ENGINEERING TRANSACTIONS • Engng. Trans. • 50, 1-2, 35–42, 2002
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.506.2002

PERIODIC SOLUTIONS FOR A KIND OF NONLINEAR OSCILLATIONS BY A NEW ASYMPTOTIC APPROACH

Ji-Huan He

Department of Mathematics,
Shanghai Donghua University,
No. 1882 Yan'an xi Lu Road, Shanghai 200051, China
jhhe@mail.shu.edu.cn

This paper proposes a new asymptotic approach to search for the periodic solutions of a kind of nonlinear oscillations. In this method the iteration technique is coupled with the traditional perturbation techniques, yielding a powerful mathematical tool for solving strongly nonlinear equations. Some examples are given to illustrate its effectiveness, convenience and accuracy. Generally, the first iteration leads to a highly accurate approximate solution which is uniformly valid for the whole solution domain. The new asymptotic approach is named the iteration-perturbation method.

1. Introduction

The past three decades have witnessed an explosive growth of numerical simulation for nonlinear problems by finite element methods and meshfree particle methods [1]. Notwithstanding this fierce competition, approximate analytical methods are still continued to develop, catching much more attention of both the scientists and engineers. Many researchers feel that the introduction of some mathematical tools (such as variational theory, homotopy technique, numerical technique) in classic perturbation methods will certainly provide a more advantageous method in the future. The recent developments of the homotopy perturbation method [2], variational iteration method [3], and linearized perturbation method [4] are some excellent examples.

In the present paper, we will apply the basic idea of iteration technique, which is widely applied in numerical simulation, to the perturbation method, and call the new asymptotic approach the iteration-perturbation method [5].

2. Outline of the iteration-perturbation method

In this paper, we consider the following nonlinear oscillations with odd nonlinearity:

$$\frac{d^2u}{dt^2} + \varepsilon u^n = 0, n = \cdots, -3, -1, 3, 5, 7, \cdots, \tag{2.1}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$. We rewrite Eq. (2.1) in the following form:

$$\frac{d^2u}{dt^2} + \varepsilon u^{2m}u = 0, \quad (m \text{ is an integer}), \tag*{(2.2)$_1$}$$

or

$$\frac{d^2u}{dt^2} = -\varepsilon u^{2m}u, \tag*{(2.2)$_2$}$$

[acceleration – restoring force]

so Eq. $(2.2)_2$ describes an oscillation: when the displacement u is positive, the acceleration u'' is negative; and conversely, when u is negative, the acceleration is positive. This just resembles a simple pendulum, showing the opposite signs of the displacement u and acceleration u'' throughout the motion. So Eq. (2.1) has a periodic solution.

It is obvious that the traditional perturbation methods can not be applied to this problem, since the unperturbed equation becomes

$$\frac{d^2u}{dt^2} = 0, \tag{2.3}$$

which can not lead to a periodic solution.

We approximate Eq. (2.1) by [5]

$$\frac{d^2u}{dt^2} + \varepsilon u_0^{n-1}u = 0, \tag{2.4}$$

where u_0 is an initial approximate solution, and we often start with $u_0 = A\cos\omega t$, where ω is the angular frequency of oscillation.

2.1. *Example 1*

Consider the motion of a ball-bearing oscillating in a glass tube which is bent into a curve such that the restoring force depends upon the cube of the displacement u (Fig. 1).

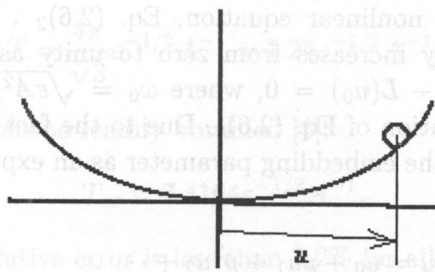

FIG. 1. A ball-bearing oscillating in a smooth tube bent to produce a restoring force proportional to the cube of displacement

The governing equation, ignoring frictional losses, is [6, 7]

$$\frac{d^2u}{dt^2} + \varepsilon u^3 = 0, \tag{2.5}$$

In our study, the parameter ε is not necessarily small, i.e. it satisfies the inequlity $0 < \varepsilon < +\infty$. Eq. (2.5) can be approximated by

$$\frac{d^2u}{dt^2} + \varepsilon A^2 u \cos^2 \omega t = 0, \tag{2.6_1}$$

or

$$\frac{d^2u}{dt^2} + \frac{1}{2}\varepsilon A^2 u + \frac{1}{2}\varepsilon A^2 u \cos 2\omega t = 0, \tag{2.6_2}$$

which is of the Mathieu type.

