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CREEP BUCKLING OF A WEDGE-SHAPED FLOATING ICE PLATE

R. Staroszczyk, B. Hedzielski

Institute of Hydroengineering,
Polish Academy of Sciences
ul. Waryńskiego 17, 71-310 Szczecin, Poland

The paper is concerned with the problem of creep buckling of a floating ice plate pressing against a rigid, vertical-walled, engineering structure of a finite length. The plate is modelled as a truncated wedge of a semi-infinite length and constant thickness, resting on a liquid base and subjected to transverse bending due to the elastic reaction of the base and in-plane axial compression due to wind and water drag forces. The ice is treated as a viscous material, with the viscosity varying with the depth of the ice cover. The results of numerical calculations, carried out by the finite-element method, show the evolution of creep buckles in the plate, and also illustrate the behaviour of the ice cover at different levels of the in-plane axial loading, at different temperatures across the ice, and for different geometries of the wedge-shaped plate.

Key words: floating ice, creep, buckling.

NOTATIONS

$b(x)$ plate width at x,
b_0 structure width,
g gravitational acceleration,
h plate thickness,
L_c , L_0 critical and dominant buckle half-wavelengths,
M bending moment per unit width of a plate,
N axial in-plane force per unit width of a plate,
P total axial force in a plate cross-section,
Q transverse shear force per unit width of a plate,
q distributed transverse load intensity,
R parameter defining flexural viscous response of a plate,
T ice temperature,
t time,
w plate deflection,
x, y, z rectangular Cartesian coordinates,

α wedge angle,
ϵ_{xx} infinitesimal normal strain,
κ plate deflection curvature,
μ_0 ice viscosity,
ϱ water density,
σ_{xx} normal stress.

1. Introduction

When a floating sea, lake, or river ice cover interacts with engineering structures, such as dams, breakwaters and off-shore rigging platforms, it is usually subjected to in-plane forces resulting from the wind and water current action as well as thermal expansion. These, approximately horizontal, forces can lead to the failure of the ice sheet by either its out-of plane buckling or crushing (brittle fracture) of ice, as observed in both the laboratory (Sodhi *et al.* [16]) and field (Sanderson [12]) conditions. Theoretical and experimental analyses show that the brittle fracture of ice is a dominant failure mechanism in ice sheets of thicknesses exceeding about 0.5 m. Elastic buckling, investigated by Kerr [6], Nevel [11] and Staroszczyk [18], is possible for only relatively thin ice sheets, of thicknesses usually not exceeding 0.3 to 0.4 m, depending on the geometry of the floating plate and the type of boundary conditions at the ice–structure contact zone. There is, however, a vast field evidence Sanderson [12] that under some conditions, in particular at very low horizontal velocities of the floating ice cover, the ice significantly thicker than 0.5 m, and even more than 1.0 m, is also susceptible to the out-of-plane buckling. Typically, during late Arctic spring, when ice becomes softer and undergoes thermal expansion, buckles form in floating ice sheets over the periods of up to several days, until tensile cracks develop at the upper surface of ice, leading to its gradual failure. Similar buckle features occur when ice is pushed against a vertical structure at very low loading levels.

The reason for such a behaviour of floating ice is its creep, which is substantial comparing with other materials encountered in civil engineering. For instance, at an axial stress of 1 MPa, which can be regarded as a typical stress magnitude in the ice-structure interaction events, the time required for creep strains to exceed those due to the elastic response of ice is less that one minute (Mellor [7], Sanderson [12]). This indicates that not only elastic, but also, and first of all, creep (viscous) effects in ice must be taken into account when attempting to determine realistic contact forces between the floating ice and an engineering structure, and also to describe properly the behaviour of the ice cover itself. Despite its importance, however, the mechanism of creep buckling, unlike elastic buckling, has not attracted much research interest as yet. Among very few examples of theoretical investigations of this phenomenon are the analyses by Sjölind [14] and (Sanderson [12]). Both are devoted to the problem of creep buckling of a floating ice plate of a uniform width and thickness, with a finite length in the first paper, and a semi-infinite length in the other. In the present work we extend those two analyses by considering the creep buckling of a floating ice plate that in the horizontal plane has a shape of a truncated wedge. Such a geometry reflects the conditions frequently occurring in nature, when radial cracks propagating from the vertical edges of the structure develop in the ice plate, bounding in this way the domain of ice which interacts with the structure. The plate, of a constant thickness, is assumed to be non-homogeneous along the vertical direction to account for the possible variation of the anisotropic properties of ice with its depth, and also to take account of the strong influence of temperature gradients across the plate on the creep rate of the material.

The analysis is carried out by applying the standard theory of thin plates resting on a liquid foundation and undergoing in-plane axial loading. The material response of ice is supposed to be viscous, with the ice viscosity strongly depending on temperature, but, unlike the common ice mechanics approach, independent of stress (such a simplifying assumption is adopted because of relatively low stress magnitudes occurring in the problem under consideration, see remarks in Sec. 2.2). The numerical results, obtained by means of the finite-element method, illustrate the evolution of the plate creep behaviour. In particular, the effect of different load levels on the plate deflection history and the time at which the plate fails due to reaching its bearing capacity is examined. In addition, the role which such factors as the structure width, the plate thickness and its in-plane shape, as well as ice temperature, play in creep buckling of the ice cover, is investigated.

2. Properties of floating ice

Before formulating a theoretical model for the ice plate undergoing creep buckling due to transverse and in-plane loading, we give a short summary of the floating ice properties that are most relevant to the analysis carried out in Secs. 3 and 4. First of all, we discuss the creep, or viscous, behaviour of floating ice which is influenced by the three main factors: (1) the material anisotropy of ice, which usually varies with the depth of ice cover, (2) the stress dependence of the ice viscosity, and (3) the strong effect of temperature on the rate of creep. These factors are discussed in the three subsequent subsections, followed by a subsection on the flexural strength of the floating ice.

