ENGINEERING TRANSACTIONS • *Engng. Trans.* • **52**, 3, 175–193, 2004
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.462.2004

ALGORITHMS OF THE METHOD OF STATICALLY ADMISSIBLE DISCONTINUOUS STRESS FIELDS (SADSF)

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Summary of the whole paper: By now, the SADSF method is practically the only tool of shape design of complex machine elements that provides an effective solution even to the problems of 3D distribution of the material, and at the same time it is still enough user friendly to be useful for engineers. This unique property of the method is due to the existence of its simple, application version. When using it, a design engineer does not need to solve by oneself any statically admissible field – which could be very difficult – but obtains such a solution by assembling various ready-made particular solutions. The latter are in general obtained by means of individual and complex analyses and provided to a designer in a form of libraries.

The algorithms presented in this paper break up with the individual approach to a particular field. The algorithms are the first ones of general character, as they apply to the fundamental problems of the method. The algorithms aid solving practically any boundary problem that one encounters in the tasks of construction of 2D statically admissible, discontinuous stress fields, first of all the limit fields. In the presented approach, one deals first with the fields arising around isolated nodes of stress discontinuity lines (Parts II and III), then integrates these fields into 2D complex fields (Part IV).

The software, created on the basis of the algorithms, among other things, allows one to quickly find all the existing solutions of the discontinuity line systems and present them in a graphical form. It gives the possibility of analysing, updating and correcting these systems. In this way, it overcomes the greatest difficulty of the SADSF method following from the fact that the systems of discontinuity lines are not known a priori, and appropriate relationships are not known either, so that they could only be found in an arduous way by postulating the line systems, and verifying them.

Application version of the SADSF method is not described in this paper; however, a reference is given to inform the reader where it can be found.

Key words: shape design, limit analysis, numerical methods.

Notations

α, β, γ – indexes of homogeneous regions stress state on physical plane,

$\{a\}$ – global plane system,

$\overset{(w)}{a_j}$ – nodal points co-ordinates of stress discontinuity lines $\mathcal{L}$ ($j = 1, 2$; w – index of node),

$\{\xi\}^{\alpha}$ – local plane system associated with principal stress directions in homogeneous region α,

S – contour of the discontinuous stress field,

S_p – loaded part of contour S,

S_u – supported part of contour S,

σ_{pl} – yield point,

$\mathcal{L}$ – stress discontinuity line,

$\mathcal{L}^{\alpha\beta}$ – line which separates the adjacent homogeneous regions α and β,

σ – stress tensor,

$\overset{(\alpha)}{\sigma}_{ij}$ – stress tensor components in homogeneous region α $(i, j = 1, 2)$,

$\overset{(\alpha)}{\sigma}_{i}$ – principal stresses in homogeneous region α $(i = 1, 2)$,

$\mathbf{p}$ – stress vector,

$\mathbf{p}^{\alpha,\beta}$ – load applied on line $\mathcal{L}^{\alpha,\beta}$,

ω – stress parameter ($\omega \in [0, 2\pi]$ or $\omega \in [0, \pi]$),

$\overset{(\alpha)}{\omega}$ – stress parameter ω in homogeneous region α,

$\{\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega}\}$ – parametric spaces,

$\overset{(\alpha)}{m}$ – stress multiplier in homogeneous region α $(m \in [0, 1])$,

$\overset{(\alpha)}{\phi}$ – angle of principal stresses in homogeneous region α, defined in system $\{a\}$,

$\mathbf{e}^{\alpha,\beta}$ – unit vector normal to line $\mathcal{L}^{\alpha,\beta}$ directed outside of region α $(\mathbf{e}^{\alpha,\beta} = -\mathbf{e}^{\beta,\alpha})$,

$\nu^{\alpha,\beta}$ – angle determining of direction of $\mathbf{e}^{\alpha,\beta}$, defined in system $\{a\}$,

$\mathbf{t}^{\alpha,\beta}$ – versor of stress discontinuity line $\mathcal{L}^{\alpha,\beta}$,

$Q^{\alpha,\beta} = 1, 2;\ q^{\alpha,\beta} = 1, 2, 3, 4$ – families and subfamilies of stress discontinuity line $\mathcal{L}^{\alpha,\beta}$, $q^{\alpha,\beta} = 1, 3$ are assigned to $Q^{\alpha,\beta} = 1$, and $q^{\alpha,\beta} = 2, 4$ are assigned to $Q^{\alpha,\beta} = 2$,

$\Delta\gamma = \Delta\gamma(\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega}, q^{\alpha,\beta})$ – function expressing the angular parameter determining versor $\mathbf{e}^{\alpha,\beta}$ normal to line $\mathcal{L}^{\alpha,\beta}$, defined in local system $\{\xi\}^{(\alpha)}$ $(\Delta\gamma \in [0, 2\pi])$; β – region adjacent to α;

$\Delta\hat{\gamma}(\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega})$ – term of function $\Delta\gamma(\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega}, q^{\alpha,\beta})$ $\left(\Delta\hat{\gamma} \in \left[0, \dfrac{\pi}{2}\right]\right)$;

$\Delta\phi = \Delta\phi(\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega}, Q^{\alpha,\beta}) \equiv \overset{(\beta)}{\phi} - \overset{(\alpha)}{\phi}$ – function expressing differences between the angles of principal stresses in adjacent homogeneous regions α and β, defined in local system $\{\xi\}^{\alpha}$ $\left(\Delta\phi \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right)$,

$\Delta\hat{\phi}(\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega})$ – term of function $\Delta\phi(\overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega}, q^{\alpha,\beta})$, $\left(\Delta\hat{\phi} \in \left[0, \dfrac{\pi}{2}\right]\right)$,

$\Delta\overset{(\alpha)}{\nu} \equiv \nu^{\alpha,\alpha+1} - \nu^{\alpha-1,\alpha}$ – angle between lines $\mathcal{L}^{\alpha-1,\alpha}$ and $\mathcal{L}^{\alpha,\alpha+1}$,

