} are given by

$$\eta_{ij} = \frac{1}{2}(V_{i;j} + V_{j;i}), \qquad \Omega_{ij} = \frac{1}{2}(V_{i;j} - V_{j;i}). \tag{3.9}$$

The paremeter ϑ defines the order of magnitude of the rotations (in radians) of the normals to the midsurface. Roughly speaking, $(\vartheta)^2$ is an infinitesimal quantity but ϑ is not.

From (3.7) and (3.9), [1, 4], one obtain:

$$E_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{2}\Omega_{3\alpha}\Omega_{3\beta} + \frac{1}{2}(\eta_{3\alpha}\Omega_{3\beta} + \eta_{3\beta}\Omega_{3\alpha}) + O\left((\vartheta)^4\right), \tag{3.10}$$

$$E_{\alpha 3} = \eta_{\alpha 3} + \frac{1}{2}\Omega_{\lambda 3}\Omega^\lambda_\alpha + \frac{1}{2}(\eta_{\lambda\alpha}\Omega^\lambda_3 + \eta_{33}\Omega_{3\alpha}) + O\left((\vartheta)^4\right), \tag{3.11}$$

$$E_{33} = \eta_{33} + \frac{1}{2}\Omega_{\lambda 3}\Omega^\lambda_3 + \eta_{\lambda 3}\Omega^\lambda_3 + O\left((\vartheta)^4\right). \tag{3.12}$$

To specify these equations for cylindrical shells, we identify curvilinear coordinates Θ^1, Θ^2, Θ^3 with cylindrical ones, x, l, t, respectively. This identification may be represented in the form of a table:

(3.13)

Θ^1	Θ^2	Θ^3
x	l	t

;

the coordinate x is chosen along a generating line of a cylindrical shell, Fig. 1, the coordinate t is measured in the direction normal to the midsurface, and the coordinate l runs along the cross-section of the midsurface; for circular (or elliptic) shells, the second cylindrical coordinate is usually identified with the angular variable φ ($\varphi = (l/2\pi) \cdot 360°$).

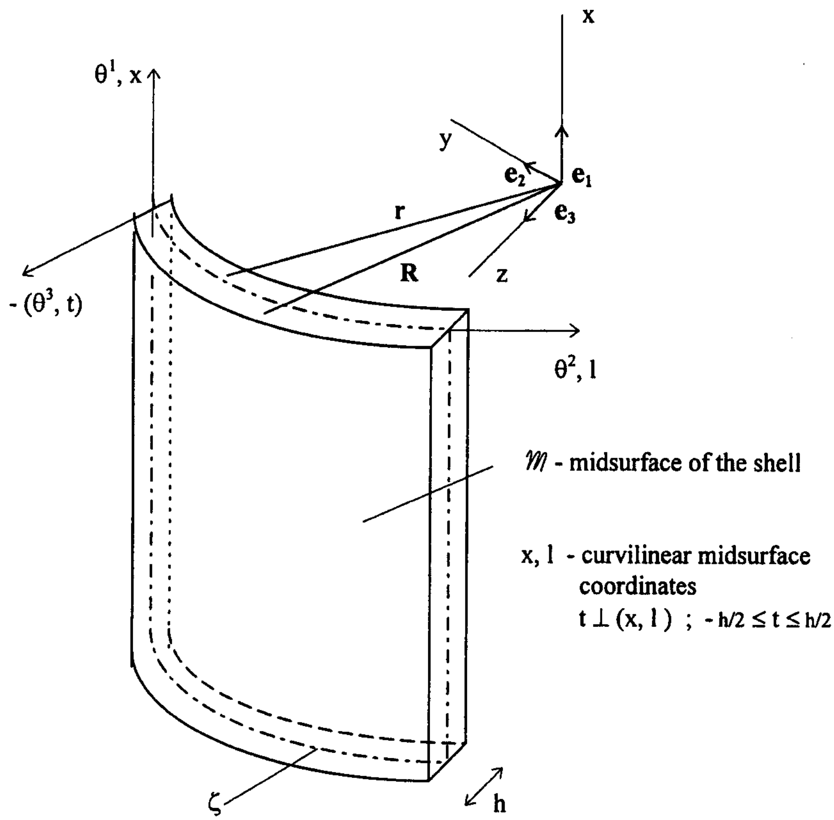

FIG. 1. Geometry and the coordinate systems for a shell.

The membrane components of the Green strain tensor for cylindrical shells are

$$E_{xx} = \eta_{xx} + \frac{1}{2}(\Omega_{tx})^2 + \eta_{tx}\Omega_{tx} + O\left((\vartheta)^4\right), \tag{3.14}$$

$$E_{xl} = \eta_{xl} + \frac{1}{2}\Omega_{tx}\Omega_{tl} + \frac{1}{2}\left(\eta_{tx}\Omega_{tl} + \eta_{tl}\Omega_{tx}\right) + O\left((\vartheta)^4\right), \tag{3.15}$$

$$E_{ll} = \eta_{ll} + \frac{1}{2}(\Omega_{tl})^2 + \eta_{tl}\Omega_{tl} + O\left((\vartheta)^4\right), \qquad E_{lx} = E_{xl}. \tag{3.16}$$

Next, we should derive formulas for the shear components $E_{\alpha 3}$ in the cylindrical cordinate system.

After some calculations we obtain:

$$(3.17)\qquad \begin{aligned} E_{xt} &= \eta_{xt} + \Omega_{lt}\Omega_{lx}a^{22} + \frac{1}{2}\eta_{xx}\Omega_{xt} + \frac{1}{2}\eta_{lx}\Omega_{lt}a^{22} + \eta_{tt}\Omega_{tx} + O\left((\vartheta)^4\right),\\ E_{tx} &= E_{xt}, \end{aligned}$$

$$(3.18)\qquad E_{lt} = \eta_{lt} + \frac{1}{2}\Omega_{xt}\Omega_{xl} + \frac{1}{2}\eta_{xl}\Omega_{xt} + \frac{1}{2}\eta_{ll}\Omega_{lt}a^{22} + \eta_{tt}\Omega_{tl} + O\left((\vartheta)^4\right).$$

The last cylindrical component of the Green strain tensor is

$$(3.19)\qquad E_{tt} = \eta_{tt} + \frac{1}{2}(\Omega_{xt})^2 + \frac{1}{2}(\Omega_{xl})^2a^{22} + \eta_{xt}\Omega_{xt} + \eta_{lt}\Omega_{lt}a^{22} + O\left((\vartheta)^4\right).$$

4. Rate equilibrium equations of cylindrical elastic-plastic shells in quasi-static processes

We shall consider an equilibrium of shells subjected to loading that increases monotonically in time from zero. Before irreversible deformations and plastic hardening appear, the shell will behave as an elastic structure. Next, plastic strains will occur in the shell, and under quasi-static loading the state of equilibrium will be steady.

