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STUDY ON NORMAL STRESSES IN COMPOSITE CURVED BEAMS SUBJECTED TO UNSYMMETRICAL BENDING

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Based on the formula for normal stresses in curved beams of one material, the formulae for normal stresses in composite curved beams with general cross-sectional shape under complicated loads can be derived. The novel formulae presented in this paper can be used in evaluating the normal stresses of connecting rod bush coated with bearing metal for internal combustion engine. These normal stress formulae, in special case, are reduced to those for composite curved beams subjected to symmetrical bending. A numerical example is given for verification at the end of this paper.

1. Introduction

The formulae for normal stresses, shearing stresses and radial stresses of composite curved beams subjected to symmetrical bending can be found in details in [1, 2]. However, in engineering applications, some composite curved beams with general sectional shape are subjected to complicated loads, resulting in two bending moments and shearing forces in the corresponding vertical direction, apart from the normal force on any section of the beams. Therefore in this paper, the formulae for normal stresses in composite curved beams are derived in a more common sense.

2. The loads and cross-section of a composite curved beam

For the purpose of convenience of discussion, we take the example of a curved coplanar beam with the constant cross-section consisting of two different materials and suppose that the two materials are connected with each other firmly, as shown in Fig. 1a, and the composite curved beam is subjected to complicated loads. We denote the y and z components of the external forces per unit length arc by $p_y(s)$ and $p_z(s)$, respectively, and assume that twisting of any cross-section is zero or negligible [3]. In the curvilinear system (s, y, z), s is the arc length measured along the geometric axis, y is a radial coordinate directed toward the center of curvature of this axis, and the direction of z is normal to the plane of the beam. A section of the beam with two different materials I and II is shown in Fig. 1b. The interface between material I and II is parallel to the z axis, and G is the centroid of beam's section.

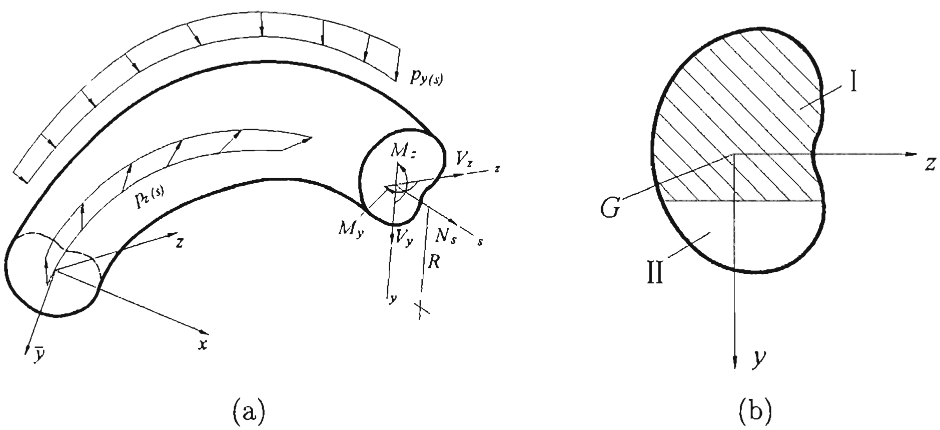

FIG. 1. Composite curved coplanar beam and its cross-section.

3. The derivation of the formulae for normal stresses in composite curved beams

Navier's assumption is introduced here. The studied composite curved beam is subjected to distributed loads of intensity $p_y(s)$ and $p_z(s)$ as shown in Fig. 1a, and on any cross-section of the beam the axial force, shearing forces and bending moments are assumed to be positive when they act in the directions shown in Fig. 2. Therefore, the normal stress at every point on the material I and II of beam's section can be written in the form [3]

$$\begin{aligned} \sigma_1 &= E_1 \varepsilon_1 = \frac{E_1}{1 - y/R}(A + By + Cz), \\ \sigma_2 &= E_2 \varepsilon_2 = \frac{E_2}{1 - y/R}(A + By + Cz). \end{aligned} \tag{3.1}$$