2.1. *Anisotropy of floating ice*

As ice is formed on the free surface of sea, lake or river water, it is made up of a conglomerate of randomly oriented ice grains, of the size ranging from

1 to 3 mm. Such ice, known in ice mechanics as granular T-1 ice, is macroscopically isotropic, and constitutes the upper layer of floating ice sheets, with a typical depth of this layer of about 30 cm (SANDERSON [12]). During the subsequent growth of ice further downwards, large, regular and vertically elongated crystals of the size of up to 10 cm are formed. Such ice, referred to as columnar S-2 ice, is transversely isotropic about the vertical axis, and develops in sea water. When, however, fresh water freezes under calm conditions typical of lakes or reservoirs, another type of columnar ice, called columnar S-1 ice, forms. This ice is also transversely isotropic, with the rotational symmetry axis coinciding again with the vertical, and its viscous macroscopic behaviour resembles very much that of a single ice crystal. Under some conditions, where uniform directional water currents occur, for instance in rivers, another type of ice, known as columnar S-3 ice, can develop in the bottom layer of floating ice cover (WEEKS and GOW [20], STANDER and MICHEL [17]). This type of ice is orthotropic in its mechanical properties, with one axis of the material symmetry aligned with the water current direction.

The creep properties of all types of anisotropic columnar ice vary considerably with direction. In the extreme case of S-1 ice, maximum and minimum axial viscosities can differ by a factor of 2 to 3, and the shear viscosities can differ by a factor of about 5. When, however, the loading is applied in the horizontal plane, as is the case in this study, the differences between various types of ice in terms of its creep response in the horizontal plane are smaller. Compared to the isotropic T-1 ice viscosity, the relevant axial viscosities for columnar ice are larger by a factor of about 1.3 to 1.5.

2.2. *Stress dependence of ice viscosity*

In general, the creep response of ice is nonlinear, with the viscosity of ice depending on the deviatoric stress magnitude. At high stress levels, within the range between about 0.1 and 1.5 MPa (HUTTER [5]), a conventional approach in ice mechanics is to use the GLEN [4] power flow law, being a form of the Norton-type creep law known in metallurgy, with the power exponent equal to about 3 (that means that, in one-dimensional case, the rate of deformation is proportional to the third power of the stress). Some experimental results (MELLOR and TESTA [8]) indicate, however, that Glen's law significantly overestimates the magnitude of the ice viscosity in lower stress regimes. Therefore, a number of alternative flow laws which agree better with experiments performed at low stresses have subsequently been proposed, for instance a formulation by SMITH and MORLAND [15]. The latter describes the stress dependence of the isotropic ice viscosity in terms of a normalised deviatoric stress invariant by means of a polynomial representation, with coefficients determined by correlation with empirical data.

There is some evidence, however, for instance the experimental results reported by DOAKE and WOLFF [2], indicating that at low stresses, roughly below 0.2 MPa, ice creeps in a nearly linear manner. This is supported by theoretical considerations due to HUTTER [5] and MORLAND [10], who argue that nonlinear viscous behaviour of ice would imply infinitely large instantaneous viscosities at zero stresses, which seems to be physically unsound. Hence, it has been suggested that at low stresses, up to about 0.2 MPa, ice creep behaviour can be satisfactorily approximated by a linear viscous flow law, that means that ice can be treated as a Newtonian fluid, with a constant (stress-independent) viscosity.

2.3. Temperature effect

Temperature has a significant influence on creep behaviour of polycrystalline ice. A common approach in glaciology is to describe this influence by an Arrhenius type law. Such a relationship describes well the viscous properties of cold ice, at temperatures below, say, $-20°$ C. At higher temperatures, however, especially very close to the ice melting point, that is within the temperature range relevant to floating ice, some experimental evidence indicates that the Arrhenius dependence is inappropriate, and hence other relations have been formulated. We use a representation proposed by SMITH and MORLAND [15], obtained by fitting to the experimental data by MELLOR and TESTA [9], which is given in the form

$$\frac{\mu_0(T)}{\mu_0(T_m)} = \left[0.68\,\exp(12\bar{T}) + 0.32\,\exp(3\bar{T})\right]^{-1}, \tag{2.1}$$

where μ_0 is the isotropic ice viscosity, T and T_m denote, respectively, the ice current and melting point temperatures, and $\bar{T}$ is a dimensionless temperature defined by $\bar{T} = (T - T_m)/[20°\text{ C}]$. It follows from (2.1) that the viscosity of ice at $-1°$ C is about 3.5 times lower than that at $-5°$. As temperature variations of such order are quite usual in floating ice due to diurnal (24 hours) cycles of heating and cooling, this indicates how substantially the creep properties of ice can change over relatively short time scales. However, these rapid changes in ice viscosity affect only the upper part of the ice plate. The solution of the heat conduction equation

$$k\frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t}, \tag{2.2}$$

where z is the vertical coordinate, t is time, and $k = 1.15 \times 10^{-6}$ m^2 s^{-1} is the thermal diffusivity coefficient for ice, shows that a free surface temperature perturbation during 24-hours temperature cycles is attenuated by a factor of 10 at a depth of ice of about 0.41 m; more rapid temperature variations decrease with depth even stronger.

2.4. *Flexural strength of floating ice*

Experimental tests conducted on simply-supported or cantilever beams made of ice, the results of which have been summarised by SCHWARZ and WEEKS [13] and HUTTER [5], demonstrate that the flexural strength (calculated from simple elastic beam analyses) of floating sea ice depends very significantly on its porosity. For pure, poreless ice, the flexural strength varies between 0.7 and 1.0 MPa, but it decreases very rapidly with increasing ice porosity, so that at about 0.1 (10%) volume porosity the strength of ice reduces to about 0.2 MPa, and, according to [13], does not change considerably for porosities larger than 0.1. Since sea ice is practically always porous, it seems sensible to ignore in applications the influence of porosity on the flexural strength of floating ice and hence to assume a constant value of 0.2 MPa.