We can apply the perturbation techniques to find the approximate solution of Eq. $(2.6)_2$. In Ref. [5], we assume that

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{2.7}$$

$$\frac{1}{2}\varepsilon A^2 = \omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots. \tag{2.8}$$

When we are studying the system where the parameter ε might tend to infinite, i.e. $\varepsilon \to \infty$, we feel that it would be more reasonable not to use the parameter ε as an expansion parameter. To this end, we introduce an artificial parameter [4] in $(2.6)_2$, and we obtain

$$\frac{d^2u}{dt^2} + \frac{1}{2}\varepsilon A^2 u + \frac{1}{2}p\varepsilon A^2 u \cos 2\omega t = 0, \quad p = 1. \tag{2.9}$$

It is obvious that when $p = 0$, Eq. (2.9) becomes a linear equation; when $p = 1$, it becomes the nonlinear equation, Eq. $(2.6)_2$. The embedding parameter p monotonically increases from zero to unity as the initial solution $u(t) = A\cos\omega_0 t \quad L(v) - L(u_0) = 0$, where $\omega_0 = \sqrt{\varepsilon A^2/2}$, continuously deformed to the exact solution of Eq. $(2.6)_2$. Due to the fact that $0 < p \leq 1$, it is more reasonable to use the embedding parameter as an expansion parameter, so we can assume that [8]

$$u = u_0 + pu_1 + p^2u_2 + \cdots . \tag{2.10}$$

$$\frac{1}{2}\varepsilon A^2 = \omega^2 + pc_1 + p^2c_2 + \cdots . \tag{2.11}$$

Substituting (2.10) and (2.11) into $(2.6)_2$, and equating the coefficients of the same powers of p, we have the following two differential equations for u_1:

$$u_1'' + \omega^2 u_1 + c_1 u_0 + \frac{1}{2}A^2\cos 2\omega t\, u_0 = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0, \tag{2.12}$$

where $u_0 = A\cos\omega t$. Substituting u_0 into Eq.(2.12), the differential equation for u_1 becomes

$$u_1'' + \omega^2 u_1 + A(c_1 + \frac{1}{4}\varepsilon A^2)\cos\omega t + \frac{1}{4}\varepsilon A^3\cos 3\omega t = 0. \tag{2.13}$$

The requirement of no secular term requires that

$$c_1 = -\frac{1}{4}\varepsilon A^2. \tag{2.14}$$

Solving Eq. (2.13), subject to the initial conditions $u_1(0) = 0$ and $u_1'(0) = 0$, yields the result:

$$u_1 = \frac{1}{32\omega^2}\varepsilon A^3(\cos 3\omega t - \cos\omega t). \tag{2.15}$$

We, therefore, obtain its first-order approximate solution by setting p=1, which reads

$$u(t) = u_0(t) + u_1(t) = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t), \tag{2.16}$$

where the angular frequency is determined from the relations (2.11) and (2.14); we obtain

$$\omega = \frac{\sqrt{3}}{2}\varepsilon^{1/2}A. \tag{2.17}$$

Its period, therefore, can be written as

$$T=\frac{4\pi}{\sqrt{3}}\varepsilon^{-1/2}A^{-1}=7.25\varepsilon^{-1/2}A^{-1}. \tag{2.18}$$

Its exact value can be readily obtained [6]:

$$T_{ex}=7.4164\varepsilon^{-1/2}A^{-1}. \tag{2.19}$$

The maximal relative error is less than 2.2% for all $\varepsilon>0$!

We can obtain the same results if we begin with the assumptions of Eqs. (2.7) and (2.8). As pointed out in Ref. [5], we always stop at the first-order approximation, since high-order approximations of Eq. $(2.6)_2$ will not lead to high accurate solutions for the original one. For example, the second-order period of Eq. $(2.6)_2$ is $T=7.6867\varepsilon^{-1/2}A^{-1}$, which is not more accurate than the first-order one. To obtain approximate solutions with higher accuracy, we replace initial approximate solution by the following one

$$u_0=A\cos\omega t+\frac{A}{24}(\cos 3\omega t-\cos\omega t). \tag{2.20}$$

So the original equation (2.1) can be approximated by the linear equation

$$\frac{d^2u}{dt^2}+\varepsilon\left[A\cos\omega t+\frac{A}{24}(\cos 3\omega t-\cos\omega t)\right]^2u=0. \tag{2.21}$$

In the same manner, we can identify the angular frequency $\omega=0.8475\varepsilon^{1/2}A$, and we obtain the approximate period $T=7.413\varepsilon^{-1/2}A^{-1}$. The relative error is about 0.05% !

2.2. Example 2

A problem of some importance in plasma physics concerns an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. The beam is injected parallel to the axis, but the magnetic field bends the path towards the axis.