Λ – region of existence line $\mathcal{L}$,

$\Lambda_{1,N}^{\alpha,\alpha+1}$ – admissible subregion of variability $\overset{(\alpha)}{\omega}$, $\overset{(\alpha+1)}{\omega}$ obtained for the settled values: $\overset{(1)}{\omega}, \overset{(N)}{\omega}, N$,

Γ – interval of existence line $\mathcal{L}$,
$\Gamma^{(\alpha)}_{1,N}$ – admissible interval of variability $\overset{(\alpha)}{\omega}$ obtained for the settled values: $\overset{(1)}{\omega}$, $\overset{(N)}{\omega}$, N,
$\mathbf{P}^{\alpha,\beta}$ – point-image of line $\mathcal{L}^{\alpha,\beta}$ in A;
D ≡ ITN, IST, ISN, INN, INL – structural objects of stress discontinuity line networks which respectively pertain to the incidences of the following types: triangle – numbers of its nodes, segment of line $\mathcal{L}$ – the triangles separated it segment of line $\mathcal{L}$, segment of line $\mathcal{L}$ – its nodes, node – adjacent nodes, and node – local numbers of lines that originate from the node,
$\{\boldsymbol{\sigma}, \mathbf{a}, \mathbf{D}\}$ – set of internal parameters of plane complex field, defined in the system $\{a\}$.
$\{\boldsymbol{\omega}\}_w, \{\boldsymbol{\phi}\}_w, \{\boldsymbol{\upsilon}\}_w$ – matrices of parameters ω, ϕ, ν determined in field around node w,
χ – angle between principal directions of stresses in regions α and β for $\beta - \alpha > 1$,
$\mathbf{n}$ – unit vector normal to outer line $\mathcal{L}$,
δ_1, δ_2 – angles determined the half-plane in which are contained stress discontinuity lines of field around node ($\delta_2 > \delta_1$),
δ – angle between outer stress discontinuity lines of field around node ($\delta \leq \delta_2 - \delta_1$),
T – number of triangular homogeneous regions in complex plane field,
W – number of nodal points in complex plane field,
N – number of homogeneous regions in field around node,
L – number of internal lines of the field,
$\mathbb{A}, \mathbb{B}, \mathbb{C}$ – modules of algorithms.

PART I

IDEA OF THE SADSF METHOD, BASIC CONDITIONS AND APPLICATION VERSION

Summary of Part I: As a form of introduction, the author presents in a broad outline the general concept of the SADSF method, its fundamentals, basic conditions, typical formulation of boundary problems and the essence of difficulties in problem solving. These are not only nonlinearities, singularities, and conditions expressed by complex functions (i.e. given in a form of algorithms), but first of all the structure of the condition system – that is not *a priori* known – which cause difficulties. As it turns out, the structure depends on boundary conditions, but the relations that express this dependence are not known. In the physical space, it refers to the problems with unknown discretization that is dependent on boundary conditions.

Also, as a form of introduction, it will be shown that the investigation of mentioned dependence is possible, anyway. However, it is possible not for two-dimensional fields as a

whole, but at least for their component fields around nodes. Thereby fundamental ideas of algorithms are presented in broad outline. Such algorithms are discussed in detail in Parts II and III.

The application version of this method, attractive for practical purposes, is briefly described in this part. Numerical examples, also included here, illustrate the position and the possibilities of the method in solving practical problems of shape design of complex thin-wall structures.

1. Introduction

1.1. Introductory remarks

The method of statically admissible discontinuous stress fields (SADSF) has its grounds in the theory of limit analysis, and it uses a model of the rigid – perfectly plastic material. The method is based on the conclusions following from the lower-bound theorem [1]. According to this theorem, the limit load of a structure, whose contours are determined based on statically admissible stress field, is equal or greater than the limit load that has been assumed in this field. The problems of optimum shape design can then be, for example, formulated as the task of seeking such an admissible stress field, which determines the minimum volume [5].

In the version of SADSF method presented in this work, one assumes that, among other things, the fields are plane, or fragments of the fields are plane (see Figs. 1 and 3), and the lines of discontinuity $\mathcal{L}$ are segments of straight lines. This implies homogeneity of the stress state in each mesh of the net created in this way, and it means that the solutions are to some extent approximate.

The set of values describing such a field is usually presented in a manner that could be mapped in a graphic form [5, 8]:

$$(1.1)\qquad \{\overset{(\alpha)}{\sigma}_{mn},\ \overset{(\omega)}{\mathbf{a}}_j,\ \mathbf{D};\ \alpha = 1..T;\ m,n = 1,2;\ j = 1,2;\ w = 1..W\},$$

where:

- $\overset{(w)}{a}_j$ – nodal points co-ordinates of stress discontinuity lines $\mathcal{L}$,
- $\overset{(\alpha)}{\sigma}_{mn}$ – stress tensor components in homogeneous region α,
- $\mathbf{D}$ – structural object describing the topology of line system $\mathcal{L}$, including the number of lines and the link system,
- T – number of triangular homogeneous regions,
- W – number of nodal points.

The systems of lines $\mathcal{L}$ usually do not exhibit any regularity; even the individual meshes could be polygons of different numbers of sides. For that reason in application, in order to obtain triangular meshes and ensure topological uniformity of the network, one introduces additional, artificial division lines (in this work these are drawn as dashed lines). The object **D** is then given a form of a matrix whose rows, representing the triangles of consecutive numbers, specify the indices of nodes at triangle vertices.

The general idea of a problem formulated by means of the SADSF method is presented, in an illustrative form, in Fig. 1. The data are (Fig. 1a): limit load **p** applied on the part S_p of the contour S, geometry of the part S_p and S_u (for example S_u – supported part), and the material of the designed element of the structure.