On the other hand, the ice plate failure mechanism can be investigated by applying the methods of fracture mechanics, that is by analysing the problem of formation and subsequent propagation of tensile cracks from the free surface towards the depth of the plate, as a result of a combined action of bending moments and in-plane compressive axial forces. An example of such an analysis, restricted to the case of a beam (instead of a plate), can be found in SANDERSON [12]. Following that analysis, a critical stress σ_{cr} at which the beam of ice starts to fail depends on the depth of surface cracks, the beam thickness, and the magnitudes of the bending and compressive forces. For crack depths assumed to be small compared to the beam thickness, and adopting ice fracture toughness of the value 0.1 MPa $m^{1/2}$ (SANDERSON [12]), a simplified analysis yields the values of the critical stress equal to $\sigma_{cr} \sim 0.35$ MPa for ice 0.2 m thick, and $\sigma_{cr} \sim 0.22$ MPa for ice 0.5 m thick, both based on the assumption that ice starts to break when a surface crack reaches a depth of 1/10 of the ice beam. These stress magnitudes, given many uncertainties associated with a proper description of micro-mechanisms involved in ice failure, agree reasonably well with the above-mentioned values determined, indirectly, from experiments. Accordingly, in this work we adopt the value of 0.2 MPa to define the flexural strength of natural floating ice.

3. GOVERNING EQUATIONS

The geometry of the problem under consideration and the adopted coordinate system are depicted in Fig. 1. A coherent plate of floating ice in the form of a truncated wedge (Fig. 1a) is pushed by wind and water drag forces towards an engineering structure situated at $x = 0$, modelled as a vertical and rigid wall of the length b_0. We idealise the problem by assuming that the floating wedge-shaped plate is symmetric about the x-axis and extends to infinity in the positive direction of x, so its geometry is fully defined by the structure length b_0 and the angle α. Further, we also assume that the loading exerted on the plate is symmetric about the x-axis, so that the net horizontal reaction of the structure, denoted by P, acts in the direction of the x-axis. The plate of a uniform thickness h (Fig. 1b), with its upper face at $z = 0$ and its lower face at $z = h$, is supposed to be in perfect contact with the underlying water, hence no lift-off of the plate can take place. At the ice-structure contact area $x = 0$ the plate edge is supposed to be simply supported.

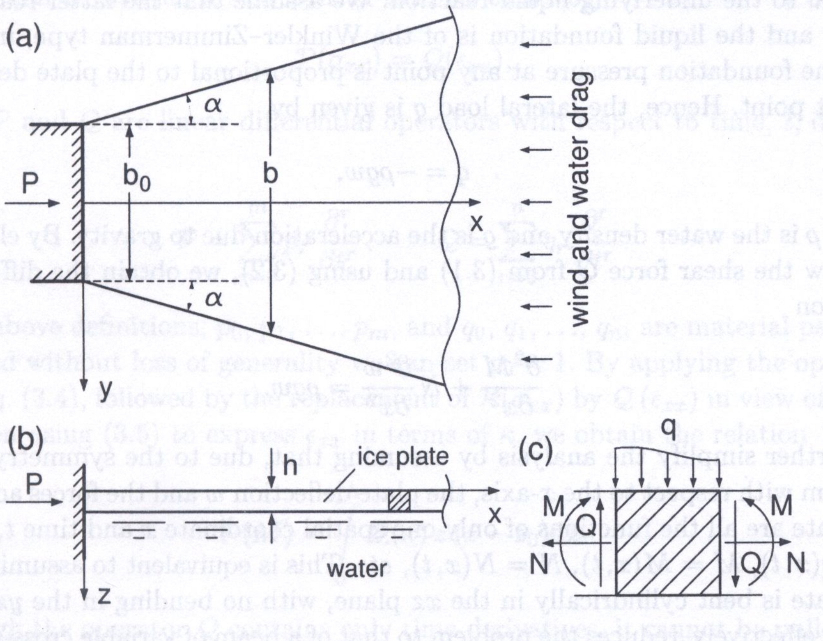

FIG. 1. Geometry of a wedge-shaped plate of floating ice interacting with a rigid structure of the width b_0: (a) plane view, (b) plate cross-section, (c) definition of internal forces.

The problem is solved by applying the classical theory of thin plates (TIMOSHENKO and WOINOWSKY–KRIEGER [19]), assuming that the plate deflections are small (that is of the order of the plate thickness), and the plate cross-sections that are normal to the middle plane in the undeformed state also remain plane and normal to the middle surface when the plate is deflected. In our case, the plate is bent by transverse loads coming from the elastic reaction of the liquid base when the ice is either lifted or depressed from its floating equilibrium state. Besides the bending, the plate is also subjected to in-plane compressive stresses along the x-axis, resulting from the wind and water drag forces.

Denote the plate deflection along the z-axis by w, and define the plate internal forces acting per unit width: the bending moment M, the vertical shear force Q, and the normal (tensile) force N, in the way shown in Fig. 1c. Then, neglecting the plate own weight, the equilibrium balances of forces in the z-axis direction and the bending moments acting on an infinitesimal plate element cut by a pair of planes parallel to the yz coordinate planes yield the two following relations:

$$\frac{\partial Q}{\partial x} + N\frac{\partial^2 w}{\partial x^2} = -q, \qquad \frac{\partial M}{\partial x} = Q. \tag{3.1}$$

In (3.1), q is the transverse distributed load acting in the z-axis direction, and is equal to the underlying liquid reaction. We assume that the latter reaction is elastic and the liquid foundation is of the Winkler–Zimmerman type, in which case the foundation pressure at any point is proportional to the plate deflection at that point. Hence, the lateral load q is given by

$$q = -\rho g w, \tag{3.2}$$

where ρ is the water density and g is the acceleration due to gravity. By eliminating now the shear force Q from (3.1) and using (3.2), we obtain the differential equation

$$\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 w}{\partial x^2} = \rho g w. \tag{3.3}$$

We further simplify the analysis by assuming that, due to the symmetry of the problem with respect to the x-axis, the plate deflection w and the forces acting on the plate are all the functions of only one spatial coordinate x and time t, that is $w = w(x,t)$, $M = M(x,t)$, $N = N(x,t)$, *etc.* This is equivalent to assuming that the plate is bent cylindrically in the xz plane, with no bending in the yz plane, which, effectively, reduces the problem to that of a beam of variable cross-section, resting on an elastic base and undergoing bending and axial compression.