where

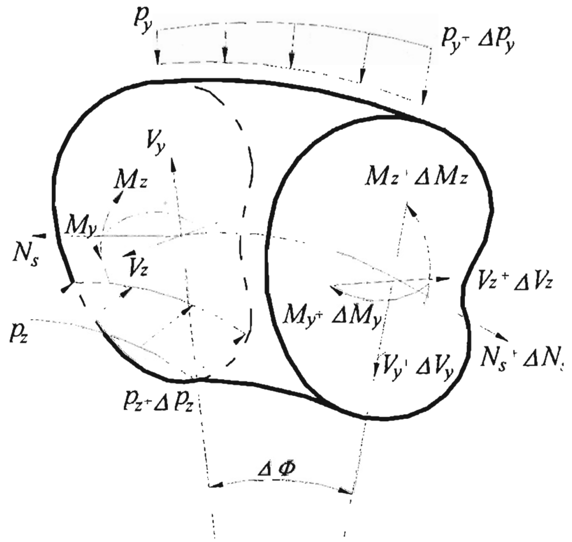

FIG. 2. Internal forces acting upon the typical beam unit.

E_1, E_2 – modulus of elasticity of material I and II,

$\varepsilon_1, \varepsilon_2$ – longitudinal strain of any fiber in material I and II of the beam.

The problem now reduces to the determination of A, B, and C. When the beam is subjected to complicated loads, which on any section result in normal force N_s, and bending moments M_y and M_z, the following equations can be obtained:

$$\begin{aligned}
N_s = \int_{A_1} \sigma_1 dA_1 + \int_{A_2} \sigma_2 dA_2 = AE_1 \int_{A_1} \frac{dA_1}{1 - y/R} \\
+ BE_1 \int_{A_1} \frac{ydA_1}{1 - y/R} + CE_1 \int_{A_1} \frac{zdA_1}{1 - y/R} + AE_2 \int_{A_2} \frac{dA_2}{1 - y/R} \\
+ BE_2 \int_{A_2} \frac{ydA_2}{1 - y/R} + CE_2 \int_{A_2} \frac{zdA_2}{1 - y/R},
\end{aligned}$$

(3.2)

$$\begin{aligned}
M_z = \int_{A_1} \sigma_1 ydA_1 + \int_{A_2} \sigma_2 ydA_2 = AE_1 \int_{A_1} \frac{ydA_1}{1 - y/R} \\
+ BE_1 \int_{A_1} \frac{y^2 dA_1}{1 - y/R} + CE_1 \int_{A_1} \frac{yzdA_1}{1 - y/R} + AE_2 \int_{A_2} \frac{ydA_2}{1 - y/R} \\
+ BE_2 \int_{A_2} \frac{y^2 dA_2}{1 - y/R} + CE_2 \int_{A_2} \frac{yzdA_2}{1 - y/R},
\end{aligned}$$

$$M_y = \int_{A_1} \sigma_1 z dA_1 + \int_{A_2} \sigma_2 z dA_2 = AE_1 \int_{A_1} \frac{z dA_1}{1-y/R} + BE_1 \int_{A_1} \frac{yz dA_1}{1-y/R} + CE_1 \int_{A_1} \frac{z^2 dA_1}{1-y/R} + AE_2 \int_{A_2} \frac{z dA_2}{1-y/R} + BE_2 \int_{A_2} \frac{yz dA_2}{1-y/R} + CE_2 \int_{A_2} \frac{z^2 dA_2}{1-y/R}, \tag{3.2 [cont.]}$$

where A_1, A_2 are the areas of material I and II, respectively. Integral coefficients are obviously functions of the geometry of the cross-section. For simplification, the following notations are introduced:

$$\begin{aligned} J_{y1} &= \int_{A_1} \frac{z^2}{1-y/R} dA_1, \quad J_{yz1} = \int_{A_1} \frac{yz}{1-y/R} dA_1, \\ J_{z1} &= \int_{A_1} \frac{y^2}{1-y/R} dA_1, \quad J_{y2} = \int_{A_2} \frac{z^2}{1-y/R} dA_2, \\ J_{yz2} &= \int_{A_2} \frac{yz}{1-y/R} dA_2, \quad J_{z2} = \int_{A_2} \frac{y^2}{1-y/R} dA_2. \end{aligned} \tag{3.3}$$