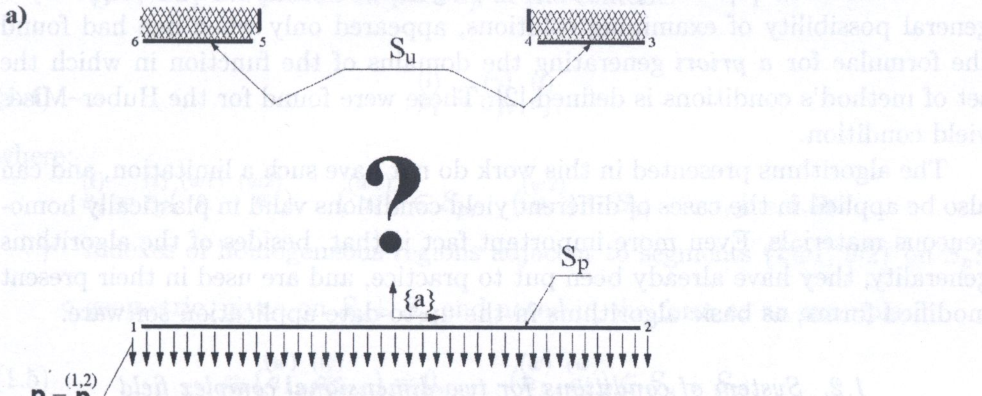

FIG. 1. Field of the 'f 90' type solved for the Huber–Missed yield condition with the data: $\{\overset{(1,2)}{\mathbf{p}} = [0.0000, -0.8369] \cdot k,\ \overset{(3,4)}{\mathbf{p}} = [0.0000, \sqrt{3}] \cdot k,\ \overset{(5,6)}{\mathbf{p}} = [0.0000, \sqrt{3}] \cdot k,\ \overset{(1)}{\mathbf{a}} = (-80, 0),\ \overset{(2)}{\mathbf{a}} = (80, 90),\ \overset{(3)}{\mathbf{a}} = (80, 90),\ \overset{(6)}{\mathbf{a}} = (-80, 90)\ [\mathrm{mm}]\}$, $k = \sigma_{\mathrm{pl}}\sqrt{3}$. a) boundary conditions and graphical formulation of the problem, b) plane, statically admissible discontinuous stress field (nodes and homogeneous region numbers, object **D** and principal stresses marked with arrows – illustrate the set of parameters given in formula (1.1)).

The problem consists in finding a statically admissible stress field (described by set (1.1), cf. Fig. 1b), which fulfils the above boundary conditions, and satisfies the uniform equivalent stress condition in limit state over the entire volume[1)]

The aim of this problem is then finding a certain limit stress field that determines economical contours (although the contours might not necessarily be the most economical ones [5]). The problem in which the criterion is the condition of equivalent stress might not have a unique solution.

The fundamentals of the SADSF method are given in the work by W. SZCZEPIŃSKI [1]. The possibility of algorithmization, and consequently a general possibility of examining solutions, appeared only when one had found the formulae for *a priori* generating the domains of the function in which the set of method's conditions is defined [2]. These were found for the Huber–Mises yield condition.

The algorithms presented in this work do not have such a limitation, and can also be applied in the cases of different yield conditions valid in plastically homogeneous materials. Even more important fact is that, besides of the algorithms generality, they have already been put to practice, and are used in their present modified forms, as basic algorithms in the up-to-date application software.

1.2. System of conditions for two-dimensional complex field

For solving the problem presented in the previous section, we have at our disposal a set of equations and inequalities that includes [1, 2, 5]:

- equilibrium equations for each line $\mathcal{L}^{\alpha,\beta}$ that separates the adjacent homogeneous regions α and β (one assumes that: $\overset{(\alpha)}{\sigma}_{12} = \overset{(\alpha)}{\sigma}_{21}$, $\overset{(\beta)}{\sigma}_{12} = \overset{(\beta)}{\sigma}_{21}$):

$$(1.2)\qquad (\overset{(\alpha)}{\sigma}_{ij} - \overset{(\beta)}{\sigma}_{ij})\,\overset{(l)}{e}_i = 0, \qquad (i,j = 1,2; \quad \alpha,\beta = 1..T; \quad l = 1..L; \quad \alpha \neq \beta),$$

where: L – number of internal lines of the field;

$$\overset{(l)}{\mathbf{e}} = \overset{(l)}{\mathbf{e}}(\overset{(w1)}{\mathbf{a}}, \overset{(w2)}{\mathbf{a}}) \text{ – unit vector normal to } \mathcal{L}^{\alpha,\beta}\,{}^{2)};$$

internal equilibrium conditions in regions α and β are identically satisfied, due to homogeneity of the field;

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1) References [1, 9] give a solution to this problem found with the methods that have existed till now. An example of approach utilising the algorithms described in this paper is presented in Part IV, and illustrated in Fig. 19. It is worth noticing that the structure of stress discontinuity line system is not mentioned among the data of the problem presented in Fig. 1a (see Fig. 1b for more details).

2) The association of indexes $\{l, \alpha, \beta\}$ and $\{l, w1, w2\}$ determines the object $\mathbf{D}$, which is uniquely mapped onto the objects of incidence that assign, for example, numbers of lines l to their adjacent triangles, numbers of segments of line l to their ending nodes, etc.

• yield condition that should be satisfied in each homogeneous region α:

$$\Phi(\overset{(\alpha)}{\sigma}_{kl},\ \sigma_{\mathrm{pl}}) = 0, \qquad (k,l = 1,2; \qquad \alpha = 1..T); \tag{1.3}$$

in the case of unlimited field this condition takes the form of weak inequality ($\leq$); the structures made of isotropic materials are analysed in the work;

• boundary conditions:

∘ loading, given on segments l of line $\mathcal{L}$, whose starting and ending nodes $\{w1,\ w2\}$ are placed on part S_p of the contour:

$$\overset{(l)}{p}_i = \overset{(\gamma)}{\sigma}_{ji}\ \overset{(l)}{e}_j, \tag{1.4}$$

where:

$$\overset{(l)}{e}_j = \overset{(l)}{e}_j(\overset{(w1)}{\mathbf{a}}, \overset{(w2)}{\mathbf{a}}), \quad (\overset{(w1)}{a_i}) \in S_p, \quad (\overset{(w2)}{a_i}) \in S_p; \qquad i,j = 1,2;$$

γ– indexes of homogeneous regions adjacent to segments $\{l, w1,\ w2\}$ on S_p;

∘ geometric, given on $S_u + S_p$ and noted in the form as an example:

$$\eta_k(\overset{(w)}{a_1}, \overset{(w)}{a_2}\ ...) = 0, \qquad (\overset{(w)}{a_1}, \overset{(w)}{a_2}) \in S_u + S_p; \tag{1.5}$$

the conditions are determined at coordinate sets of the nodal points w placed at $S_u + S_p$.