A method of treating such a problem, presented in [4], uses Neal's [3] variational principle. It is based on the functional

$$(4.1)\qquad I(\dot{\mathbf{V}}) = \int_{\mathcal{V}} \left(\frac{1}{2}\dot{s}^{ij}(\dot{\mathbf{V}})\dot{E}_{ij}(\dot{\mathbf{V}}) + \frac{1}{2}s^{ij}(\dot{\mathbf{V}})\ddot{E}_{ij}(\dot{\mathbf{V}}) - \varrho\dot{F}^i\dot{V}_i\right) d\mathcal{V} - \int_{\mathcal{A}} {}^*\dot{t}^k\dot{V}_k\, d\mathcal{A},$$

with certain subsidiary conditions on the shell boundary

$$\dot{V}_\alpha = {}^*\dot{V}_\alpha, \qquad \dot{V}_t = {}^*\dot{V}_t;$$

$\dot{\mathbf{V}}$ is here an independent variable subjected to the variation. For quasi-static processes it is assumed that $\ddot{\mathbf{V}} = 0$. The notations used in (4.1) are defined as follows:

- s^{ij} – the second Piola–Kirchhoff stress tensor,
- ϱF^i – the components of the body force vector,
- $\dot{E}_{ij}, \ddot{E}_{ij}$ – the components of the first and the second time derivatives of the Green strain tensor,
- ${}^*t^i = t^{ji}n_j$ – the components of the prescribed external stress vector,
- t^{ji} – the components of the first Piola–Kirchhoff stress tensor,
- n_j – the components of the unit outward vector on $\mathcal{A}$ – the boundary surface of the shell, and $\mathcal{V}$ – the shell volume: $\partial\mathcal{V} = \mathcal{A}$.

An essential assumption of this theory is the kinematical hypothesis, [4, 2]:

$$v_\alpha = \overset{0}{v}_\alpha + t\overset{1}{v}_\alpha, \qquad \alpha = x, l, \qquad v_t = \overset{0}{v}_t + t\overset{1}{v}_t, \tag{4.2}$$

which will reduce the 3-D problem considered to the 2-D one. To perform this transition, the resultant quantities have to be defined:

$$\overset{n}{L}{}^{ij} = \int_{-h/2}^{h/2} t^n \mu s^{ij}\, dt, \tag{4.3}$$

n-th order stress resultants, $i, j = x, l, t$ (the quantity $\overset{0}{L}{}^{tx}$ is usually denoted by Q^x and called a shear force),

$$\overset{n}{B}{}^\alpha = \int_{-h/2}^{h/2} t^n \mu\varrho\, \mu^\alpha_\beta\, F^\beta\, dt, \qquad \beta = x, l, \qquad \overset{n}{B}{}^t = \int_{-h/2}^{h/2} t^n \mu\varrho F^t\, dt, \tag{4.4}$$

n-th order body couples,

$$\overset{n}{p}{}^\alpha = \left(t^n \mu\, \mu^\alpha_\beta\, {}^*t^{t\beta}\right)\Big|_{-h/2}^{h/2}, \qquad \alpha = x, l, \qquad \overset{n}{p}{}^t = \left(t^n \mu\, {}^*t^{tt}\right)\Big|_{-h/2}^{h/2}, \tag{4.5}$$

n-th order couples of the surface loads on $\mathcal{S}^+$ and $\mathcal{S}^-$ (where $\mathcal{S}^+$ and $\mathcal{S}^-$ – top and bottom surfaces),

$${}^*\overset{n}{L}{}^{\alpha\beta} = \int_{-h/2}^{h/2} t^n \mu\mu^\alpha_\lambda\, {}^*t^{\beta\lambda}\, dt, \qquad {}^*\overset{n}{L}{}^{t\beta} = \int_{-h/2}^{h/2} t^n \mu\, {}^*t^{\beta t}\, dt, \tag{4.6}$$

n-th order couples of the boundary loads on $\mathcal{B}_s$ (where $\mathcal{B}_s$ – a part of the lateral boundary surface).

The notation μ used in these formulas is defined as follows:

$$\begin{aligned} \mu &= \sqrt{g/a}, \qquad g = \det[g_{ij}], \qquad i, j = x, l, t, \\ a &= \det[a_{\alpha\beta}], \qquad \alpha, \beta = x, l. \end{aligned} \tag{4.7}$$

Let us derive the exact formula for μ in the case of cylindrical shells:

$$g = \det[g_{ij}] = \det[g_{\alpha\beta}], \qquad \begin{aligned} [g^{\alpha\beta}] &= \left[(\mu^{-1})^\alpha_\lambda\right] [a^{\lambda\omega}] \left[(\mu^{-1})^T{}_\omega{}^\beta\right], \\ \det[g^{\alpha\beta}] &= \left\{\det[(\mu^{-1})^\delta{}_\lambda]\right\}^2 \det[a^{\varrho\omega}], \\ \det[\mathbf{g}] &= (\det[\mu^\lambda_\alpha])^2 \det[\mathbf{a}], \\ \det[\mu^\alpha{}_\beta] &= 1 - Ht, \end{aligned} \tag{4.8}$$

$$a = \det[a_{\lambda\omega}] = (y')^2 + (z')^2 = (s')^2, \qquad g = (1 - Ht)^2 (s')^2, \tag{4.9}$$
$$\mu = 1 - Ht, \qquad \text{(see (3.5))}.$$

By virtue of (4.2) one can obtain, [4]

$$E_{\alpha\beta} = \sum_0^2 (t)^n \overset{n}{E}_{\alpha\beta}(x,l) + O(\vartheta^4), \qquad E_{\alpha t} = \sum_0^1 (t)^n \overset{n}{E}_{\alpha t}(x,l) + O(\vartheta^4), \tag{4.10}$$
$$E_{tt} = \overset{0}{E}_{tt} + O(\vartheta^4).$$

Four rate equilibrium equations obtained in [4] were expressed in terms of the quantities:

$$\overset{0}{S}{}^{\alpha\beta} = \overset{0}{\dot{L}}{}^{\alpha\beta} - b^\alpha_\lambda \overset{1}{\dot{L}}{}^{\lambda\beta} + \left(\overset{1}{v}{}^\lambda \overset{0}{L}{}^{\beta t}\right)^{\cdot} - \frac{1}{2}\left(\overset{1}{v}_t|^\alpha \overset{1}{L}{}^{\beta t}\right)^{\cdot}, \tag{4.11}$$

$$\overset{0}{S}{}^{t\beta} = \left(1 + \frac{1}{2}\overset{1}{v}_t\right) \overset{0}{\dot{L}}{}^{\beta t} + \left(\overset{0}{\varphi}_\lambda \overset{0}{L}{}^{\lambda\beta}\right)^{\cdot} + \left(\overset{1}{\varphi}_\lambda \overset{1}{L}{}^{\lambda\beta}\right)^{\cdot} + \frac{1}{2}\overset{1}{\dot{v}}_t \overset{0}{L}{}^{\beta t}, \tag{4.12}$$

$$\overset{1}{S}{}^{\alpha\beta} = \overset{1}{\dot{L}}{}^{\alpha\beta} - b^\alpha_\lambda \overset{2}{\dot{L}}{}^{\lambda\beta} + \left(\overset{1}{v}{}^\lambda \overset{1}{L}{}^{\beta t}\right)^{\cdot}, \qquad \overset{1}{\hat{S}}{}^{\alpha\beta} = \overset{1}{\dot{L}}{}^{\alpha\beta} - b^\alpha_\lambda \overset{2}{\dot{L}}{}^{\lambda\beta}, \tag{4.13}$$

$$\overset{1}{S}{}^{t\beta} = \overset{1}{\dot{L}}{}^{\beta t} - \frac{1}{2}\left(\overset{0}{\varphi}{}^\beta_\alpha \overset{1}{L}{}^{\lambda t}\right)^{\cdot} + \left(\overset{0}{\varphi}_\lambda \overset{1}{L}{}^{\lambda\beta}\right)^{\cdot} + \left(\overset{1}{\varphi}_\lambda \overset{2}{L}{}^{\lambda\beta}\right)^{\cdot}, \tag{4.14}$$