The bending moment per unit width of the plate, $M(x)$, is determined by integrating the normal stress $\sigma_{xx}(x,z)$ across the plate depth

$$M = \int_0^h \sigma_{xx} z \, \mathrm{d}z. \tag{3.4}$$

The normal strains along the x-axis, $\epsilon_{xx}(x,z)$, are defined by a relation following from the assumption of the plane cross-sections in the deformed state, yielding

$$\epsilon_{xx} = \kappa(z - z_0), \tag{3.5}$$

where κ is the local curvature of the plate deflection curve, and z_0 is the position of a neutral plane. The stress and the strain are connected by a constitutive law of the form depending on a specific material of which the body is made. In line with what has been said in Secs. 2.2 and 2.4, and in particular that: (1) the average stress in ice at its flexural failure is about 0.2 MPa, (2) the viscous response of ice up to the latter stress level of 0.2 MPa can be approximated by a linear relation, and additionally (3) the elastic constants for ice are practically stress-independent (cf. HUTTER [5]), it seems well justified in our problem to confine attention to the case of linear viscoelasticity. For a general linearly viscoelastic material, being in a simple stress state, the constitutive equation relating σ_{xx} and ϵ_{xx} can be written in the form (FINDLEY *et al.* [3])

$$\mathcal{P}(\sigma_{xx}) = \mathcal{Q}(\epsilon_{xx}), \tag{3.6}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are linear differential operators with respect to time, t, defined by

$$\mathcal{P} = \sum_{r=0}^{m} p_r \frac{\partial^r}{\partial t^r}, \qquad \mathcal{Q} = \sum_{r=0}^{n} q_r \frac{\partial^r}{\partial t^r}. \tag{3.7}$$

In the above definitions, $p_0, p_1, \ldots, p_m$, and $q_0, q_1, \ldots, q_n$, are material parameters, and without loss of generality we can set $p_0 = 1$. By applying the operator $\mathcal{P}$ to Eq. (3.4), followed by the replacement of $\mathcal{P}(\sigma_{xx})$ by $\mathcal{Q}(\epsilon_{xx})$ in view of (3.6), and then using (3.5) to express ϵ_{xx} in terms of κ, we obtain the relation

$$\mathcal{P}(M) = \int_0^h \mathcal{Q}(\kappa)\, z\,(z - z_0)\mathrm{d}z. \tag{3.8}$$

Although the operator $\mathcal{Q}$ contains only time derivatives, it cannot be pulled out of the integral since the material properties, described by the parameters q_0 to q_n, may in general be functions of the depth z. For small deflections w and the derivatives $(dw/dx)^2 \ll 1$, the plate curvature κ (regarded positive if it is convex downwards) can be approximated in terms of w by $\kappa = -\partial^2 w/\partial x^2$. By using the latter relation for κ in (3.8), next differentiating (3.8) twice with respect to x, and combining the resulting equation with (3.3) (after the application of the operator $\mathcal{P}$), we arrive at the following equation:

$$\int_0^h \mathcal{Q}\left(\frac{\partial^4 w}{\partial x^4}\right) z\,(z - z_0)\mathrm{d}z - N\,\mathcal{P}\left(\frac{\partial^2 w}{\partial x^2}\right) + \rho g\,\mathcal{P}(w) = 0. \tag{3.9}$$

This equation describes the behaviour of the whole class of viscoelastic plates resting on a liquid base and subjected to axial loading along the x-axis. The equations for particular materials of which the plate is made (for instance elastic solid, Maxwell fluid, Kelvin solid, Burgers four-parameter fluid, *etc.*) can be derived by an appropriate choice of rheological parameters p_r and q_r appearing in (3.7) and defining a given material model. In the problem analysed here we deal with the behaviour of the material in which the irreversible deformations due to creep exceed elastic strains by two to three orders of magnitude. For this reason, ignoring the purely elastic (instant) response of ice, as well as transient creep effects, we assume that the response of ice to stress is that of a purely viscous material. Such a material is described by means of only two rheological parameters, namely p_0 (equal to unity) and q_1, with the other equal to zero. Accordingly, the constitutive relation reads

$$\sigma_{xx} = q_1\,\dot{\epsilon}_{xx}\,, \tag{3.10}$$

where the superposed dot denotes the time derivative. After performing on (3.9) the operations prescribed by $\mathcal{P}$ and $\mathcal{Q}$, we obtain the relation

$$R\,\frac{\partial^4 \dot{w}}{\partial x^4} - N\,\frac{\partial^2 w}{\partial x^2} + \rho g w = 0, \tag{3.11}$$

where R is given by

$$R = \int_0^h q_1(z)\, z\,(z - z_0)\,\mathrm{d}z. \tag{3.12}$$

Obviously, the parameter q_1 is related to the viscosity of ice. To establish this relation, we recall that the viscous behaviour of creeping materials is commonly described in terms of deviatoric stresses, reflecting the fact that for many materials, including ice, the mean (hydrostatic) pressure has negligibly small effect on the rate of creep. Hence, we express the viscous response of ice by

$$S_{ij} = 2\mu_0 \dot{\epsilon}_{ij}, \qquad S_{ij} = \sigma_{ij} - \frac{1}{3}\,\sigma_{kk}\delta_{ij}\,, \tag{3.13}$$

where σ_{ij}, S_{ij}, and $\dot{\epsilon}_{ij}$ are components of the stress, deviatoric stress, and strain-rate tensors, respectively, σ_{kk} denotes the mean pressure, δ_{ij} is the Kronecker symbol, and μ_0 is viscosity. If we assume that in our case the ice plate, loaded along the x-axis, is not constrained in the lateral direction along the y-axis, then $S_{xx} = (2/3)\,\sigma_{xx}$, and hence $\sigma_{xx} = 3\mu_0\dot{\epsilon}_{xx}$. By comparing the latter expression with (3.10) we find that $q_1 = 3\mu_0$. If, however, the plate is constrained in the y direction, that is $\dot{\epsilon}_{yy} = 0$, then $S_{xx} = (1/2)\,\sigma_{xx}$, and $q_1 = 4\mu_0$.