The above conditions, written perhaps in a different form, are commonly quoted in literature. However, if one takes into account that equilibrium Eqs. (1.2) – set on each line $\mathcal{L}^{\alpha,\beta}$– create a homogeneous system, one must also add the conditions of existence of its solution, and these take the form[3] [2]:

$$0 \leq \frac{\overset{(\alpha)}{\sigma}_{AA} - \overset{(\beta)}{\sigma}_{AA}}{\overset{(\alpha)}{\sigma}_{ii}\ \overset{(\beta)}{\sigma}_{ii}} \leq 1, \qquad \det|\overset{(\alpha)}{\sigma}_{ij} - \overset{(\beta)}{\sigma}_{ij}| = 0, \qquad \overset{(\alpha)}{\sigma} \neq \overset{(\beta)}{\sigma}, \quad \alpha \neq \beta, \tag{1.6}$$

$i, j, A = 1,2$; A – is not summed.

Moreover, each homogeneous region α of the obtained field must take such a place that is realizable on the physical 2D space, precisely – its area must not be negative. It leads to obvious inequalities:

$$- \begin{vmatrix} \overset{(w1)}{a_1} & \overset{(w1)}{a_2} & 1 \\ \overset{(w2)}{a_1} & \overset{(w2)}{a_2} & 1 \\ \overset{(w3)}{a_1} & \overset{(w3)}{a_2} & 1 \end{vmatrix}^{(\alpha)} > 0, \qquad (\alpha = 1..T), \tag{1.7}$$

[3] Here we have two double-sided inequalities, for $A = 1$ and $A = 2$.

where: $w1, w2, w3$ – are the indices of vertices of triangle α, numbered here clockwise.

These are the geometric conditions for the existence of solution.

1.3. *The problem of existence*

When reviewing the above–specified set of conditions, one should have already noticed that the unknowns in the problem could be, in general, all the elements of set (1.1), including the components of the structural (topological) object **D**.

However:

- among the conditions (1.2)÷(1.7) we could hardly find any one that would be defined on these components; although we have the conditions (1.6), (1.7), they only allow us to determine whether the previously assumed topology is preserved, or not;
- for arbitrarily assumed structures (topologies), the solutions of the systems (1.2)÷(1.7) might generally not exist, and often they do not exist; they exist only for some particular structures (link systems), depending on boundary conditions; however, the relation between the structure and the boundary conditions have not been found yet.

Then, at the starting point, we do not know the number of unknowns and even the dimension of the system of conditions (1.2)÷(1.7) which must be created for the solution of the boundary problem. In the physical space, this can refer to a problem with unknown discretization. Here it means that the structure of division of the field into homogeneous regions is unknown, and additionally, the structure depends on the boundary conditions.

This is the most essential difficulty in solving each discontinuous and limit stress field – the effect of which is that the structure must be *a priori* assumed, and then verified. However, the verification is only possible at the final stage of the process of solving the whole system. It can be best illustrated by the example of condition (1.7), whose verification requires the co-ordinates of all nodal points to be known.

One can possibly prove that the solution does not exist within the assumed structure, but this finding brings practically no hints on how to correct the structure.

The structure of the system of stress discontinuity lines $\mathcal{L}$, for which a solution of the field exists in the physical 2D space for the given boundary conditions, is called an admissible structure for these boundary conditions.

2. The essence of the problem and the solution method [4)]

2.1. *Fundamental assumptions*

Because it has been found that there are no conditions that would be defined on the variables describing the field structure, the problem of searching for the relation between the structure and the boundary condition seems to be difficult or even impossible. Such a problem commonly appears in practical application when one has to solve discontinuous stress fields. Nevertheless, as it will be shown in this work, one can tackle the problem and examine it. However, this will not be possible when the problem is formulated for a field assumed as a set of homogeneous states. One needs to assume that the fundamental component units of it are the fields around the nodes. If there exist the solutions of fields around all the nodes of a complex field, then the global solution for the field also exists.

The idea of one of typical problems that can be formulated for a field around a node is illustrated in Fig. 2a. The data are the components of limit stress states $\overset{(1)}{\sigma}_{ij}, \overset{(N)}{\sigma}_{ij}$ in the outer regions. One must find the states of stress $\overset{(\alpha)}{\sigma}_{ij}$ in the regions 2...N-1, the parameters $\nu^{\alpha,\alpha+1}$ determining the direction of lines $\mathcal{L}^{\alpha,\alpha+1}$, and the number N of homogenous regions for which the solution of this field does exist.