$$\overset{1}{\hat{S}}{}^{t\beta} = \left(\overset{0}{\varphi}_\lambda \overset{1}{\dot{L}}{}^{\lambda\beta}\right)^{\cdot} + \left(\overset{1}{\varphi}_\lambda \overset{2}{\dot{L}}{}^{\lambda\beta}\right)^{\cdot}, \tag{4.15}$$

$$\overset{1}{R}{}^{tt} = -\overset{0}{\dot{L}}{}^{tt} - \frac{1}{2}\left\{\left(\overset{1}{v}_\alpha - \overset{0}{\varphi}_\alpha\right)\overset{0}{L}{}^{\alpha t}\right\}^{\cdot}, \tag{4.16}$$

$$\overset{1}{R}{}^{\beta t} = -\left\{\left(1 - \frac{1}{2}\overset{1}{v}_t\right)\right) \overset{0}{\dot{L}}{}^{\beta t} + \left(\overset{0}{\varphi}{}^\beta_\lambda \overset{0}{L}{}^{\lambda t}\right)^{\cdot} + \left(\overset{1}{v}{}^\beta|_\lambda \overset{1}{L}{}^{\lambda t}\right)^{\cdot}$$
$$+ \left(\overset{1}{v}{}^\beta \overset{0}{L}{}^{tt}\right)^{\cdot} - \frac{1}{2}\overset{1}{\dot{v}}_t \overset{0}{L}{}^{\beta t}\Big\}, \tag{4.17}$$

in which the time derivative

$$(\,)^{\cdot} \overset{\text{df}}{=} \frac{d(\,)}{dt}$$

has been used, and for $n = 0, 1$, $\alpha, \beta = x, l$, the quantities

$$\overset{n}{\varphi}_{\alpha\beta} = \overset{n}{v}_{\alpha|\beta} - b_{\alpha\beta}\overset{n}{v}_t\,, \tag{4.18}$$

$$\overset{n}{\varphi}_\alpha = \overset{n}{v}_{t,\alpha} + b^\lambda_\alpha \overset{n}{v}_\lambda \tag{4.19}$$

were defined with the use of the notations: $(\cdot)_|$ – covariant differentiation, $(\cdot)_,$ – partial differentiation in the undeformed midsurface.

The equilibrium equations for elastic-plastic shells under quasi-static loadings are derived in a general form in [4] from the variational principle $\delta I(\dot{\mathbf{v}}) = 0$, with $I(\dot{\mathbf{v}})$ taken in the form that is cited here in the Appendix as the formula (*). They are the Euler equations for this principle.

The first of equilibrium equations of [4] was:

$$\overset{0}{S}{}^{\alpha\beta}{}_{|\beta} - b^{\alpha}_{\beta}\overset{0}{S}{}^{t\beta} + \overset{0}{\dot{B}}{}^{\alpha} + \overset{0}{\dot{p}}{}^{\alpha} = 0. \tag{4.20}$$

Let us write it down in the cylindrical frame (x, l, t); of course we will obtain two equations for $\alpha = x, l$. So, we have

$$\begin{aligned} \overset{0}{S}{}^{xx}_{,x} + \overset{0}{S}{}^{xl}{}_{,l} + \Gamma^{x}_{\lambda l}\overset{0}{S}{}^{\lambda l} + \Gamma^{l}_{\lambda l}\overset{0}{S}{}^{x\lambda} + \overset{0}{\dot{B}}{}^{x} + \overset{0}{\dot{p}}{}^{x} &= 0,\\ \overset{0}{S}{}^{xx}{}_{,x} + \overset{0}{S}{}^{xl}{}_{,l} + \Gamma^{l}_{ll}\overset{0}{S}{}^{xl} + \overset{0}{\dot{B}}{}^{x} + \overset{0}{\dot{p}}{}^{x} &= 0, \end{aligned} \tag{4.21}$$

where

$$\Gamma^{l}_{ll} = \frac{1}{2}\left(\ln\left((y')^2 + (z')^2\right)\right)' = (\ln s')' = \frac{d^2 s}{dl^2}\Big/\frac{ds}{dl}, \qquad (\cdot)' = \frac{d(\cdot)}{dl}. \tag{4.22}$$

If the cylindrical coordinate l is proportional to the arc s directed along the shell cross-section, then $\Gamma^{l}_{ll} = 0$, and we obtain

$$\delta\overset{0}{\dot{v}}_x: \qquad \frac{\partial\overset{0}{S}{}^{xx}}{\partial x} + \frac{\partial\overset{0}{S}{}^{xl}}{\partial l} + \overset{0}{\dot{B}}{}^{x} + \overset{0}{\dot{p}}{}^{x} = 0 \tag{4.23}$$

(this equation results from the variation of the functional (4.1), or rather its counterpart (*) in the Appendix, with respect to $\overset{0}{\dot{v}}_x$). Similarly, we obtain the next equilibrium equation for cylindrical shells:

$$\delta\overset{0}{\dot{v}}_l: \qquad \frac{\partial\overset{0}{S}{}^{lx}}{\partial x} + \frac{\partial\overset{0}{S}{}^{ll}}{\partial l} + 2s''/s'\,\overset{0}{S}{}^{ll} - H\overset{0}{S}{}^{tl} + \overset{0}{\dot{B}}{}^{l} + \overset{0}{\dot{p}}{}^{l} = 0, \tag{4.24}$$

where $s(l)$ is the natural parameter, i.e. the length of an arc, and $(1/2)H$ is the mean curvature of the shell, (3.7). Here also the third term in (4.24) vanishes when $s = s(l) = \mathrm{const}\cdot l$. ($s$ is proportional to l). The successive equation is

$$\delta\overset{0}{\dot{v}}_t: \qquad \overset{0}{S}{}^{t\beta}{}_{,\beta} + b_{\alpha\beta}\overset{0}{S}{}^{\alpha\beta} + \Gamma^{\beta}_{\alpha\beta}\overset{0}{S}{}^{t\alpha} + \overset{0}{\dot{B}}{}^{t} + \overset{0}{\dot{p}}{}^{t} = 0, \tag{4.25}$$

where

$$b_{ll} = H\left(\frac{ds}{dl}\right)^2, \qquad \frac{ds}{dl} = \sqrt{y'^2 + z'^2}, \qquad b_{\alpha\beta}\overset{0}{S}{}^{\alpha\beta} = H\left(\frac{ds}{dl}\right)^2\overset{0}{S}{}^{ll}. \tag{4.26}$$

Equation (4.25) may be written also in the form:

$$(4.27)\qquad \delta\overset{0}{\dot v}_t:\qquad \frac{\partial \overset{0}{S}{}^{tx}}{\partial x}+\frac{\partial \overset{0}{S}{}^{tl}}{\partial l}+s''/s'\,\overset{0}{S}{}^{tl}-H(s')^2\overset{0}{S}{}^{ll}+\overset{0}{\dot B}{}^t+\overset{0}{\dot p}{}^t=0.$$