In Eqs. (3.2), some surface integrals in material I can be written as

$$\begin{aligned} \int_{A_1} \frac{1}{1-y/R} dA_1 &= A_1 + \frac{1}{R} \int_{A_1} y dA_1 + \frac{1}{R^2} J_{z1}, \\ \int_{A_1} \frac{z}{1-y/R} dA_1 &= \int_{A_1} z dA_1 + \frac{1}{R} J_{yz1}, \\ \int_{A_1} \frac{y}{1-y/R} dA_1 &= \int_{A_1} y dA_1 + \frac{1}{R} J_{z1}. \end{aligned}$$

The same is true for some surface integrals in material II. Hence, Eqs. (3.2) can be reduced to

$$\begin{aligned} N_s &= Aa + Bb + Cd, \\ M_y &= Ad + Bf + Cg, \\ M_z &= Ab + Be + Cf, \end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
a &= E_1\left(A_1 + \frac{1}{R}\int\limits_{A_1} y dA_1 + \frac{1}{R^2}J_{z1}\right) \\
&\qquad\qquad + E_2\left(A_2 + \frac{1}{R}\int\limits_{A_2} y dA_2 + \frac{1}{R^2}J_{z2}\right), \\
b &= E_1\left(\int\limits_{A_1} y dA_1 + \frac{1}{R}J_{z1}\right) + E_2\left(\int\limits_{A_2} y dA_2 + \frac{1}{R}J_{z2}\right), \\
d &= E_1\left(\int\limits_{A_1} z dA_1 + \frac{1}{R}J_{yz1}\right) + E_2\left(\int\limits_{A_2} z dA_2 + \frac{1}{R}J_{yz2}\right), \\
e &= E_1 J_{z1} + E_2 J_{z2}, \\
f &= E_1 J_{yz1} + E_2 J_{yz2}, \\
g &= E_1 J_{y1} + E_2 J_{y2}.
\end{aligned}
\tag{3.5}
$$

Solving Eqs. (3.4), we find

$$
\begin{aligned}
A &= \frac{1}{H}\left[\left(eg - f^2\right)N_s - (bg - df)\,M_z + (bf - de)\,M_y\right], \\
B &= \frac{1}{H}\left[-(bg - df)\,N_s + \left(ag - d^2\right)M_z - (af - bd)\,M_y\right], \\
C &= \frac{1}{H}\left[(bf - de)\,N_s - (af - bd)\,M_z + \left(ae - b^2\right)M_y\right],
\end{aligned}
\tag{3.6}
$$

in which

$$
H = aeg - af^2 - b^2 g + 2bdf - d^2 e.
$$

Finally, substituting Eqs. (3.6) into Eqs. (3.1), we obtain

$$
\begin{aligned}
\sigma_1 &= \frac{E_1}{H(1 - y/R)}\big\{[(eg - f^2)N_s - (bg - df)M_z + (bf - de)M_y] \\
&\qquad - [(bg - df)N_s - ag - d^2)M_z + (af - bd)M_y]y \\
&\qquad + [(bf - de)N_s - (af - bd)M_z + (ae - b^2)M_y]z\big\}, \\
\sigma_2 &= \frac{E_2}{H(1 - y/R)}\big\{[(eg - f^2)N_s - (bg - df)M_z + (bf - de)M_y] \\
&\qquad - [(bg - df)N_s - (ag - d^2)M_z + (af - bd)M_y]y \\
&\qquad + [(bf - de)N_s - (af - bd)M_z + (ae - b^2)M_y]z\big\}.
\end{aligned}
\tag{3.7}
$$

Equations (3.7) are general formulae for normal stresses in composite curved beams. By simply equating of Eqs. (3.7) to zero, we obtain the equation for the neutral axis, which reads

$$
\begin{aligned}
&\left[\left(eg-f^2\right)N_s-(bg-df)\,M_z+(bf-de)\,M_y\right]\\
&\qquad -\left[(bg-df)\,N_s-\left(ag-d^2\right)M_z+(af-bd)\,M_y\right]y\\
&\qquad +\left[(bf-de)\,N_s-(af-bd)\,M_z+\left(ae-b^2\right)M_y\right]z=0\,.
\end{aligned}\tag{3.8}
$$

Given the value of s, it is obvious that neutral axis doesn't generally pass through the centroid of the cross-section.