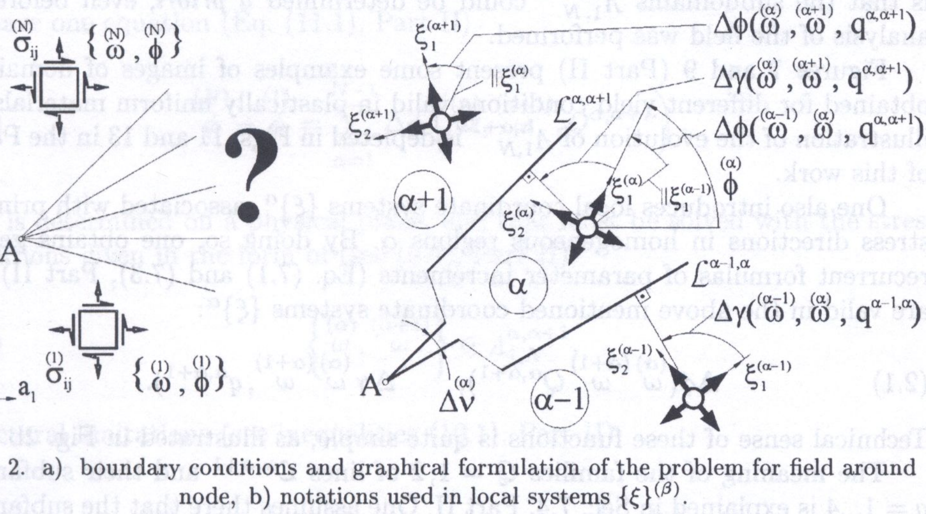

Fig. 2. a) boundary conditions and graphical formulation of the problem for field around node, b) notations used in local systems $\{\xi\}^{(\beta)}$

There could also be other cases of data and unknowns, for which a different formulation would be proper (see Part III), but those will not be presented in this preliminary study.

[4)] More details will be presented in the following parts of the work.

2.2. Parametric description and its implications

In order to practically solve the presented boundary problem for a field around a node, one has assumed that the states of stress in its homogeneous regions are defined by the parameters: $\{\overset{(\alpha)}{\omega}, \overset{(\alpha)}{\phi}\}$, where $\overset{(\alpha)}{\phi}$ is the angle of principal stresses, and $\overset{(\alpha)}{\omega}$ is the stress parameter. States of stress given by parametrisation satisfy yield condition for each value of $\overset{(\alpha)}{\omega}$ [1].

The result of it is a reduction of condition system, but even more important effect of this assumption is that, in parametric spaces $\{\overset{(\alpha)}{\omega}\ \overset{(\alpha+1)}{\omega}\}$, one can create, first of all, an illustrative image of the domain where an individual line $\mathcal{L}^{\alpha,\alpha+1}$ exists – the image, denoted here with the symbol Λ that represents the conditions (1.6). Then, one derives convenient formulas of evolution of the subdomains $\Lambda_{1,N}^{\alpha,\alpha+1}$ which pertain to the case when the conditions (1.6) are set up simultaneously for all the field discontinuity lines around the node, and the values of elements of the set $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, N\}$ are arbitrarily assumed. If, with the data $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, N\}$, the condition $\{\overset{(\alpha)}{\omega}\ \overset{(\alpha+1)}{\omega}\} \in \Lambda_{1,N}^{\alpha,\alpha+1}$ holds for each line $\mathcal{L}^{\alpha,\alpha+1}$ of the field, the field's discontinuity lines exist.

The fact important for a technically useful solution of the formulated problem is that the subdomains $\Lambda_{1,N}^{\alpha,\alpha+1}$ could be determined *a priori*, even before any analysis of the field was performed.

Figures 7 and 9 (Part II) present some examples of images of domains Λ obtained for different yield conditions valid in plastically uniform materials. An illustration of the evolution of $\Lambda_{1,N}^{\alpha,\alpha+1}$ is depicted in Figs. 11 and 13 in the Part II of this work.

One also introduces local coordinate systems $\{\xi\}^{\alpha}$, associated with principal stress directions in homogeneous regions α. By doing so, one obtains general recurrent formulas of parameter increments (Eq. (7.1) and (7.3), Part II) that are valid in the above mentioned coordinate systems $\{\xi\}^{\alpha}$:

$$(2.1) \qquad \Delta\phi(\overset{(\alpha)}{\omega}\ \overset{(\alpha+1)}{\omega}\ Q^{\alpha,\alpha+1}), \qquad \Delta\gamma\, \overset{(\alpha)}{\omega}\ \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}).$$

Technical sense of these functions is quite simple, as illustrated in Fig. 2b.

The meaning of the families $Q = 1, 2$ of lines $\mathcal{L}^{\alpha,\alpha+1}$ and their subfamilies $q = 1...4$ is explained in Sec. 7.4, Part II. One assumes there that the subfamilies $q = 1, 3$ are related to the family $Q = 1$, while these of $q = 2, 4$ are related to $Q = 2$.

Employing obvious geometric relations, one can easily transform the function values $\Delta\phi$, $\Delta\gamma$ into the global system $\{a\}$, and at the same time find another function, important for the field description (Fig. 2b, Eqs. (10.2), Part II).

$$(2.2) \qquad \Delta\overset{(\alpha)}{\nu} \equiv \nu^{\alpha,\alpha+1} - \nu^{\alpha-1,\alpha} \equiv \Delta\nu(\overset{(\alpha-1)}{\omega}, \overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha-1,\alpha}, q^{\alpha,\alpha+1}).$$

The function values determine the angles between consecutive discontinuity lines $\mathcal{L}^{\alpha,\alpha+1}$.

The formula of $\Delta\phi$ is derived from the equality condition of existence $(1.6)_2$, while the formula of $\Delta\gamma$ is based on the equilibrium conditions (1.2). The application of the functions $\Delta\gamma$, $\Delta\phi$ implies then that the conditions $(1.6)_2$ and (1.2) are identically fulfilled.

2.3. *System of conditions*

Finally, for the problem illustrated in Fig. 2, the data are:

$$\left\{ \overset{(1)}{\omega}, \overset{(1)}{\phi}, \overset{(N)}{\omega}, \overset{(N)}{\phi} \right\}, \tag{2.3}$$

but the number of regions N is not known in advance. In order to find unknown field parameters

$$\left\{ N, \overset{(2)}{\omega}, ..., \overset{(N-1)}{\omega}, q^{1,2}, q^{2,3}, ..., q^{N-1,N} \right\}, \tag{2.4}$$

we have one equation (Eq. (11.1), Part II)

$$\overset{(N)}{\phi} - \overset{(1)}{\phi} = \sum_{\alpha=1}^{N-1} \Delta\phi \left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, Q^{\alpha,\alpha+1} \right), \tag{2.5}$$

that is determined on a physical plane, and that must be solved with the stress conditions given in the form of (see (9.4), Part II):

$$\left\{ \overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega} \right\} \in \Lambda_N^{\alpha,\alpha+1}, \tag{2.6}$$

structural limitations (see inequalities (10.1), Part II):

$$\Delta\overset{(\alpha)}{\nu} > 0 \qquad (\alpha = 1, 2, ..N), \qquad \delta = \sum_{\alpha=1}^{N} \Delta\overset{(\alpha)}{\nu} \leq 2\pi, \tag{2.7}$$

and geometric conditions, needed in some cases, that specify, for example, the requirement that the whole field be contained within a previously given half-plane.