From the third equilibrium equation, obtained in [4] as a result of variation of the functional (4.1) (or rather $I(\dot{\mathbf v})$ cited in Appendix as (*)), with respect to $\overset{1}{\dot v}_l$, we get two equations:

$$(4.28)\qquad \delta\overset{1}{\dot v}_x:\qquad \frac{\partial \overset{1}{S}{}^{xx}}{\partial x}+\frac{\partial \overset{1}{S}{}^{xl}}{\partial l}+s''/s'\overset{1}{S}{}^{xl}+\overset{1}{R}{}^{xt}+\overset{1}{\dot B}{}^l+\overset{1}{\dot p}{}^l=0,$$

$$(4.29)\qquad \delta\overset{1}{\dot v}_l:\qquad \frac{\partial \overset{1}{S}{}^{lx}}{\partial l}+\frac{\partial \overset{1}{S}{}^{ll}}{\partial l}+2s''/s'\overset{1}{S}{}^{ll}-H\hat{S}^{tl}+\overset{1}{R}{}^{lt}+\overset{1}{\dot B}{}^l+\overset{1}{\dot p}{}^l=0.$$

From the fourth equilibrium equation of [4] we obtain the last rate equilibrium equation for cylindrical shells,

$$(4.30)\qquad \delta\overset{1}{\dot v}_t:\qquad \frac{\partial \overset{1}{S}{}^{lx}}{\partial x}+\frac{\partial \overset{1}{S}{}^{tl}}{\partial l}+s''/s'\overset{1}{S}{}^{tl}+H(s')^2\overset{1}{\hat S}{}^{ll}+\overset{1}{R}{}^{tt}\,\overset{1}{\dot B}{}^t+\overset{1}{\dot p}{}^t=0,$$

where, as we remember,

$$(\cdot)'=\frac{d(\cdot)}{dl}\qquad\text{and}\qquad H=\frac{y''z'-y'z''}{(y'^2+z'^2)^{3/2}}.$$

5. Geometrically nonlinear circular cylindrical shells – rate equilibrium equations

Now, let us write down the whole system of the rate equilibrium equations for geometrically nonlinear circular cylindrical shells of radius r_0; they will have a simple form resulting from:

$$(5.1)\qquad s=r_0\cdot l,\qquad s'=r_0,\qquad s''=0,\qquad H=-\frac{1}{r_0}$$

(the sign – in the expression for H concerns the outside normal to $\mathcal{M}$, which is often used in the shell theory).

$$(5.2)\qquad \begin{aligned} &\delta\overset{0}{\dot v}_x: && \overset{0}{S}{}^{xx}{}_{,x}+\overset{0}{S}{}^{xl}{}_{,l}+\overset{0}{\dot B}{}^x+\overset{0}{\dot p}{}^x=0,\\ &\delta\overset{0}{\dot v}_l: && \overset{0}{S}{}^{lx}{}_{,x}+\overset{0}{S}{}^{ll}{}_{,l}+\frac{1}{r_0}\overset{0}{S}{}^{tl}+\overset{0}{\dot B}{}^l+\overset{0}{\dot p}{}^l=0,\end{aligned}$$

$$
\begin{aligned}
&\delta\overset{0}{\dot{v}}_t: && \overset{0}{S}{}^{tx}{}_{,x} + \overset{0}{S}{}^{tl}{}_{,l} - r_0\overset{0}{S}{}^{ll} + \overset{0}{\dot{B}}{}^t + \overset{0}{\dot{p}}{}^t = 0,\\
&\delta\overset{1}{\dot{v}}_x: && \overset{1}{S}{}^{xx}{}_{,x} + \overset{1}{S}{}^{xl}{}_{,l} + \overset{1}{R}{}^{xt} + \overset{1}{\dot{B}}{}^x + \overset{1}{\dot{p}}{}^x = 0,\\
&\delta\overset{1}{\dot{v}}_l: && \overset{1}{S}{}^{lx}{}_{,x} + \overset{1}{S}{}^{ll}{}_{,l} + \frac{1}{r_0}\overset{1}{\hat{S}}{}^{tl} + \overset{1}{R}{}^{lt} + \overset{1}{\dot{B}}{}^l + \overset{1}{\dot{p}}{}^l = 0,\\
&\delta\overset{1}{\dot{v}}_t: && \overset{1}{S}{}^{tx}{}_{,x} + \overset{1}{S}{}^{tl}{}_{,l} - r_0\overset{1}{\hat{S}}{}^{ll} + \overset{1}{R}{}^{tt} + \overset{1}{\dot{B}}{}^t + \overset{1}{\dot{p}}{}^t = 0.
\end{aligned}
\tag{5.2 [cont.]}
$$

5.1. *Circular cylindrical shells – rotationally symmetric deformation*

At the end of our analysis of the rate equilibrium problem for elastic-plastic circular cylindrical shells, we can derive from (5.2) the corresponding equations for the case of shells under rotationally symmetric conditions of loading and deformation.

So, we are dealing now with the case

$$
v_l = 0 \quad (\overset{0}{v}_l = 0, \quad \overset{1}{v}_l = 0), \qquad \overset{n}{L}_{xl} = \overset{n}{L}_{lx} \equiv 0, \qquad n = 0,\ 1,
\tag{5.3}
$$

$\overset{0}{L}{}^l_l = \alpha_1$, $\overset{1}{L}{}^l_l = \alpha_2$, $\overset{0}{L}{}^x_x = \alpha_3$, $\alpha_i =$ constant for $i = 1,2,3$ along a circumference of the cylindrical shell, and $\overset{0}{L}{}^x_x$ is constant along the shell length as well.

The stress and strain tensors $\mathbf{s}$ and $\mathbf{E}$ are defined as follows, cf (6.3):

$$
\begin{aligned}
\mathbf{s}^T &= \left[s^{xx},\ s^{ll},\ s^{tt},\ 0,\ s^{xt},\ 0\right],\\
\mathbf{E}^T &= \left[E^{xx},\ E^{ll},\ E^{tt},\ 0,\ E^{xt},\ 0\right].
\end{aligned}
\tag{5.4}
$$

Using the notations (4.18), (4.19) and making use of (2.19), (2.20), we get:

$$
\begin{aligned}
&\overset{n}{\varphi}_x = \overset{n}{v}_{t,x}, \qquad \overset{n}{\varphi}_l = 0, \qquad \overset{n}{\varphi}{}^x_l = \overset{n}{\varphi}_{xl}, \qquad \overset{n}{\varphi}_{xx} = \overset{n}{v}_{x,x},\\
&\qquad \overset{n}{\varphi}_{xl} = \overset{n}{\varphi}_{lx} = 0, \qquad \overset{n}{\varphi}_{ll} = -b_{ll}\,\overset{n}{v}_t,
\end{aligned}
\tag{5.5}
$$

for $n = 0, 1$. The above relationships enable us to express the components of the strain tensor $\mathbf{E}$ by means of the formulas:

$$
\begin{aligned}
\overset{0}{E}{}^x_x &= \overset{0}{E}_{xx} = \overset{0}{v}_{x,x} + \frac{1}{2}\left(\overset{0}{v}_{t,x}\right)^2 + O(\vartheta^4),\\
\overset{1}{E}{}^x_x &= \overset{1}{E}_{xx} = \overset{1}{v}_{x,x} + \overset{0}{v}_{t,x}\,\overset{1}{v}_{t,x} + O(\vartheta^4/h),\\
\overset{2}{E}{}^x_x &= \overset{2}{E}_{xx} = \frac{1}{2}\left(\overset{1}{v}_{t,x}\right)^2 + O(\vartheta^4/h^2),
\end{aligned}
\tag{5.6}
$$

$$\overset{0}{E}_{ll} = -b_{ll}\overset{0}{v}_t + O(\vartheta^4), \qquad \overset{1}{E}_{ll} \cong \overset{0}{v}_t + r_0\overset{1}{v}_t,$$