4. Discussion

Equations (3.7) can be reduced to simpler forms in some special cases:

a. Unsymmetrical curved beam loaded in its plane; In this case, p_z and M_y are both equal to zero, and Eqs. (3.7) become

$$
\begin{aligned}
\sigma_1&=\frac{E_1}{H\,(1-y/R)}\Big\{\left[\left(eg-f^2\right)N_s-(bg-df)\,M_z\right]-\left[(bg-df)\,N_s\right.\\
&\qquad\left.-\left(ag-d^2\right)M_z\right]y+\left[(bf-de)\,N_s-(af-bd)\,M_z\right]z\Big\},\\
\sigma_2&=\frac{E_2}{H\,(1-y/R)}\Big\{\left[\left(eg-f^2\right)N_s-(bg-df)\,M_z\right]-\left[(bg-df)\,N_s\right.\\
&\qquad\left.-\left(ag-d^2\right)M_z\right]y+\left[(bf-de)\,N_s-(af-bd)\,M_z\right]z\Big\},
\end{aligned}\tag{4.1}
$$

b. If the cross-section is symmetrical with respect to the y axis, then

$$
\begin{aligned}
S_{y1}&=\int_{A_1} z\,dA_1=0, & S_{y2}&=\int_{A_2} z\,dA_2=0,\\
J_{yz1}&=\int_{A_1}\frac{yz}{1-y/R}\,dA_1=0, & J_{yz2}&=\int_{A_2}\frac{yz}{1-y/R}\,dA_2=0,\\
d&=f=0, & H&=aeg-b^2g.
\end{aligned}
$$

Equations (3.7) reduce to

$$
\begin{aligned}
\sigma_1&=\frac{E_1}{1-y/R}\left[\frac{(N_se-M_zb)}{(ae-b^2)}-\frac{(N_sb-M_za)\,y}{(ae-b^2)}+\frac{M_yz}{g}\right],\\
\sigma_2&=\frac{E_2}{1-y/R}\left[\frac{(N_se-M_zb)}{(ae-b^2)}-\frac{(N_sb-M_za)\,y}{(ae-b^2)}+\frac{M_yz}{g}\right],
\end{aligned}\tag{4.2}
$$

c. Symmetrical curved beam loaded in its plane; now M_y is zero, and $d = f = 0$, Eqs. (9) can be further written in the form:

$$\sigma_1 = \frac{E_1}{1 - y/R}\left[\frac{(N_s e - M_z b) - (N_s b - M_z a)\, y}{(ae - b^2)}\right],$$
$$\sigma_2 = \frac{E_2}{1 - y/R}\left[\frac{(N_s e - M_z b) - (N_s b - M_z a)\, y}{(ae - b^2)}\right]. \tag{4.3}$$

For symmetrical curved beam loaded in its plane, it follows $M_y = 0$, and $d = f = 0$. By Eqs. (4.3), we obtain the results for connecting rod bush of internal combustion engine subjected to radial, uniformly distributed load on gauge dia, which are almost the same as those in [4].

When the curved beam is made of the same material, it holds $E_1 = E_2 = E$. And we have

$$a = EA + \frac{E}{R^2}J_z, \qquad b = \frac{E}{R}J_z, \qquad d = \frac{E}{R}J_{yz},$$
$$e = EJ_z, \qquad f = EJ_{yz}, \qquad g = EJ_y,$$
$$\left(aeg - af - b^2 g + 2bdf - d^2 e\right) = E^3 A J_y J_z - E^3 A J_{yz}^2.$$

Introducing the above results into Eqs. (3.7), we obtain:

$$\sigma_1 = \sigma_2 = \sigma_s = \frac{N_s}{A} - \frac{M_z}{RA} + \frac{M_z J_y - M_y J_{yz}}{J_y J_z - J_{yz}^2}\frac{y}{1 - y/R} + \frac{M_y J_z - M_z J_{yz}}{J_y J_z - J_{yz}^2}\frac{z}{1 - y/R} \tag{4.4}$$

which is the formula in [3].