The conditions (2.7) refer to the conditions type (1.7).

2.4. Method of solution

It is evident that, as the number N is not *a priori* known, neither the number of unknowns, nor the number of components of the sum of Eq. (2.5) is known beforehand. The same refers to the numbers of conditions (2.6) and (2.7). On top of that, the system is nonlinear and contains several singularities. The functions that describe it are very complicated (see also the Eqs. (7.1) and (7.3), Part II) and, because of their complicity, are usually given in the form of algorithms.

It limits the possibilities of finding the system's solutions, and practically only numerical solutions remain. For this reason, the control over the content of variables in the domain becomes so important for the problem investigation possibility. As one could easily notice, the domains of functions $\Delta\phi$, $\Delta\gamma$, $\Delta\nu$, in which the conditions of type (2.5), (2.6), (2.7) are set, can be determined basing on the formulas of evolution of subdomain $\Lambda_{1,N}^{\alpha,\alpha+1}$. It is also worth underlining once more that these subdomains are *a priori* determined, when one only knows $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, N\}$, and in the case when, with an assumed yield condition, the domain Λ is known.

There is an enormous variety of possible cases of solution of the systems type (2.5), (2.6), (2.7), sometimes surprising and difficult for interpretation, and because of that the attempt of classifying or ordering them is ineffective. For the time being, we can only notice that the number of unknown parameters ω is identical with the number of equations only if the sought field around the node contains $N = 3$ homogeneous regions.

On the other hand, it is also known that the solutions not always exist for all the data values $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \overset{(1)}{\phi}, \overset{(N)}{\phi}\}$ and $N = 3$. In such cases, one usually assumes the number of regions $N = 4$ or greater, and introduces additional conditions to find the solution. However, there could be such sets of data values for which the increase of N does not lead to finding the solution, but N should be decreased instead.

These cases are described in Sec. 14.2 (Part III).

One must take into account the previously mentioned variety of possible sets of data and unknowns, limited possibilities of the analysis due to the lack of balance between the number of unknowns and the number of equations, possible existence of solutions in isolated points of the data space only, etc. Therefore, the categories of condition systems type (2.5), (2.6), (2.7), appearing in practical applications, have been juxtaposed in the form of a set of appropriately selected elementary problems. These are presented in Sec. 13 (Part III), while Sec. 14 describes particular properties of the fields.

Obviously, in a complex field consisting of many nodes, the already revealed effects will multiplicate. This might explain the reason why the problems considered in this work have remained unrecognised for such a long time.

3. Tasks of the presented algorithms

The problem of existence of the system of stress discontinuity lines around a node, briefly presented in the previous section, has been reduced to the examination of the existence of solutions for a system of conditions type (2.5), (2.6), (2.7). The criteria of existence have there been defined on a physical plane. It is then the criterion, whose fulfilment guarantees that the solution exists, because one can map it onto a complete field around the node. Therefore, one can treat the solution algorithm of the systems type (2.5)–(2.7) as an algorithm of searching for all the solutions of the field around the node under given boundary conditions.

It is worth noticing that the algorithm still requires that the number of regions N be assumed. Nevertheless, the implementations built upon it allow us to conclude effectively and almost instantaneously whether, for an assumed number N, the solution of a given boundary problem exists or not. It also makes it possible to represent all the solutions in an illustrative graphical form, and analyse different variants of systems of lines $\mathcal{L}$ without employing any individual relationships. Application of these implementations has then many attributes of a direct approach.

The algorithms presented in this work are the first ones, with the exception of perhaps those of work [2], that tackle the above-described problem of existence, and solve it for the field around the nodes. For the time being, it is done only for an isolated node, which makes the algorithms more universal, and at the same time a bit more wearisome when one uses the algorithms in each task of constructing complex 2D fields. We assume then that the algorithms are primarily predestined for finding new solutions, for which the permissible structures of line systems $\mathcal{L}$ are not yet known.

If at least one solution of a complex field is known for certain types of boundary conditions, and only the data values of boundary parameters change, the approach based on the conditions (1.2)÷(1.7) may be more effective for the application. However, one must also assume parametric description of the field in order be able to exploit the formulas of subdomain evolution $\Lambda_{1,N}^{\alpha,\alpha+1}$ and retain control over the content of variables in the subdomains that would guarantee obtaining the solutions in a numerical way [3]. The algorithm constructed in this way becomes, however, a specific algorithm for calculation of parameters of a particular field. It can be useful, for example, when we create libraries of solutions used within the framework of the so-called application version that will be briefly described in Sec. 5.

The latest concepts, already confirmed by successful tests, go even further. These exploit the possibility of recording the consecutively executed procedures, without making use of any individually created algorithms, but using only those fundamental ones that have been described in this work. All individual features of a particular solution are coded in the set of indices of the above mentioned procedures, in the structural object $\mathbf{D}$, in the sets of indices of subfamilies q, in indices of roots ω and in a form of some additional entities.

For the needs of this work, we also assume that the process of assembling the fields goes on independently around each node. Owing to this fact, specific features of the created complex field appear only in the integration algorithms. The latter govern the component fields around nodes, and integrate these fields into complex ones according to the sketch of the line system structure $\mathcal{L}$ that can be drawn manually (using computer mouse) on the monitor screen. The sketch is automatically transformed into the set of coordinates of the object $\mathbf{D}$.

