PERIODIC SOLUTIONS FOR A KIND OF NONLINEAR OSCILLATIONS
BY A NEW ASYMPTOTIC APPROACH

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This paper proposes a new asymptotic approach to search for the periodic solutions of
a kind of nonlinear oscillations. In this method the iteration technique is coupled with the
traditional perturbation techniques, yielding a powerful mathematical tool for solving strongly
nonlinear equations. Some examples are given to illustrate its effectiveness, convenience and
accuracy. Generally, the first iteration leads to a highly accurate approximate solution which
is uniformly valid for the whole solution domain. The new asymptotic approach is named the
iteration-perturbation method.

1. INTRODUCTION

The past three decades have witnessed an explosive growth of numerical sim-
ulation for nonlinear problems by finite element methods and meshfree particle
methods [1]. Notwithstanding this fierce competition, approximate analytical
methods are still continued to develop, catching much more attention of both
the scientists and engineers. Many researchers feel that the introduction of some
mathematical tools (such as variational theory, homotopy technique, numerical
technique) in classic perturbation methods will certainly provide a more advan-
tageous method in the future. The recent developments of the homotopy pertur-
bation method [2], variational iteration method [3], and linearized perturbation
method [4] are some excellent examples.

In the present paper, we will apply the basic idea of iteration technique,
which is widely applied in numerical simulation, to the perturbation method,
and call the new asymptotic approach the iteration-perturbation method [5].
2. Outline of the Iteration-Perturbation Method

In this paper, we consider the following nonlinear oscillations with odd nonlinearity:

\[
\frac{d^2u}{dt^2} + \varepsilon u^n = 0, \quad n = \cdots, -3, -1, 3, 5, 7, \cdots
\]

with initial conditions \(u(0) = A\) and \(u'(0) = 0\). We rewrite Eq. (2.1) in the following form:

\[
(2.2)_1 \quad \frac{d^2u}{dt^2} + \varepsilon u^{2m}u = 0, \quad (m \text{ is an integer}),
\]

or

\[
(2.2)_2 \quad \frac{d^2u}{dt^2} = -\varepsilon u^{2m}u,
\]

[acceleration – restoring force]

so Eq. (2.2)_2 describes an oscillation: when the displacement \(u\) is positive, the acceleration \(u''\) is negative; and conversely, when \(u\) is negative, the acceleration is positive. This just resembles a simple pendulum, showing the opposite signs of the displacement \(u\) and acceleration \(u''\) throughout the motion. So Eq. (2.1) has a periodic solution.

It is obvious that the traditional perturbation methods can not be applied to this problem, since the unperturbed equation becomes

\[
(2.3) \quad \frac{d^2u}{dt^2} = 0,
\]

which can not lead to a periodic solution.

We approximate Eq. (2.1) by [5]

\[
(2.4) \quad \frac{d^2u}{dt^2} + \varepsilon u_0^{n-1}u = 0,
\]

where \(u_0\) is an initial approximate solution, and we often start with \(u_0 = A \cos \omega t\), where \(\omega\) is the angular frequency of oscillation.

2.1. Example 1

Consider the motion of a ball-bearing oscillating in a glass tube which is bent into a curve such that the restoring force depends upon the cube of the displacement \(u\) (Fig. 1).
The governing equation, ignoring frictional losses, is \[6, 7\]

\[
\frac{d^2u}{dt^2} + \varepsilon u^3 = 0, \tag{2.5}
\]

In our study, the parameter \(\varepsilon\) is not necessarily small, i.e. it satisfies the inequality \(0 < \varepsilon < +\infty\). Eq. (2.5) can be approximated by

\[
\frac{d^2u}{dt^2} + \varepsilon A^2 u \cos^2 \omega t = 0, \tag{2.6}_1
\]
or

\[
\frac{d^2u}{dt^2} + \frac{1}{2} \varepsilon A^2 u + \frac{1}{2} \varepsilon A^2 u \cos 2\omega t = 0, \tag{2.6}_2
\]

which is of the Mathieu type.