Equation (3.11) describes the deflection $w(x,t)$ of the plate of a unit width. In order to derive the relation for the wedge-shaped plate, with its width b varying with x, we multiply both sides of (3.11) by $b(x)$ defined by

(3.14) $$b(x) = b_0 + 2x \tan\alpha,$$

to obtain the relation for the creep behaviour of a floating ice sheet under the combined action of bending and axial compression:

(3.15) $$Rb(x)\frac{\partial^4 \dot{w}}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + \rho g b(x) w = 0.$$

In the above equation, $P = -Nb$ is the total compressive load that is transmitted along the x-axis direction through the whole cross-section b of the sheet. The load P is assumed to be independent of x in the vicinity of the rigid wall located at $x = 0$, that is in the region where, for $\alpha > 0$, the failure of the plate due to creep buckling occurs. In doing so, we neglect the influence of the wind and water drag forces on the local variation of the total axial force in the plate. This is because the change with x of the axial stresses in the plate due to drag tractions is very small compared to the magnitude of P near $x = 0$. Indeed, even supposing extreme weather conditions, say a wind of the speed 30 ms^{-1} and a current of the speed of 1 ms^{-1}, their combined effect will be a horizontal distributed load of the intensity in the region of 5 to 10 Pa (SANDERSON [12]). This translates, in the case of a plate as thin as 0.1 m, in an increase with x of the axial stress by about 100 Pa, which is about three orders of magnitude less than the stress levels during the plate failure (see Sec. 2.4). The differential equation (3.15) is solved with the boundary conditions at $x = 0$ representing the case of a simply-supported plate, and expressed by $w(0,t) = 0$ and $\partial^2 w/\partial x^2(0,t) = 0$, since other types of boundary conditions seem to be less likely to occur in the field (SANDERSON [12]). At $x \to \infty$, the regularity conditions are applied.

In derivation of Eq. (3.15) we have assumed that the ice response is linearly viscoelastic. Thus, strictly, our theoretical model should not be applied to non-linearly viscoelastic materials, for instance when the floating ice cover is subjected to stress levels which are significantly in excess of 0.2 MPa and nonlinear creep effects become important. In some cases, however, as argued by CALLADINE [1] and SANDERSON [12], even for nonlinear constitutive viscous flow laws the relation (3.15) can still be used, provided that there are only small variations in stresses and strain-rates about their respective mean values, and hence the relations between the bending moment and the plate curvature-rate remain linear. Although such an approach cannot be considered to be a rigorous treatment of the problem, and instead the governing equations should be derived by applying the methods of nonlinear viscoelasticity from the very outset, the method of combining equations analogous to (3.15) and nonlinear creep laws has been employed by a number of investigators, for example by SJÖLIND [14] and SANDERSON [12].

4. Plate of uniform width

Because of the presence of the variable coefficient $b(x)$, no exact closed-form analytic solution of (3.15) is available for the general case of a wedge-shaped ice sheet defined by $\alpha > 0$, and hence a numerical method must be employed. Only in a particular case of a plate of uniform width, when $\alpha = 0$ and $b(x) = b_0$, which simplifies (3.15) to the equation with constant coefficients

$$Rb_0 \frac{\partial^4 \dot{w}}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} + \rho g b_0 w = 0, \tag{4.1}$$

can some analytic results be obtained (Sjölind [14], Sanderson [12]). Before proceeding further, however, it is useful to realise two essential differences between the elastic and creep buckling mechanisms. First, elastic buckling occurs instantly after a critical load has been reached and is followed by unstable failure of ice, with the presence of underlying water having a destabilising effect on the floating plate [14]. In contrast, creep buckling is a time-dependent process which can occur at any load level, and leads to the failure of ice only if sufficiently large strain-rates (and hence stresses) in the plate have been reached. Increasing a load magnitude will simply result in increasing the rate of creep buckling. And second, unlike elastic buckling, creep buckling requires an initial perturbation in the plate deflection; this perturbation will subsequently evolve under applied loading. However, not any initial buckle $w(x, 0)$ will grow with time under a given load level P. In order to demonstrate this, we re-write Eq. (4.1) in the form

$$Rb_0 \frac{\partial^4 \dot{w}}{\partial x^4} = p(x,t), \tag{4.2}$$

where

$$p(x,t) = -\left(P \frac{\partial^2 w}{\partial x^2} + \rho g b_0 w \right). \tag{4.3}$$

The expression $p(x,t)$ can be treated in (4.2) as a transverse load depending on the axial force P and the current deflection $w(x,t)$. The existing deflection will grow with time only if $p(x,t) > 0$, and, reversely, it will decay with time if $p(x,t) < 0$. A stationary state, with w not evolving, occurs for $p(x,t) = 0$, that is when the expression in parentheses in Eq. (4.3) becomes zero. Assuming that (4.3) can be solved by separation of variables, and taking account of the boundary conditions $w(0,t) = 0$ and $\partial^2 w/\partial x^2(0,t) = 0$, we adopt a general solution in the form

$$w(x,t) = A(t) \sin(\pi x/L), \tag{4.4}$$

where $A(t)$ is a time-dependent buckle amplitude, and L is an arbitrary half-wavelength. By substituting (4.4) into (4.3), we find a critical length of a buckle half-wave, denoted by L_c, given by the following relation:

$$L_c = \pi \sqrt{\frac{P}{\rho g b_0}}. \tag{4.5}$$

The critical length L_c determines the longest buckling half-wave whose amplitude can increase with time. Any existing buckles for which $L > L_c$ will decrease with time (provided that P is not increasing).