The governing equation for the path $u(x)$ of the electrons is [6]

$$\frac{d^2u}{dt^2}+\frac{c}{u}=0. \tag{2.22}$$

We approximate the above equation by

$$\frac{d^2u}{dt^2}+\frac{c}{A^2\cos^2\omega t}u=0, \tag*{$(2.23)_1$}$$

or

$$u'' + \frac{2c}{A^2}u + u'' \cos 2\omega t = 0. \tag{2.23$_2$}$$

We can also introduce an artificial parameter in Eq. (2.23)$_2$:

$$u'' + \frac{2c}{A^2}u + pu'' \cos 2\omega t = 0 \tag{2.24}$$

and assume that the solution of Eq. (2.24) and the coefficient of u can be expressed as a series of the artificial parameter p. Of course, this will lead to an ideal result [5]. In this example, we apply another heuristic approach to Eq. (2.24). Rewrite Eq. (2.24) in the for

$$u'' + \frac{2c}{A^2}u + 1 \cdot u'' \cos 2\omega t = 0, \tag{2.25}$$

and suppose that the solution and the coefficients $2c/A$ and 1 in Eq. (2.25) can be also expressed by a series of the artificial parameter p,which does not appear in Eq. (2.25):

$$u = u_0 + pu_1 + p^2 u_2 + \cdots . \tag{2.26}$$

$$\frac{2c}{A^2} = \omega^2 + pc_1 + p^2 c_2 + \cdots , \tag{2.27}$$

$$1 = pa_1 + p^2 a_2 + \cdots . \tag{2.28}$$

By a similar operation, we obtain the differential equation for u_1:

$$u_1'' + \omega^2 u_1 + c_1 u_0 + a_1 u_0'' \cos 2\omega t = 0, \tag{2.29}$$

where $u_0 = A \cos \omega t$. Substitution of u_0 into Eq. (2.29) leads to

$$\begin{aligned} u_1'' + \omega^2 u_1 &= -Ac_1 \cos \omega t + a_1 A\omega^2 \cos \omega t \cos 2\omega t \\ &= (-Ac_1 + \frac{a_1 A\omega^2}{2}) \cos \omega t + \frac{a_1 A\omega^2}{2} \cos 3\omega t. \end{aligned} \tag{2.30}$$

The requirement of no secular term requires

$$c_1 = \frac{a_1 \omega^2}{2}. \tag{2.31}$$

As we mentioned above, we always stop at the first-order approximation for Eq. (2.25), so from Eqs. (2.27) and (2.28), omitting the higher-order terms, and setting $p = 1$, we obtain

$$\frac{2c}{A^2} = \omega^2 + \frac{a_1\omega^2}{2} \quad \text{and} \quad a_1 = 1.$$

Therefore, we obtain the approximate value of frequency, which reads

$$\omega = \sqrt{\frac{4c}{3A^2}}. \tag{2.32}$$

The approximate value of the period can be written as

$$T = \frac{\sqrt{3}\pi A}{c^{1/2}} = \frac{5.44A}{c^{1/2}}. \tag{2.33}$$

ACTON and SQUIRE[6], using the method of weighted residuals, obtained the following result:

$$T = \frac{16A}{3c^{1/2}} = \frac{5.33A}{c^{1/2}}. \tag{2.34}$$

CONCLUSION

In conclusion of this paper, we can write an iteration-perturbation equation for Eq. (2.1):

$$\frac{d^2u_{k+1}}{dt^2} + \varepsilon u_k^{n-1}u_{k+1} = 0, \tag{2.35}$$

where u_k is the obtained k-th approximate solution, and u_{k+1} can be solved by various new perturbation techniques.

ACKNOWLEDGEMENTS

The work is supported by the National Key Basic Research Special Fund of China (No.G1998020318) and National Natural Foundation of China.

REFERENCES

1. J.H. HE, *Treatment shocks in transonic aerodynamics in Meshless Methods, Part I Lagrange multiplier approach*, Int. J. Turbo and Jet-Engines, **16**, 1, 19-26, 1999.
2. J.H. HE, *Homotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering, **178**, 257-262, 1999.

3. J.H. He, *Variational iteration method: a kind of nonlinear analytical technique: some examples*, International Journal of Nonlinear Mechanics, **34**, 4, 699-708, 1999.

4. J.H. He, *A new perturbation technique which is also valid for large parameters*, Journal of Sound and Vibration, **229**, 5, 1257-1263, 1999.

5. J.H. He, *Iteration perturbation method for strongly nonlinear oscillations*, Journal of Vibration and Control, **7**, 5, 631-642, 2001.

6. J.R. Acton, P.T. Squire, *Solving Equations with Physical Understanding*, Adam Hilger Ltd, Bristol and Boston 1985.

7. I. Andrianov, J. Awrejcewicz, *Construction of periodic solution to partial differential equations with nonlinear boundary conditions*, International Journal of Nonlinear Sciences and Numerical Simulation, **1**, 4, 327-332, 2000.

8. J.H. He, *A review on some new recently developed nonlinear analytical techniques*, International Journal of Nonlinear Sciences and Numerical Simulation, **1**, 1, 51-70, 2000.

Received August 10, 2000; revised version December 18, 2000.