(5.7) $$\overset{0}{E}{}^l_l = a^{ld}\overset{0}{E}_{dl} = a^{ll}\overset{0}{E}_{ll} = \frac{\overset{0}{v}_t}{r_0} + O(\vartheta^4), \qquad \overset{1}{E}{}^l_l \cong \frac{\overset{0}{v}_t}{r_0^2} + \frac{\overset{1}{v}_t}{r_0},$$

$$\overset{2}{E}{}^l_l = \frac{\overset{1}{v}_t}{r_0^2} + O(\vartheta^4/h^2),$$

(5.8) $$\overset{0}{E}_{xl} \equiv 0, \qquad \overset{1}{E}_{xl} \equiv 0, \qquad \overset{0}{E}_{lt} \equiv 0, \qquad \overset{1}{E}_{lt} \equiv 0,$$

(5.9) $$\overset{0}{E}_{xt} = \overset{0}{E}{}^t_x = \frac{1}{2}\left(\overset{0}{v}_{t,x} + \overset{1}{v}_x\right) + \frac{1}{2}\overset{1}{v}_x\overset{0}{v}_{x,x} + \frac{1}{4}\overset{1}{v}_t\left(\overset{0}{v}_{t,x} - \overset{1}{v}_x\right) + O(\vartheta^4),$$

$$\overset{1}{E}_{xt} = \overset{1}{E}{}^t_x = \frac{1}{2}\overset{1}{v}_{t,x} + \frac{1}{2}\overset{1}{v}_x\overset{1}{v}_{x,x} - \frac{1}{4}\overset{1}{v}_{t,x}\overset{0}{v}_{x,x} + O(\vartheta^4/h),$$

(5.10) $$\overset{0}{E}_{tt} = \overset{0}{E}{}^t_t = \overset{1}{v}_t + \frac{1}{2}\left(\overset{1}{v}_x\right)^2 + O(\vartheta^4).$$

In the literature concerning the geometrical nonlinearity of thin shells [2, 7], one often assumes that

(5.11) $$v_t = \overset{0}{v}_t, \qquad \text{i.e.} \qquad \overset{1}{v}_t = 0.$$

Let us substitute this assumption to (5.6) – (5.10). For the sake of comparison with the results obtained by other authors, we can use now physical components of the strain tensor. Then we get:

(5.12) $$\overset{0}{E}{}^x_x = \overset{0}{v}_{x,x} + \frac{1}{2}\left(\overset{0}{v}_{t,x}\right)^2 + O(\vartheta^4), \qquad \overset{1}{E}{}^x_x = \overset{1}{v}_{x,x} + O(\vartheta^4/h),$$

(5.13) $$\overset{0}{E}{}^l_l = \frac{1}{r_0}\overset{0}{v}_t + O(\vartheta^4), \qquad \overset{1}{E}{}^l_l = \frac{1}{r_0^2}\overset{0}{v}_t + O(\vartheta^4/h),$$

(5.14) $$\overset{0}{E}{}^t_t = \frac{1}{2}\left(\overset{1}{v}_x\right)^2 + O(\vartheta^4),$$

(5.15) $$\overset{0}{E}{}^x_l = \overset{1}{E}{}^x_l = 0,$$

(5.16) $$\overset{0}{E}{}^x_t = \frac{1}{2}\left(\overset{0}{v}_{t,x} + \overset{1}{v}_x\right) + \frac{1}{2}\overset{1}{v}_x\overset{0}{v}_{x,x} + O(\vartheta^4), \qquad \overset{1}{E}{}^x_t = \frac{1}{2}\overset{1}{v}_x\overset{1}{v}_{x,x} + O(\vartheta^4/h).$$

Of course, in the present problem we have $\overset{n}{\dot{B}}{}^l \equiv 0$, $\overset{n}{\dot{p}}{}^l \equiv 0$, $n = 0, 1$.

Let us assume the validity of the assumption (5.11) in equilibrium equations of axisymmetrically loaded circular shells. Then, (starting from (5.2)), these equations can be written down successively as follows:

The first of them is

(5.17) $$\overset{0}{\dot{L}}{}^x_{x,x} + \left(\overset{1}{v}_x\overset{0}{L}{}^{xt}\right)^{\cdot}_{,x} + \overset{0}{\dot{B}}{}^x + \overset{0}{\dot{p}}{}^x = 0.$$

The second of the equations (5.2) in this case is fulfilled identically.

In the next equation we have to take into account that

$$\overset{n}{L}{}^{ll} = a^{xl}\overset{n}{L}{}^{l}_{x} + a^{ll}\overset{n}{L}{}^{l}_{l} = 1/(r_0)^2\overset{n}{L}{}^{l}_{l}. \tag{5.18}$$

Then, we obtain

$$r_0\overset{0}{\dot{L}}{}^{xt}{}_{,x} + r_0\left(\overset{0}{v}_{t,x}\overset{0}{L}{}^{x}_{x}\right)^{\cdot}_{,x} - \overset{0}{\dot{L}}{}^{l}_{l} + 1/r_0\overset{1}{\dot{L}}{}^{l}_{l} + \overset{0}{\dot{B}}{}^{t} + \overset{0}{\dot{p}}{}^{t} = 0, \tag{5.19}$$

that is a known equation, see [2, 6], for geometrically nonlinear circular cylindrical shells; the second term in the above equation indicates its non-linearity.

Further, putting $\overset{1}{L}{}^{t}_{x} = 0$, we obtain from the fourth of equations (5.2):

$$\overset{1}{\dot{L}}{}^{x}_{x,x} - \overset{0}{\dot{L}}{}^{xt} + \left(\overset{0}{v}_{x,x}\overset{0}{L}{}^{xt}\right)^{\cdot} + \left(\overset{1}{v}_{x}\overset{0}{L}{}^{tt}\right)^{\cdot} = 0. \tag{5.20}$$

For simplicity we have assumed here

$$\overset{1}{\mathbf{B}} = \mathbf{0}, \qquad \overset{1}{\mathbf{p}} = \mathbf{0}.$$

The fifth of equations (5.2) for axisymmetrically loaded circular cylindrical shells is satisfied identically. To conclude, let us derive the last of equations for the considered problem. We should start from the equation

$$\overset{1}{S}{}^{tx}{}_{,x} - r_0\overset{1}{\hat{S}}{}^{ll} + \overset{1}{R}{}^{tt} = 0. \tag{5.21}$$

Due to the relation $\overset{1}{L}{}^{ll} = 1/(r_0)^2\overset{1}{L}{}^{l}_{l}$, we obtain

$$r_0\left(\overset{0}{v}_{t,x}\overset{1}{L}{}^{xx}\right)^{\cdot}_{,x} - r_0\overset{0}{\dot{L}}{}^{tt} - \frac{1}{2}r_0\left(\left(\overset{1}{v}_{x} - \overset{0}{v}_{t,x}\right)\overset{0}{L}{}^{xt}\right)^{\cdot} - \overset{1}{\dot{L}}{}^{l}_{l} = 0, \tag{5.22}$$

that is the last equilibrium equation we had to obtain in this case of the shell problem.