We can apply the perturbation techniques to find the approximate solution of Eq. (2.6)_2. In Ref. [5], we assume that

\[
u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{2.7}
\]

\[
\frac{1}{2} \varepsilon A^2 = \omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots. \tag{2.8}
\]

When we are studying the system where the parameter \(\varepsilon\) might tend to infinite, i.e. \(\varepsilon \to \infty\), we feel that it would be more reasonable not to use the parameter \(\varepsilon\) as an expansion parameter. To this end, we introduce an artificial parameter [4] in (2.6)_2, and we obtain

\[
\frac{d^2u}{dt^2} + \frac{1}{2} \varepsilon A^2 u + \frac{1}{2} p \varepsilon A^2 u \cos 2\omega t = 0, \quad p = 1. \tag{2.9}
\]
It is obvious that when $p = 0$, Eq. (2.9) becomes a linear equation; when $p = 1$, it becomes the nonlinear equation, Eq. (2.6)_2. The embedding parameter $p$ monotonically increases from zero to unity as the initial solution $u(t) = A \cos \omega_0 t \quad L(v) - L(u_0) = 0$, where $\omega_0 = \sqrt{\epsilon A^2 / 2}$, continuously deformed to the exact solution of Eq. (2.6)_2. Due to the fact that $0 < p \leq 1$, it is more reasonable to use the embedding parameter as an expansion parameter, so we can assume that [8]

\begin{equation}
(2.10) \quad u = u_0 + pu_1 + p^2 u_2 + \cdots.
\end{equation}

\begin{equation}
(2.11) \quad \frac{1}{2} \epsilon A^2 = \omega^2 + pc_1 + p^2 c_2 + \cdots.
\end{equation}

Substituting (2.10) and (2.11) into (2.6)_2, and equating the coefficients of the same powers of $p$, we have the following two differential equations for $u_1$:

\begin{equation}
(2.12) \quad u_1'' + \omega^2 u_1 + c_1 u_0 + \frac{1}{2} A^2 \cos 2\omega t u_0 = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0,
\end{equation}

where $u_0 = A \cos \omega t$. Substituting $u_0$ into Eq.(2.12), the differential equation for $u_1$ becomes

\begin{equation}
(2.13) \quad u_1'' + \omega^2 u_1 + A(c_1 + \frac{1}{4}\epsilon A^2) \cos \omega t + \frac{1}{4} \epsilon A^3 \cos 3\omega t = 0.
\end{equation}

The requirement of no secular term requires that

\begin{equation}
(2.14) \quad c_1 = -\frac{1}{4} \epsilon A^2.
\end{equation}

Solving Eq. (2.13), subject to the initial conditions $u_1(0) = 0$ and $u_1'(0) = 0$, yields the result:

\begin{equation}
(2.15) \quad u_1 = \frac{1}{32 \omega^2} \epsilon A^3 (\cos 3\omega t - \cos \omega t).
\end{equation}

We, therefore, obtain its first-order approximate solution by setting $p=1$, which reads

\begin{equation}
(2.16) \quad u(t) = u_0(t) + u_1(t) = A \cos \omega t + \frac{\epsilon A^3}{32 \omega^2} (\cos 3\omega t - \cos \omega t),
\end{equation}

where the angular frequency is determined from the relations (2.11) and (2.14); we obtain

\begin{equation}
(2.17) \quad \omega = \frac{\sqrt{3}}{2} \epsilon^{1/2} A.
\end{equation}
Its period, therefore, can be written as

\[(2.18) \quad T = \frac{4\pi}{\sqrt{3}} \varepsilon^{-1/2} A^{-1} = 7.25\varepsilon^{-1/2} A^{-1}.\]

Its exact value can be readily obtained [6]:

\[(2.19) \quad T_{ex} = 7.4164\varepsilon^{-1/2} A^{-1}.\]

The maximal relative error is less than 2.2% for all \(\varepsilon > 0\)!

We can obtain the same results if we begin with the assumptions of Eqs. (2.7) and (2.8). As pointed out in Ref. [5], we always stop at the first-order approximation, since high-order approximations of Eq. (2.6)_2 will not lead to high accurate solutions for the original one. For example, the second-order period of Eq. (2.6)_2 is \(T = 7.6867\varepsilon^{-1/2} A^{-1}\), which is not more accurate than the first-order one. To obtain approximate solutions with higher accuracy, we replace initial approximate solution by the following one

\[(2.20) \quad u_0 = A \cos \omega t + \frac{A}{24} (\cos 3\omega t - \cos \omega t).\]

So the original equation (2.1) can be approximated by the linear equation

\[(2.21) \quad \frac{d^2 u}{dt^2} + \varepsilon \left[ A \cos \omega t + \frac{A}{24} (\cos 3\omega t - \cos \omega t) \right]^2 u = 0.\]

In the same manner, we can identify the angular frequency \(\omega = 0.8475\varepsilon^{1/2} A\), and we obtain the approximate period \(T = 7.413\varepsilon^{-1/2} A^{-1}\). The relative error is about 0.05%!