The above-mentioned algorithms, although very difficult in realisation, are relatively simple as a concept. However, they only have an auxiliary meaning for the principal result of this work, and for that reason they will be presented in a descriptive form only (Part IV).

Obviously, the set of conditions (2.5)–(2.7), may also be used in the investigation of infinitesimally small neighbourhood of nodes of convergence of stress discontinuity curves. It pertains to the case when the curves separate variable stress fields.

4. The concept of development

The sets of formulas, presented in this work, which give basis for creation of the described algorithms, are relatively complicated, especially those in Part II. It follows from the nature of the problems, and the difficulty is even aggravated due to the fact that they cannot be illustrated by physical phenomena. Discontinuous statically admissible fields are the fields fulfilling only the static conditions of the problem, and it proves to be very far from reality.

For this reason, to facilitate reading, the author will present graphical illustration of the transformations, instead of quoting appropriate formulas. The examples will in turn be selected from such fields, whose solutions are already known from the papers [1] or [9], although the methods applied there have not taken advantage of the use of a computer. Consequently, the problem of solving the fields around a single node will be illustrated by examples of fields created around a convex or concave corner. Similarly, the solution of a complex field is illustrated with the known field shown in Fig. 1, called in literature the field type 'f 90'. Using this example, we have also illustrated (in Fig. 19 Part IV) the method of integration of fields around nodes.

In the descriptions, one consequently applies the rule that only the information necessary for understanding fundamental algorithms is given. Derivations and more exhaustive comments are almost completely omitted. In the cases of functions given in the form of algorithms, one accepted the principle that only the headers and the specified formal parameters will be available for the reader.

One assumes that, from the moment of creating the algorithmic realisation, the specific content of sets of algorithm describing the formulas is not important any longer.

This work is then limited to describing the specificity of computer approaches and the fundamental problems of the method. That is why the author does not give references to the results obtained by the previously used methods, including numerous results of numerical investigations on already shaped elements of structures and experimental research results. Almost all the formulas, algorithms and their implementations presented in this work do not have any equivalents in literature, with a partial exception of paper [2].

5. Application version of the SADSF method

5.1. *The essential idea*

The problems solved by means of algorithms presented in this work have a fundamental character. Another class of algorithms, in a way complementary but still distinct, is that related to the programs oriented on direct practical applications. These are constructed in such a way that the user does not need to solve by oneself any systems of stress discontinuity lines, but can exploit libraries of ready-made solutions instead (see Fig. 3b). Using these libraries, one can select the component fields and then integrate them (Fig. 3c) into various admissible configurations in order to fulfil the given boundary conditions (Fig. 3a).

In other words, the libraries are collections of ready-made particular solutions of low or medium degree of complicity, treated as partly autonomous subsystems of complex fields.

The approach to the problems of the SADSF method described here is called *the application version*. Its natural simplicity is obtained by hiding all the difficulties of the method in the phase of constructing the library fields, and is reached at the expense of universality, as the set of solutions is limited to those fields that can be constructed from the ready-made elements. However, due to its simplicity, the approach brings about very attractive applications of the SADSF method, facilitating the practical tasks of complex machine elements design, and the importance of the method becomes comparable only with that of the Finite Element Method (FEM).

5.2. *Applications*

The basic field of application of the SADSF method includes the tasks of approximate shaping of elements of structures characterised by very complex geometry.

In more simple cases, the applications usually concern the tasks with unknown free boundary, one of examples of that is the solution shown in Fig. 1.