Passing to the more particular case of a zero shear strain E_{xt} and a constant axial force $\overset{0}{L}{}^{x}_{x}$ along the shell (when $\overset{0}{p}{}^{x} = \overset{0}{B}{}^{x} = 0$) we obtain

$$\overset{0}{\dot{L}}{}^{x}_{x,x} - \left(\overset{0}{v}_{t,x}\overset{0}{L}{}^{xt}\right)^{\cdot}_{,x} = 0, \tag{5.23}$$

$$r_0\overset{0}{\dot{L}}{}^{xt}{}_{,x} + r_0\left(\overset{0}{v}_{t,x}\overset{0}{L}{}^{x}_{x}\right)^{\cdot}_{,x} - \overset{0}{\dot{L}}{}^{l}_{l} + \overset{0}{\dot{B}}{}^{t} + \overset{0}{\dot{p}}{}^{t} = 0, \tag{5.24}$$

$$\overset{1}{\dot{L}}{}^{xx}{}_{,x} - \overset{0}{\dot{L}}{}^{xt} = 0, \tag{5.25}$$

which is in agreement with the results obtained in [2] (for the case of moderate deflections), where geometrical nonlinearity has been introduced in the thin shell theory in a different way, which, however, turns out to be equivalent to the method used in [4] (cf. a proof of this fact given in [6]).

6. Constitutive relations for elastic-plastic material

Let us take into consideration an orthotropic elastic-plastic material for which the additivity assumption (in rates) is true:

$$\dot{E}_{ij} = \dot{E}^e_{ij} + \dot{E}^p_{ij}\,, \tag{6.1}$$

where $\dot{E}^e_{ij}$ is an elastic part and $\dot{E}^p_{ij}$ a plastic part of the strain rate $\dot{E}_{ij}$.

A. Elastic behaviour of the material

We assume now that in the elastic phase of deformation of the shells, their material is guided by a linear law

$$\dot{s}^{ab} = H^{abcd}\dot{E}^e_{cd}\,, \tag{6.2}$$

where s is the Piola–Kirchhoff stress tensor.

It must be stressed that the constitutive relations for both (elastic and plastic) phases, are given in [4] in a Cartesian frame. So, for our further application it will be necessary to transform them to the cylindrical system. Indices (a, b, c, d) are there the Cartesian ones. For an orthotropic material described in the Cartesian reference frame (what is often used in experiments) with axes coinciding with the axes of orthotropy, the symmetric matrix of elasticity H^{abcd} consists of 9 nonzero scalar parameters,

$$\begin{bmatrix} \dot{s}^{11} \\ \dot{s}^{22} \\ \dot{s}^{33} \\ \dot{s}^{23} \\ \dot{s}^{13} \\ \dot{s}^{12} \end{bmatrix} = \begin{bmatrix} H^{1111} & H^{1122} & H^{1133} & 0 & 0 & 0 \\ & H^{2222} & H^{2233} & 0 & 0 & 0 \\ & & H^{3333} & 0 & 0 & 0 \\ & & & H^{2323} & 0 & 0 \\ & & & & H^{1313} & 0 \\ & & & & & H^{1212} \end{bmatrix} \begin{bmatrix} \dot{E}^e_{11} \\ \dot{E}^e_{22} \\ \dot{E}^e_{33} \\ 2\dot{E}^e_{23} \\ 2\dot{E}^e_{13} \\ 2\dot{E}^e_{12} \end{bmatrix}. \tag{6.3}$$

Just for simplicity, let us assume that there is a plane of material isotropy which is parallel to the shell mid-surface. Then 5 parameters determine the **H** matrix:

$$\begin{gathered} H^{1111} = H^{2222}, \qquad H^{3333}, \qquad H^{1122}, \qquad H^{1313} = H^{2323}, \\ H^{1133} = H^{2233}, \qquad H^{1212} = \frac{1}{2}\left(H^{1111} - H^{1122}\right). \end{gathered} \tag{6.4}$$

In the matrix equation (6.3) and further on, the indices $(1,2,3)$ – Cartesian ones – concern the axes (x,y,z), respectively. In order to refer the parameters of $\mathbf{H}$ to the cylindrical frame we have to transform them according to the formula

$$H^{ijkl} = H^{abcd}\alpha^i_a\,\alpha^j_b\,\alpha^k_c\,\alpha^l_d\,, \tag{6.5}$$

where α^i_a are components of the Jacobi matrix for the transition from the Cartesian to the cylindrical frame. For the simplest case, i.e. for a circular cylindrical shells, we can perform this transformation as follows.

First we shall determine the inverse matrix $[\alpha^i_a]^{-1}$ as we start with the known change of coordinates for t: $r_0 - h/2 \le t \le r_0 + h/2$,

$$x = x(x,l,t), \qquad y = y(x,l,t) = t\cos(l), \qquad z = z(x,l,t) = t\sin(l). \tag{6.6}$$

So, $[\alpha^i_a]^{-1} = [\alpha^a_i]$ and we have:

$$[\alpha^a_i] = \begin{bmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial l} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial l} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial l} & \dfrac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -t\sin(l) & \cos(l) \\ 0 & t\cos(l) & \sin(l) \end{bmatrix}. \tag{6.7}$$

Next, we can determine the matrix $[\alpha^i_a] = [\alpha^a_i]^{-1}$, for $i = x,y,z$; $a = x,l,t$.

$$[\alpha^i_a] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -t^{-1}\sin(l) & t^{-1}\cos(l) \\ 0 & \cos(l) & \sin(l) \end{bmatrix} = \begin{bmatrix} \alpha^1_1 & \alpha^1_2 & \alpha^1_3 \\ \alpha^2_1 & \alpha^2_2 & \alpha^2_3 \\ \alpha^3_1 & \alpha^3_2 & \alpha^3_3 \end{bmatrix}. \tag{6.8}$$

It is obvious that

$$\alpha^1_2 = \alpha^2_1 = \alpha^1_3 = \alpha^3_1 = 0. \tag{6.9}$$

Taking into account (6.4), from (6.5) we obtain

$$\begin{aligned} H^{ijkl} = H^{1111}&(\alpha^i_1\alpha^j_1\alpha^k_1\alpha^l_1 + \alpha^i_2\alpha^j_2\alpha^k_2\alpha^l_2) + H^{3333}\alpha^i_3\alpha^j_3\alpha^k_3\alpha^l_3 \\ &+ H^{1122}(\alpha^i_1\alpha^j_1\alpha^k_2\alpha^l_2 + \alpha^i_2\alpha^j_2\alpha^k_1\alpha^l_1) + H^{1212}\alpha^i_1\alpha^j_2\alpha^k_1\alpha^l_2 \\ &+ H^{1133}(\alpha^i_1\alpha^j_1\alpha^k_3\alpha^l_3 + \alpha^i_3\alpha^j_3\alpha^k_1\alpha^l_1 + \alpha^i_2\alpha^j_2\alpha^k_3\alpha^l_3 + \alpha^i_3\alpha^j_3\alpha^k_2\alpha^l_2) \\ &+ H^{1313}(\alpha^i_2\alpha^j_3\alpha^k_2\alpha^l_3 + \alpha^i_1\alpha^j_3\alpha^k_1\alpha^l_3). \end{aligned} \tag{6.10}$$

Note again that the indices $(1,2,3)$ play the role of the Cartesian indices (x,y,z). At present we can determine all H^{ijkl} (for (i,j,k,l) equal to (x,l,t) each).