Now a question arises how, for $L < L_c$, the length of a buckle affects the rate of growth of its amplitude. To answer this question, we suppose that amplitudes $A(t)$ of creeping buckles increase in an exponential manner, that is

$$A(t) = w_0 \exp(t/\tau), \tag{4.6}$$

where w_0 is an initial small deflection amplitude, and τ is a time constant. On inserting the latter relation into (4.4), and then substituting the resulting expression for $w(x,t)$ into the differential Eq. (4.1), we obtain the relation

$$\frac{1}{\tau} = \frac{1}{R}\left[\frac{P}{b_0}\left(\frac{L}{\pi}\right)^2 - \rho g\left(\frac{L}{\pi}\right)^4\right], \tag{4.7}$$

which expresses the growth-rate constant τ in terms of the buckle length L and the axial load P. From among all possible perturbations of different lengths L, the fastest growing is that for which τ attains the minimum value. Hence, by differentiating (4.7) with respect to L and setting it to zero, we find that τ is minimised for the buckle half-wavelength L_0 defined by

$$L_0 = \pi \sqrt{\frac{P}{2\rho g b_0}}, \tag{4.8}$$

which, like L_c, depends on the load magnitude P but does not depend on the material viscous properties described by R. As the creep deformation of the plate proceeds from its initial state with small perturbations, the buckle of the length L_0, with the largest growth-rate of the amplitude, soon becomes the dominant buckling mode. The corresponding time constant for the dominant buckle, obtained by substituting (4.8) into (4.7), is defined by

$$\tau_0 = \frac{4R\rho g b_0^2}{P^2}. \tag{4.9}$$

By comparing the expressions (4.5) and (4.8) we note that, independently of the loading level P, the critical and dominant buckle half-wavelengths remain always at a constant ratio given by

$$\frac{L_c}{L_0} = \sqrt{2}. \tag{4.10}$$

5. Numerical solution and results

The fourth-order in space, and first-order in time, partial differential equation (3.15), which describes the creep behaviour of a floating ice sheet of variable width $b(x)$ in response to the compressive load P, has been solved numerically by applying the finite-element method. The weighted residual, or Galerkin, version of the method has been used, in which the problem equation is satisfied in an integral mean sense (Zienkiewicz and Taylor [21]). In the space domain, the plate has been discretised along the x-axis by introducing one-dimensional finite elements. At each nodal point two parameters, w and dw/dx, are used to approximate the plate deformation in order to ensure continuity of the plate deflection and its slope between elements. In the time domain, the equation has been integrated by applying an implicit θ-method with equal time step lengths.

In numerical calculations, the results of which are presented below, 400 finite elements of the same length, equal to $1.5\,h$, were used. Hence, the behaviour of a semi-infinite plate was approximated by the plate of the finite length of $600\,h$. The constant (stress-independent) isotropic viscosity of ice at the melting point was adopted to be $\mu_0 = 1\times10^{11}$ kgm^{-1} s^{-1}, with its temperature dependence described by (2.1) and the elastic constants, Young's modulus E and Poisson's ratio ν, were equal to 9.0 GPa and 0.31, respectively. The water density was assumed to be $\rho = 10^3$ kgm^{-3}, and $g = 9.81$ ms^{-2}. The initial small deflection of the plate was adopted as a sum of twenty harmonic components, given in the form

$$w_0(x) = \sum_{i=1}^{20} \pm w_0^{(i)} \sin(i\pi x/L), \tag{5.1}$$

where the signs ($\pm$) were selected at random, and all the component amplitudes $w_0^{(i)}$ were equal and such that the maximum deflection $w_0 = 0.001$ m. L, defining the length of the longest initial perturbation, was related to the dominant buckle half-wavelength L_0 for a plate of uniform width by choosing $L = 3L_0$. In this way, $w_0(x)$ includes two components which are longer than the critical half-wavelength L_c determined by (4.8), one component (the third longest) with the length equal to L_0, and the remaining harmonic components of the lengths smaller than L_0. In the simulations, the value of the compressive load P exerted on the floating plate is normalised by the magnitude of the force causing elastic buckling of the respective plate. The latter force, denoted by P_e, is calculated by using the results obtained by STAROSZCZYK [18]. The evolution of the plate deflection $w(x,t)$ from its initial state, prescribed by (5.1), is followed up to the time t_c at which the tensile stress at any point in the plate reaches the stress level $\sigma_{\mathrm{cr}} = 0.2$ MPa, the latter value chosen to be that corresponding to the ice flexural strength for porous ice. The results presented below are obtained, if not stated otherwise, for ice temperature equal to $-2°$ C at the upper surface, and $0°$ C at the lower surface of the plate.

In Fig. 2 we illustrate the time variation of the deflection $w(x,t)$ of the plate of a unit width and the thickness $h = 0.2$ m, subjected to the compressive loading $P = 0.1\,P_e$. The plots demonstrate how the plate displacements, shown at the intervals of 4500 s=1.25 hr, gradually evolve from the initial, random distribution of small perturbations, into a regular pattern, more and more dominated with increasing time by the buckling mode of the length L_0 given by (4.8).