### 2.2. Example 2

A problem of some importance in plasma physics concerns an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. The beam is injected parallel to the axis, but the magnetic field bends the path towards the axis.

The governing equation for the path \(u(x)\) of the electrons is [6]

\[(2.22) \quad \frac{d^2 u}{dt^2} + \frac{c}{u} = 0.\]

We approximate the above equation by

\[(2.23) \quad \frac{d^2 u}{dt^2} + \frac{c}{A^2 \cos^2\omega t} u = 0,\]
or

$$u'' + \frac{2c}{A^2} u + u'' \cos 2\omega t = 0.$$  \tag{2.23}_2$$

We can also introduce an artificial parameter in Eq. (2.23)_2:

$$u'' + \frac{2c}{A^2} u + pu'' \cos 2\omega t = 0$$  \tag{2.24}

and assume that the solution of Eq. (2.24) and the coefficient of $u$ can be expressed as a series of the artificial parameter $p$. Of course, this will lead to an ideal result [5]. In this example, we apply another heuristic approach to Eq. (2.24). Rewrite Eq. (2.24) in the form

$$u'' + \frac{2c}{A^2} u + 1 \cdot u'' \cos 2\omega t = 0,$$  \tag{2.25}

and suppose that the solution and the coefficients $2c/A$ and 1 in Eq. (2.25) can be also expressed by a series of the artificial parameter $p$, which does not appear in Eq. (2.25):

$$u = u_0 + pu_1 + p^2 u_2 + \cdots.$$  \tag{2.26}

$$\frac{2c}{A^2} = \omega^2 + pc_1 + p^2 c_2 + \cdots,$$  \tag{2.27}

$$1 = pa_1 + p^2 a_2 + \cdots.$$  \tag{2.28}

By a similar operation, we obtain the differential equation for $u_1$:

$$u''_1 + \omega^2 u_1 + c_1 u_0 + a_1 u''_0 \cos 2\omega t = 0,$$  \tag{2.29}

where $u_0 = A \cos \omega t$. Substitution of $u_0$ into Eq. (2.29) leads to

$$u''_1 + \omega^2 u_1 = -Ac_1 \cos \omega t + a_1 A\omega^2 \cos \omega t \cos 2\omega t$$

$$= (-Ac_1 + \frac{a_1 A\omega^2}{2}) \cos \omega t + \frac{a_1 A\omega^2}{2} \cos 3\omega t.$$  \tag{2.30}

The requirement of no secular term requires

$$c_1 = \frac{a_1 \omega^2}{2}.$$  \tag{2.31}$$
As we mentioned above, we always stop at the first-order approximation for Eq. (2.25), so from Eqs. (2.27) and (2.28), omitting the higher-order terms, and setting \( p = 1 \), we obtain

\[
\frac{2c}{A^2} = \omega^2 + \frac{a_1 \omega^2}{2} \quad \text{and} \quad a_1 = 1.
\]

Therefore, we obtain the approximate value of frequency, which reads

\[(2.32) \quad \omega = \sqrt{\frac{4c}{3A^2}}.
\]

The approximate value of the period can be written as

\[(2.33) \quad T = \frac{\sqrt{3\pi A}}{c^{1/2}} = \frac{5.44A}{c^{1/2}}.
\]

Acton and Squire[6], using the method of weighted residuals, obtained the following result:

\[(2.34) \quad T = \frac{16A}{3c^{1/2}} = \frac{5.33A}{c^{1/2}}.
\]

**Conclusion**

In conclusion of this paper, we can write an iteration-perturbation equation for Eq. (2.1):

\[(2.35) \quad \frac{d^2u_{k+1}}{dt^2} + \varepsilon u_{k}^{n-1} u_{k+1} = 0,
\]

where \( u_k \) is the obtained \( k \)-th approximate solution, and \( u_{k+1} \) can be solved by various new perturbation techniques.

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**References**


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