$$H^{xxxx} = H^{1111}(\alpha^1_1)^4 = H^{1111}, \tag{6.11}$$

$$H^{llll} = H^{1111}(\alpha_2^2)^4 + H^{3333}(\alpha_3^2)^4 + (2H^{1133} + H^{1313})(\alpha_2^2)^2(\alpha_3^2)^2, \tag{6.12}$$

$$H^{tttt} = H^{1111}(\alpha_2^3)^4 + H^{3333}(\alpha_3^3)^4 + (2H^{1133} + H^{1313})(\alpha_2^3)^2(\alpha_3^3)^2, \tag{6.13}$$

$$H^{xxll} = H^{1122}(\alpha_2^2)^2(\alpha_1^1)^2 + H^{1133}(\alpha_1^1)^2(\alpha_3^2)^2 = H^{llxx}, \tag{6.14}$$

$$H^{xxtt} = H^{1122}(\alpha_2^3)^2(\alpha_1^1)^2 + H^{1133}(\alpha_1^1)^2(\alpha_3^3)^2 = H^{ttxx}, \tag{6.15}$$

$$\begin{aligned} H^{lltt} = {} & H^{1111}(\alpha_2^3)^2(\alpha_2^2)^2 + H^{3333}(\alpha_3^2)^2(\alpha_3^3)^2 + H^{1313}\alpha_2^2\alpha_3^2\alpha_2^3\alpha_3^3 \\ & + H^{1133}\left((\alpha_2^2)^2(\alpha_3^3)^2 + (\alpha_3^2)^2(\alpha_2^3)^2\right) = H^{ttll}, \end{aligned} \tag{6.16}$$

$$\begin{aligned} H^{ltlt} = {} & H^{1111}(\alpha_2^3)^2(\alpha_2^2)^2 + H^{3333}(\alpha_3^2)^2(\alpha_3^3)^2 \\ & + H^{1313}(\alpha_2^2)^2(\alpha_3^3)^2 + 2H^{1133}\alpha_2^2\alpha_3^3\alpha_3^2\alpha_2^3, \end{aligned} \tag{6.17}$$

$$H^{xtxt} = H^{1212}(\alpha_1^1)^2(\alpha_2^3)^2 + H^{1313}(\alpha_1^1)^2(\alpha_3^3)^2, \tag{6.18}$$

$$H^{xlxl} = H^{1212}(\alpha_1^1)^2(\alpha_2^2)^2 + H^{1313}(\alpha_1^1)^2(\alpha_3^2)^2. \tag{6.19}$$

Let us return to (6.8) from which we have

$$\begin{gathered} \alpha_1^1 = 1, \qquad \alpha_2^2 = -t^{-1}\sin(l), \qquad \alpha_3^2 = t^{-1}\cos(l), \\ \alpha_2^3 = \cos(l), \qquad \alpha_3^3 = \sin(l). \end{gathered} \tag{6.20}$$

Now, we assume that $t = \text{const} = r_0$, and from (6.12) - (6.19) we obtain

$$H^{xxxx} = H^{1111}, \tag{6.21}$$

$$\begin{aligned} H^{llll} = {} & (r_0)^{-4}\Big(H^{1111}\sin^4(l) + H^{3333}\cos^4(l) \\ & + (2H^{1133} + H^{1313})\sin^2(l)\cos^2(l)\Big), \end{aligned} \tag{6.22}$$

$$\begin{aligned} H^{tttt} = {} & H^{1111}\cos^4(l) + H^{3333}\sin^4(l) \\ & + (2H^{1133} + H^{1313})\sin^2(l)\cos^2(l), \end{aligned} \tag{6.23}$$

$$H^{xxll} = (r_0)^{-2}\left(H^{1122}\sin^2(l) + H^{1133}\cos^2(l)\right) = H^{llxx}, \tag{6.24}$$

$$H^{xxtt} = H^{1122}\cos^2(l) + H^{1133}\sin^2(l) = H^{ttxx}, \tag{6.25}$$

$$\begin{aligned} H^{lltt} = {} & (r_0)^{-2}\Big(\sin^2(l)\cos^2(l)(H^{1111} + H^{3333} - H^{1313}) \\ & + H^{1133}(\sin^4(l) + \cos^4(l))\Big) = H^{ttll}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} H^{ltlt} = {} & (r_0)^{-2}\Big(\sin^2(l)\cos^2(l)(H^{1111} + H^{3333} - 2H^{1133}) \\ & + H^{1313}\sin^4(l)\Big), \end{aligned} \tag{6.27}$$

$$H^{xtxt} = H^{1212}\cos^2(l) + H^{1313}\sin^2(l), \tag{6.28}$$

$$H^{xlxl} = (r_0)^{-2}\left(H^{1212}\sin^2(l) + H^{1313}\cos^2(l)\right). \tag{6.29}$$

In this way we have got the H^{ijkl} matrix which describes the elastic properties of the shell material in the cylindrical frame, where each of the indices (i, j, k, l)

refers to cylindrical coordinates (x, l, t). It has the form (note that H^{ltlt} is not equal to H^{xtxt}):

$$(6.30)\qquad H^{ijkl} = \begin{bmatrix} H^{xxxx} & H^{xxll} & H^{xxtt} & 0 & 0 & 0 \\ & H^{llll} & H^{lltt} & 0 & 0 & 0 \\ & & H^{tttt} & 0 & 0 & 0 \\ & & & H^{ltlt} & 0 & 0 \\ & & & & H^{xtxt} & 0 \\ & & & & & H^{xlxl} \end{bmatrix}.$$

B. Plastic behaviour of the material

Following [4] we will make use of Hill's yield condition which describes the behaviour of an orthotropic plastic material with kinematical hardening

$$(6.31)\qquad \mathcal{F} = A_{abcd}({}^D s^{ab} - \beta^{ab})({}^D s^{ab} - \beta^{ab}) = \frac{2}{3}(Y_R)^2,$$

where ${}^D s^{ab}$ is a deviator of the **P–K** stress tensor, β^{ab} defines kinematic hardening of the material, and Y_R is a reference yield stress obtained in uniaxial tension (in a principal direction). All the above quantities are specified in [4]. As in the elastic part, we assume here that the material is initially isotropic in the plane (1,2), where 1, 2 are the Cartesian axes. It is characterized by

$$(6.32)\qquad \begin{aligned} A_{1111} &= A_{2222} = \frac{2}{3}, \qquad A_{3333} = \frac{2}{3}(Y_R/Y_3)^2, \\ A_{1133} &= A_{2233} = -\frac{2}{3}(Y_R/Y_3)^2, \qquad A_{1122} = -\frac{2}{3}\left(1 - \frac{1}{2}(Y_R/Y_3)^2\right) \end{aligned}$$

with $Y_R = Y_1 = Y_2$. Let us assume that $A_{2323} = A_{1313} = A_{1212} = 0$.