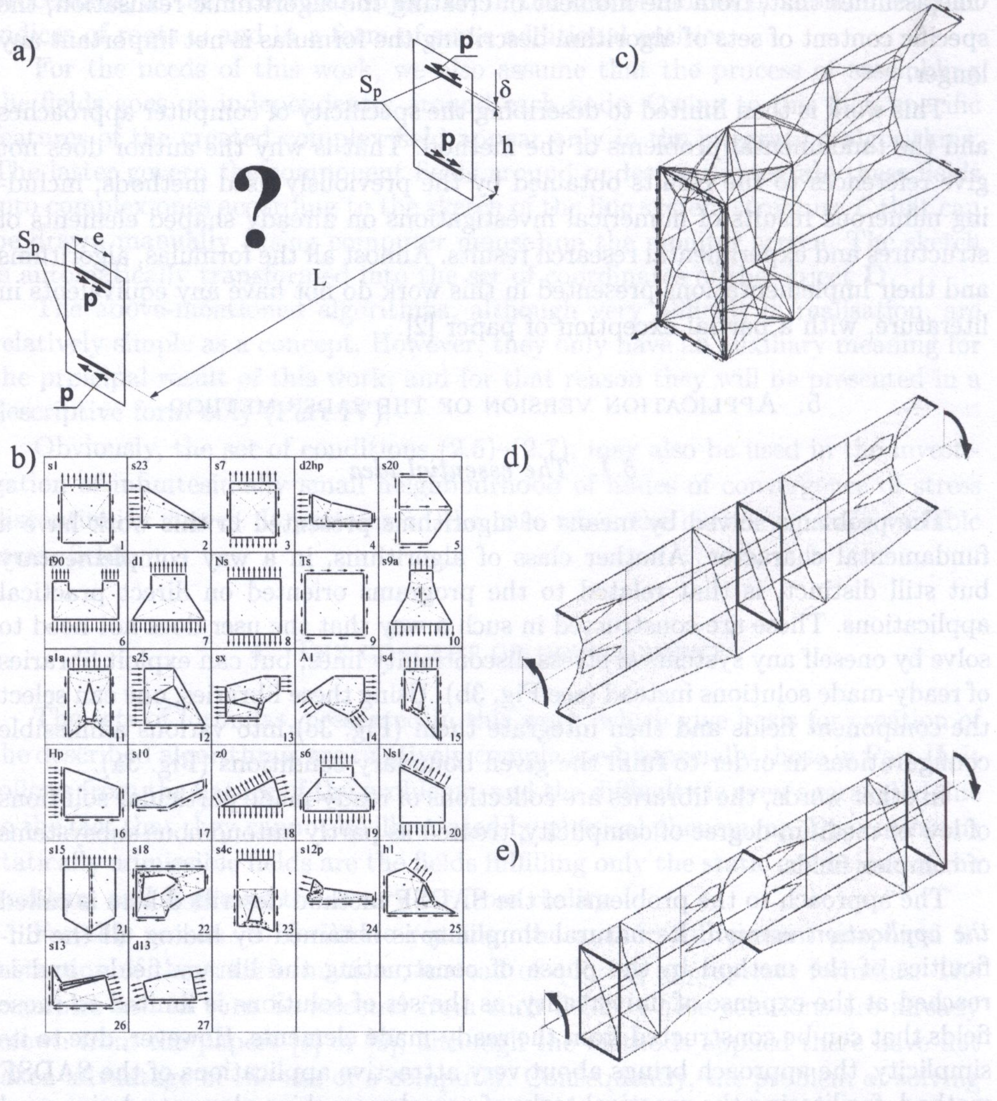

FIG. 3. Idea of the application version of the SADSF method – example of formulation and solution to the problem of design of a complex field in a torque shell of channel profile: a) boundary conditions and graphical formulation of the shape design problem, b) library of ready-made particular solutions, c) complex field determining shape of shell (the sketch shows countours of a half of symmetrical shell), d) variant 1 of field composition from Fig. 3c, e) variant 2 of field composition from Fig. 3c.

However, one could also perform tasks such as shown in Fig. 3, whose formulation and results are related to the problem of finding the best distribution of material [10, 11]. Here, only boundary conditions are given – as illustrated in Fig. 1a – and one determines not only dimensions and shape but also the structure (the layout) of shell's component elements inscribed into the area between the given boundaries. The essential detail of this solution is that, although one has searched for a shell based on channel profile contours and the resulting shell still remains "open", the obtained layout of component elements is very resistant to torsion.

The SADSF method may be a self-contained tool of approximate shaping, and can also be used as an auxiliary means for other methods, including the FEM. It can provide initial data on dimensions and shape that, with the exception of tasks undertaken in [10, 11], must be found anyway to make practical analyses of the FEM possible. In this way, one ensures that the effort of computation would result in a good quality of the structure, and the designer would not end up with ascertainment of its poor load capability. This could be, for example, the case of the shell of Fig. 3, which would exhibit poor resistance to torsion if additional elements were not found. Without them, the shell's limit load would not be several times or several dozen percent lower but dozen times lower.

We must strongly emphasize that the SADFS method and the FEM are different, use different techniques, are applicable to different problems, and then they should not be compared to each other. One can only compare the general formulation of problems characteristic of both methods, and the final results.

A designer might be concerned that the SADSF method gives only approximate solutions, obtained under the condition of equalized equivalent stress. This condition may lead to ambiguous solutions because of the assumption of piecewise-homogeneous fields, and because of lack of several other desirable properties. These imperfections, however, are of limited importance for the tasks of preliminary design. On the other hand, the conclusion drawn on this basis that the errors of approximation would be too great, turns out to be wrong. Numerous investigations, both numerical and experimental, of actual elements designed with the use of the SADSF method show surprisingly good properties of the elements, even quite a satisfactory equalization of equivalent stress in the elasticity range, although this range of loads is not the subject of analysis in the SADSF method.

The obtained shapes and dimensions can then be approved at once, or they can be subject to further improvements.

Despite good quality of elements shaped by this method, the main field of application of the SADSF method is the preliminary phase of design, specifically to the moment when only boundary conditions are given, and the knowledge about the structure is almost none. The tasks of this kind essentially pertain to the cases where distribution of the material is unknown. Such problems can at present be solved by other methods, however, for the time being, applicability of those methods is limited to two-dimensional structures [10], and even in these cases very sophisticated mathematical apparatus is needed, which makes these methods inaccessible for design engineers. These facts may explain why, in the preliminary stage of design, one has not yet employed virtually any appropriate practical methods, and only intuitive approach has been used. There have not been any such method like the SADSF in its application version, which is relatively easy to use, well confirmed by the theorems of structure design theory, allowing one to take into consideration the structural (topological) parameters and design structures of complicated geometry. This situation will not change even with the development of the methods presented, for example, in the work [10]. As one can easily guess, their application will remain limited anyway, due to a high level of difficulty.

5.3. The software

At the moment, there are two software packages for implementing application version of the SADSF method [7, 8, 12]. Both of them are user-friendly, and have well designed modules of library management. The mentioned software is not discussed in this work. We have to notice, however, that the range of applicability of software depends on the content of its libraries, and quantity of the library solutions. These in turn depend on the method of software development, specifically on the methods of solving the fundamental problems, such as those presented in this work. However, in both above-mentioned software packages on can still find fragments of such libraries, in which component fields are developed in a form of sets of analytical relationships that are valid only for specific, individual fields. Derivation of the relationships requires arduous conversions of formulas, usually so complicated that examination of them is practically impossible. The development of the method is then hindered.

As one can see, the previously mentioned up-to-date concepts are based almost exclusively on the fundamental algorithms described in this work. However, we should mention another version of implementation of the application version of the SADSF method, a conservative one, being already put into practice a few years ago [6, 12] that also employs the evolution formulas $\Lambda_{1,N}^{\alpha,\alpha+1}$. In this version, one takes advantage of a partial autonomy of the component tasks, a property that allows one to concisely describe individual characteristics of particular solutions, and relatively easily create the elements of libraries containing as many as several dozen of homogeneous regions. It turns out that the discontinuous limit stress field of such complexity exists, for example, in shape design problem of rectangular element loaded over its entire perimeter by shear stresses of values close to $0.25\sigma_{\mathrm{pl}}$ – as shown in Fig. 20 in Part IV.

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Received January 31, 2003.