5. Numerical example

As a numerical example, we take into consideration an unsymmetrical curved beam made of two materials shown in Fig. 3a. We assume that the distance from the centroid axis z of the section to the center O of curvature is R, the external side of the section is a rectangle $4a' \times a'$ for material I, and the internal side is on angle section of $4a' \times 4a' \times a'$ for material II (see Fig. 3b). The beam is loaded by equal moments $M_y = M_z = M$ and $N_s = 0$, so that Eqs. (3.7) become

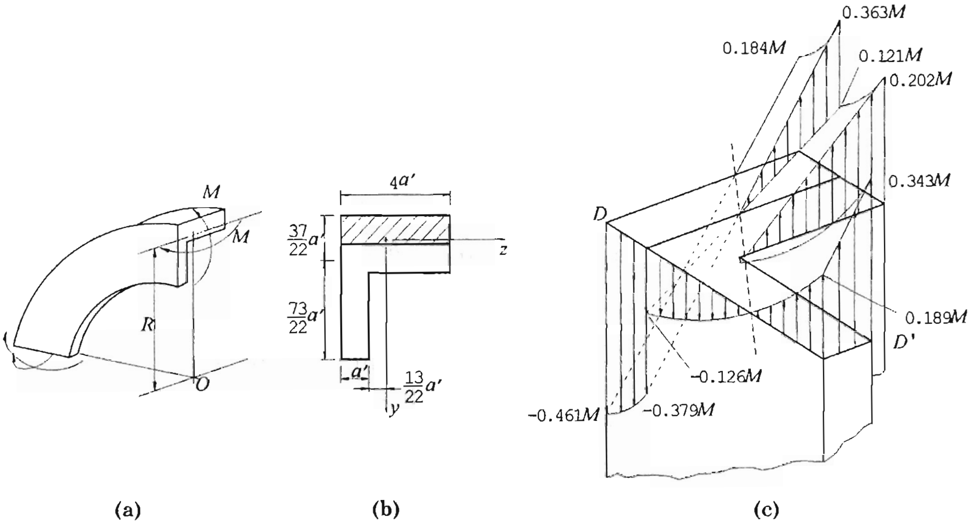

FIG. 3. Bending of the composite curved beam with unsymmetrical cross-section.

$$
\begin{aligned}
\sigma_1 &= \frac{ME_1}{H(1-y/R)}\{[(bf-de)-(bg-df)] \\
&\quad +[(ag-d^2)-(af-bd)]y+[(ae-b^2)-(af-bd)]z\}, \\
\sigma_2 &= \frac{ME_2}{H(1-y/R)}\{[(bf-de)-(bg-df)] \\
&\quad +[(ag-d^2)-(af-bd)]y+[(ae-b^2)-(af-bd)]z\}.
\end{aligned} \tag{5.1}
$$

Note that for the structure shown, the twisting moments are also necessary on some section to provide equilibrium; but again we neglect their influence on the normal stresses [3]. In order to simplify the calculation, we assume $E_1 = 3$ E, $E_2 =$ E, $a' = 1$cm, $R = 6$cm. Substitution of these values into Eqs. (3.3) yields the geometric coefficients of the section, then introduction of the obtained geometric coefficients into Eqs. (3.5) and (5.1) results in the following equations:

$$
\begin{aligned}
\sigma_1 &= \frac{3M}{1-y/R}\left(\frac{100.04+763.13y+933.34z}{13558.27}\right), \\
\sigma_2 &= \frac{M}{1-y/R}\left(\frac{100.04+763.13y+933.34z}{13558.27}\right).
\end{aligned} \tag{5.2}
$$

A plot of Eqs. (5.2) is shown in Fig. 3c. The variation is linear in z direction but hyperbolic in the y direction. Normal stresses are changed suddenly at the interface of the two materials. Since $E_1 = 3E_2$, the maximum stress doesn't occur at D′ but at D on the section. We can obtain the equation for neutral axis by simply equating σ_1 or σ_2 of Eqs. (5.2) to zero and simplifying

$$100.04 + 763.13y + 933.34z = 0$$

Conclusion

a. In this paper, the general formulae for normal stresses in curved beams made of two materials are derived; however, the results can be readily extended to multi-materials.

b. The formulae derived in this paper are valid for composite curved beams of general cross-sectional shape subjected to complicated loads. The effects of torsion can be neglected when the bars of solid section are subjected to transverse forces passing through the centroid of the section, thus the given formulae can be applied to the calculation of normal stresses for composite curved beams subjected to unsymmetrical bending, which represents a general engineering application.

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