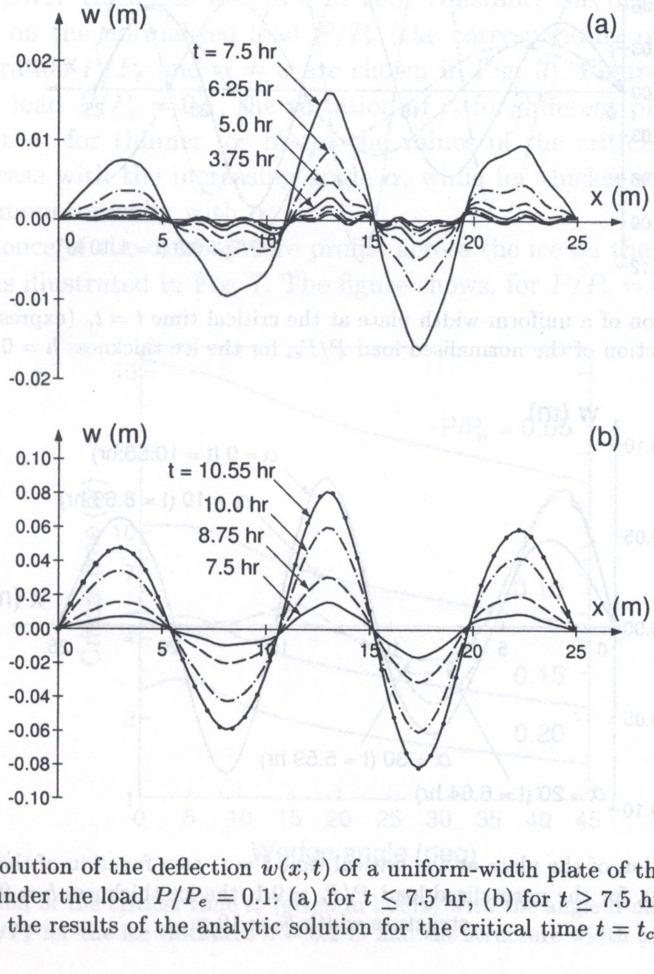

FIG. 2. Evolution of the deflection $w(x,t)$ of a uniform-width plate of the thickness $h = 0.2$ m under the load $P/P_e = 0.1$: (a) for $t \leq 7.5$ hr, (b) for $t \geq 7.5$ hr. The solid circles show the results of the analytic solution for the critical time $t = t_c = 10.55$ hr.

The solid line for the critical time $t = t_c = 37980$ s$=10.55$ hr displays the deflection of the plate at the onset of its failure. We note that the deflections $w(x, t_c)$ are of the order of $h/2$. For comparison, the results, indicated by the solid circles, of the analytic solution discussed in Sec. 4, are also presented to demonstrate the accuracy of the finite-element solution.

Figure 3 illustrates the effect of the in-plane load magnitude P/P_e on the ice displacement at the critical time t_c, when ice starts to fail; the respective values of the critical times are given in hours. The results, obtained for the plate of a unit width and the thickness $h = 0.2$ m, show that while the maximum plate deflections $w(x, t_c)$ decrease by a factor of about two with a sixfold increase in

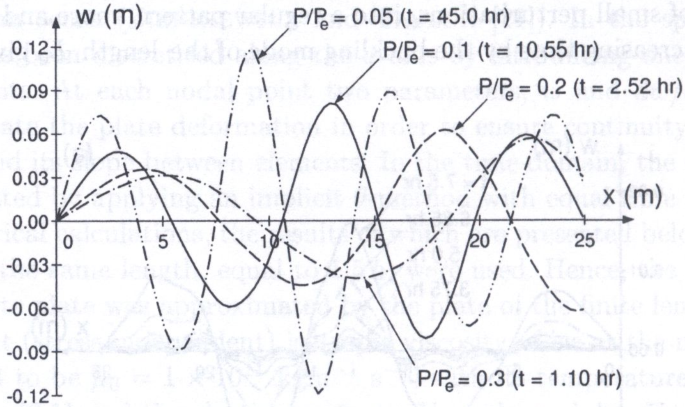

FIG. 3. Deflection of a uniform-width plate at the critical time $t = t_c$ (expressed in hours) as a function of the normalised load P/P_e, for the ice thickness $h = 0.2$ m.

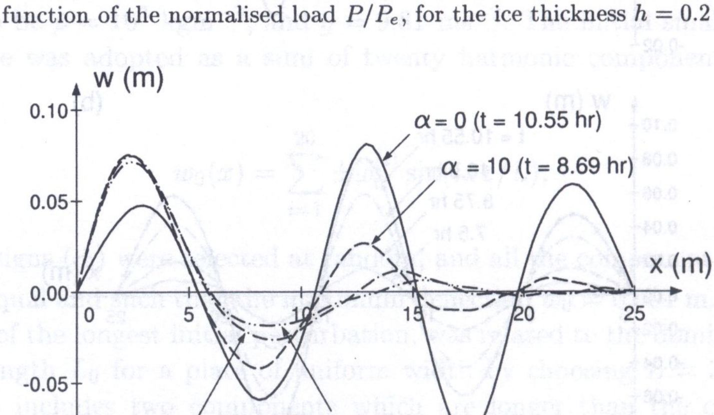

FIG. 4. Deflection of the plate at the critical time $t = t_c$ as a function of the wedge angle α (given in deg), for the normalised load $P/P_e = 0.1$, the ice thickness $h = 0.2$ m and the structure width $b_0 = 10$ m.

the load level, the critical times at which the ice cover fails change with the normalised load very substantially, decreasing by a factor of 41 for the same, sixfold increase in loading.

Figure 4 shows the deflection of a wedge-shaped plate at the onset of its failure for various angles α (given in deg) and the fixed normalised load $P/P_e = 0.1$. The results are obtained for the plate $h = 0.2$ m thick and $b_0 = 10$ m wide at the edge $x = 0$. We note that the deformation of the plate due to creep buckling for the angles $\alpha > 0$, even as small as 10°, attenuates rapidly with x, so practically only a few buckles in the neighbourhood of the structure can be discerned.