The rate constitutive equations, in the Cartesian frame of reference are

$$(6.33)\qquad \dot{s}^{ab} = K^{abcd}\dot{E}_{cd},$$

where **K** is a matrix which should be determined first by an inversion $\mathbf{K}^{-1}$ that is given below:

$$(6.34)\qquad (K^{-1})_{abcd} = (H^{-1})_{abcd} + 2\left(A_{ab11}({}^D s^{11} - \beta^{11}) + A_{ab22}({}^D s^{22} - \beta^{22}) + A_{ab33}({}^D s^{33} - \beta^{33})\right) q_{cd}.$$

The quantities q_{ab} and p are

$$q_{ab} = 1/p\left(A_{ab11}({}^Ds^1 - \beta^{11}) + A_{ab22}({}^Ds^2 - \beta^{22}) + A_{ab33}({}^Ds^3 - \beta^{33})\right), \tag{6.35}$$

$$p = 2c\Big(A_{ab11}\,A_{ab11}(11)^2 + A_{ab22}\,A_{ab22}(22)^2 + A_{ab33}\,A_{ab33}(33)^2 + 2\,(A_{ab11}\,A_{ab22}(22)\,(11) + A_{ab11}\,A_{ab33}(33)\,(11) + A_{ab22}\,A_{ab33}(22)(33))\Big). \tag{6.36}$$

where the notation $(ee) := ({}^Ds^{ee} - \beta^{ee})$ has been used to simplify the last expression. In the above and other formulas, c (see [4]) is a constant. Note that the summation convention concerns the indices (a, b). Therefore

$$\begin{aligned} p = 2c\Big((11)^2\left((A_{1111})^2 + (A_{1122})^2 + (A_{1133})^2\right) & \\ + (33)^2\left(2(A_{1133})^2 + (A_{3333})^2\right) & \\ + (22)^2\left((A_{1111})^2 + (A_{1122})^2 + (A_{1133})^2\right) & \\ + 2\,(22)\,(11)\left(2A_{1122}A_{1111} + (A_{1133})^2\right) & \\ + 2\,(33)\,(11)\,\left(...\right) + ...\Big). & \end{aligned} \tag{6.37}$$

Consequently, the indices (a, b, c, d) are referred here to the Cartesian frame. So it is necessary to transform the components A_{abcd} to the cylindrical frame using the inverse of (6.5). This enables us to express the constitutive equations in terms of cylindrical coordinates

$$\dot{s}^{ij} = K^{ijkl}\dot{E}_{kl} \tag{6.38}$$

(with the cylindrical indices i, j, k, l). It is seen that the explicit expression for p, even in the case of very simple, transversal orthotropy of the material, is very complicated and only the computer-aided procedures may be effective here, both on the symbolic and computational level.

Obviously, the matrix **K** should be interpreted as follows: it reduces to matrix **H** in the elastic stage of deformation and during an unloading process in the plastic phase.

7. Concluding remarks

The approach to geometrically nonlinear shells, known as (MRT), proposed for elastic-plastic problems in a general form in [4], has been applied here to the cylindrical shells. At first, geometrical relationships, simpler than those in the general case, have been derived. Next, while analysing the geometrical and equilibrium equations, their particular forms for the special cases of deformation and loading were presented. So, it was possible to compare these equations with the known results, obtained earlier for nonlinear plastic shells, [2]. At the end, the orthotropic constitutive relations (see [4]) have been transformed from the original Cartesian form to the cylindrical coordinates. This step is necessary for obtaining the explicit, effective form of the equilibrium equations.

All these results will be useful in the numerical problem for elastic-plastic, geometrically nonlinear, cylindrical shells. It will be the next stage of this paper. The nonlinear rate variational problem will be solved numerically with the use of an iterative treatment (as it is suggested in [4]). The variational functional for quasi-static processes (4.1) (see also Appendix) which needs to be precised for every real problem, will be used directly in the incremental process. The equilibrium equations with appropriate boundary conditions should be fulfilled on every step of the iteration procedure.

It is important that the shell theory with which this paper is concerned, admits both the geometrical nonlinearity and more realistic description of the shell material (anisotropic elastic-plastic with kinematical hardening).

Appendix

As it was mentioned before, (MRT) deals with 2-D shell problems. This means that instead of the functional (4.1), its 2-D form has been used in this theory. That is obtained, naturally, by substituting to (4.1) the resultant quantities (4.3)–(4.6) and using of the assumption (4.2). The resulting expressions read:

$$
\begin{aligned}
(*)\qquad I(\mathbf{v}) = \int\limits_{\mathcal{M}} \Bigg\{ & \frac{1}{2}\left[\sum_{n=0}^{2} \overset{n}{\dot{L}}{}^{\alpha\beta} \overset{n}{\dot{E}}_{\alpha\beta} + 2\sum_{n=0}^{1} \overset{n}{\dot{L}}{}^{\alpha t} \overset{n}{\dot{E}}_{\alpha t} + \overset{0}{\dot{L}}{}^{tt} \overset{0}{\dot{E}}_{tt}\right] \\
& + \frac{1}{2}\left[\sum_{n=0}^{2} \overset{n}{\mathring{L}}{}^{\alpha\beta} \overset{n}{\ddot{E}}_{\alpha\beta} + 2\sum_{n=0}^{1} \overset{n}{\mathring{L}}{}^{\alpha t} \overset{n}{\ddot{E}}_{\alpha t} + \overset{0}{\mathring{L}}{}^{tt} \overset{0}{\ddot{E}}_{tt}\right] \\
& - \sum_{n=0}^{1}\left[\left(\overset{n}{\dot{B}}{}^{\alpha} + \overset{n}{\dot{p}}{}^{\alpha}\right)\overset{n}{\dot{v}}_{\alpha} + \left(\overset{n}{\dot{B}}{}^{t} + \overset{n}{\dot{p}}{}^{t}\right)\overset{n}{\dot{v}}_{t}\right]\Bigg\}\, d\mathcal{M} \\
- \int\limits_{\zeta_s} \sum_{n=0}^{1} \Bigg\{ & \left[{}^{*}\overset{n}{\dot{L}}{}^{\alpha\beta}\nu_\alpha\nu_\beta \overset{n}{\dot{v}}_{v} + {}^{*}\overset{n}{\dot{L}}{}^{\alpha\beta} t_\alpha \nu_\beta \overset{n}{\dot{v}}_{t} + {}^{*}\overset{n}{\dot{L}}{}^{t\beta}\nu_\beta \overset{n}{\dot{v}}_{t}\right]\Bigg\}\, ds, \qquad \text{for} \quad \alpha,\beta = x, l.
\end{aligned}
$$

The kinematical boundary conditions on ζ_v (which is a part of $\zeta = \partial\mathcal{M}$) are

$$
\overset{n}{\dot{v}}_{\nu} = \overset{n}{\dot{v}}{}^{\alpha}\nu_\alpha = {}^{*}\overset{n}{\dot{v}}_{\nu}, \qquad \overset{n}{\dot{v}}_{t} = \overset{n}{\dot{v}}{}^{\alpha} t_\alpha = {}^{*}\overset{n}{\dot{v}}_{t}, \qquad \overset{n}{\dot{v}}_{t} = {}^{*}\overset{n}{\dot{v}}_{t}\,,
$$

where $\overset{n}{\dot{v}}_t$ and $\overset{n}{\dot{v}}_\nu$ are the velocity components in the directions of the tangent vector $\mathbf{t}$ and the unit outward normal vector $\boldsymbol{\nu}$ of ζ, Fig. 1, ($\mathbf{t} = t^\alpha \mathbf{a}_\alpha$, $\boldsymbol{\nu} = \nu^\alpha \mathbf{a}_\alpha$.

Of course during transformation of the functional (4.1) into its 2-D counterpart, some geometrical relationships have been taken into account, [4]:

$$d\mathcal{V} = \mu\, dt\, d\mathcal{M}, \qquad dS = \mu\, d\mathcal{M}, \qquad n_\alpha\, d\mathcal{B} = \nu_\alpha \mu\, dt\, ds,$$

where $d\mathcal{B}$ – the area element of the lateral boundary surface, dS – the area element at the top and bottom surfaces, t – the third cylindrical coordinate and s – runs along ζ.

Acknowledgement

This paper was partially supported by the State Committee for Scientific Research (KBN), Poland, under grant No 3 P40401306.

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e-mail: aslawian@ippt.gov.pl

Received September 26, 1997; new version February 4, 1998.