The critical time t_c, required to fail a floating ice sheet due to its creep deformation started from initial, small-amplitude imperfections, is plotted in Figs. 5 and 6 as a function of the angle α defining the in-plane geometry of the truncated wedge. Figure 5 illustrates, for the structure width $b_0 = 10$ m and the ice cover thickness $h = 0.2$ m kept constant, the dependence of the critical time on the normalised load P/P_e (the corresponding plate deflections for selected ratios P/P_e and $\alpha = 0$ are shown in Fig. 3). Figure 6 displays, at the constant load $P/P_e = 0.1$, the variation of t_c for different plate thicknesses h. We note that for thinner ice plates the values of the critical time initially slightly increase with the increasing angle α, while for thicker ice the values of t_{cr} decrease monotonically with α.

The influence of the temperature profile across the ice on the creep buckling of the plate is illustrated in Fig. 7. The figure shows, for $P/P_e = 0.1$, $h = 0.2$ m

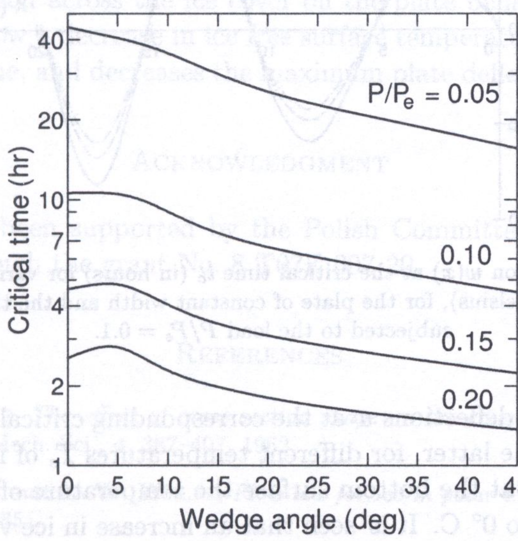

FIG. 5. Variation of the critical time t_c (given in hours) with the angle α and the normalised load P/P_e for the ice thickness $h = 0.2$ m and the structure width $b_0 = 10$ m.

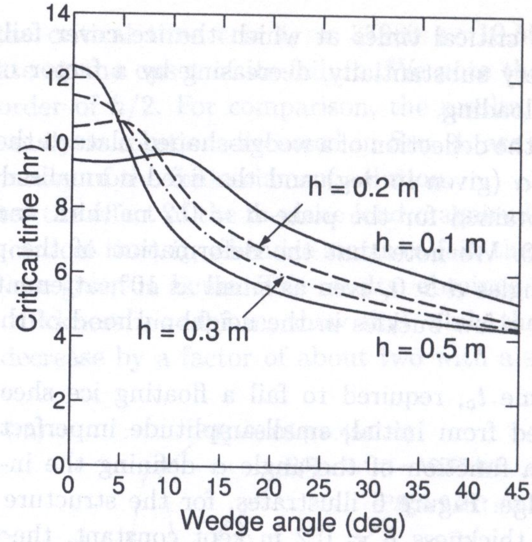

FIG. 6. Variation of the critical time t_c (given in hours) with the angle α and the ice thickness h for the normalised load $P/P_e = 0.1$ and the structure width $b_0 = 10$ m.

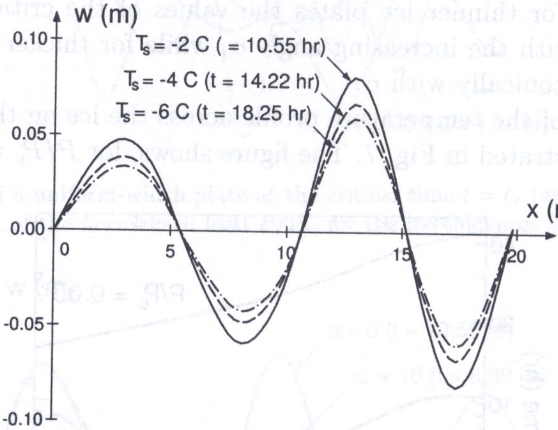

FIG. 7. Plate deflection $w(x)$ at the critical time t_c (in hours) for various ice-free surface temperatures T_s (in Celsius), for the plate of constant width and the thickness $h = 0.2$ m, subjected to the load $P/P_e = 0.1$.

and $\alpha = 0$, the plate deflections w at the corresponding critical times t_c, together with the values of the latter, for different temperatures T_s of ice at the top, free surface of the plate; at the bottom surface the temperature of ice is regarded to be constant, equal to 0° C. It is seen that an increase in ice viscosity caused by the temperature decrease in the upper part of the plate cross-section, results in an increase in the time needed to fail the plate. The plate displacements at the onset of failure, however, decrease with decreasing the ice free surface temperature: the cooling of ice from $T_s = -2°$ C to $T_s = -6°$ C leads to the reduction of the plate deflections $w(t_c)$ by nearly 30%.

6. Conclusions

The problem of creep buckling of a wedge-shaped semi-infinite floating ice plate subjected to in-plane compression has been treated numerically by using the finite-element method. An analytic solution, possible only in the simple case of a parallel-sided plate, shows that the lengths of the dominant (i.e. the fastest growing) and the critical (i.e. stationary) creep buckles are at a fixed ratio, regardless of the load magnitude and the plate thickness. The results of the numerical simulations illustrate the evolution of the plate deflection throughout the process of creep buckling of ice, from randomly distributed small-amplitude initial perturbations up to the critical time at which the ice cover fails. In particular, it is shown how the fastest-growing buckling mode develops and gradually dominates the deformation of the plate. The influence of the compressive load magnitude and the plate shape and thickness on the critical time, and the plate deflection then, has been investigated. The critical times for the cases considered have turned out to vary over a wide range of several orders of magnitude, depending on the load level and the plate geometry, whereas the plate deflections at the start of ice failure change rather moderately, with typical maximum vertical displacements equal to about 0.5 of the plate thickness. In addition, the effect of the temperature variation across the ice cover on the plate behaviour has been examined, showing how a decrease in ice free surface temperature increases the respective critical time, and decreases the maximum plate deflections at that time.

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