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COMPARISON BETWEEN IMPULSIVE AND PERIODIC NON-NEWTONIAN LUBRICATION OF HUMAN HIP JOINT

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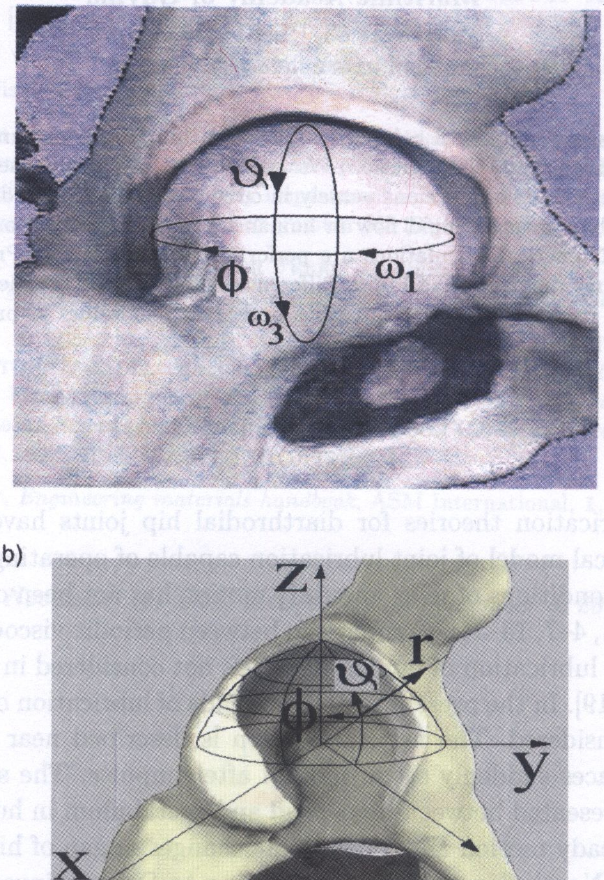
This paper shows numerical comparisons between lubrication of human hip joint gap in unsteady impulsive and periodic motion. We assume that spherical bone head in human hip joint moves at least in two directions namely in circumference and meridian directions. Basic equations describing synovial fluid flow in human hip joint are solved on the analytical and numerical way. Numerical calculations are performed in Mathcad 11 Professional Program, taking into account the method of Finite Differences. This method satisfies stability of numerical solutions of partial differential equations and gives real values of pressure and capacity forces occurring in human hip joints.

1. Introduction

Many lubrication theories for diarthrodial hip joints have been proposed, but a theoretical model of joint lubrication capable of operating under impulsive and periodic conditions of joint unsteady motion has not been completely formulated as yet: [1, 4–7, 13–15]. Comparison between periodic viscoelastic lubrication and impulsive lubrication of human joint was not considered in foregoing papers: [9, 11–12, 16–19]. In the present paper two kinds of lubrication of human hip joint have been considered. The first lubrication is described near two co-operating hip joint surfaces suddenly set in motion after impulse. The second kind of lubrication is presented between bone head and acetabulum in human hip joint for periodic unsteady motion and for periodic changes of gap of hip joints. Synovial fluid has non-Newtonian properties according to Dowson investigations [1]. For a description of such a fluid, one has used the Rivlin–Ericksen constitutive equations. Bone head has often ellipsoidal shape, but difference between semi-minor and semi-major axis can not be greater than minimal value of gap height to make possible the rotary motion [1]. Thus for normal hip joint we can assume a spherical shape of bone head. Spherical bone head can be moved by rotary motion in one or two different directions, Fig. 1. Symbol φ denotes coordinate in circumferential direction, r is coordinate in gap height direction, ϑ denotes coordinate in meridional direction. For synovial fluid flow in joint gap, three components v_φ, v_r, v_ϑ of velocity in three directions: φ, r, ϑ are considered. Pressure p depends on: φ, ϑ and time t variable. The gap height ε may be a function of three variables: φ, ϑ and t. Basic equations presenting synovial fluid flow in the gap of a human joint during impulsive and periodic motion of human limbs are solved in analytical and numerical way. Numerical calculations are performed in Mathcad 11 Professional Program taking into account the method of Finite Differences. This method satisfies stability of numerical solutions of partial dif-

a)



b)

FIG. 1. Human hip joint: a) spherical bone head, b) acetabulum.

ferential equations and gives real values of pressure and capacity forces occurring in human hip joints. The problem of impulsive and periodic lubrication of human hip joint will be solved for the human joint surfaces between bone head and acetabulum by means of equations of conservation of momentum and continuity equation. These equations and the second order approximation of the general constitutive equation given by RIVLIN and ERICKSEN can be written in the following form [10]:

$$\text{(1.1)} \qquad \operatorname{Div}\mathbf{S} = \rho\mathbf{a}, \qquad \operatorname{div}\mathbf{v} = 0,$$

$$\text{(1.2)} \qquad S = -p\mathbf{I} + \eta_0 A_1 + \alpha\mathbf{A}_1^2 + \beta\mathbf{A}_2,$$

where: $\mathbf{S}$ – is the stress tensor, p – pressure, $\mathbf{I}$ – the unit tensor, $\mathbf{A}_1$ and $\mathbf{A}_2$ – the first two Rivlin–Ericksen tensors and η_0, α, β – three material constants, where η denotes dynamic viscosity in Pas, and α, β are pseudo-viscosity coefficients in Pas^2. Tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are given by symmetric matrices defined by:

$$\text{(1.3)} \qquad \mathbf{A}_1 \equiv \mathbf{L} + \mathbf{L}^T, \qquad \mathbf{A}_2 \equiv \operatorname{grad}\mathbf{a} + (\operatorname{grad}\mathbf{a})^T + 2\mathbf{L}^T\mathbf{L}, \qquad \mathbf{a} \equiv \mathbf{L}\mathbf{v}\frac{\partial\mathbf{v}}{\partial t},$$

where: $\mathbf{L}$ – tensor of gradient fluid velocity vector in s^{-1}, $\mathbf{L}^T$ – tensor for transpose of a matrix of gradient vector of a synovial fluid in s^{-1}, $\mathbf{v}$ – velocity in m/s, t – time in s, $\mathbf{a}$ – acceleration vector $\mathrm{m/s}^2$.

Symbol grad $\mathbf{a}$ denotes tensor of rank two. The characteristic time t_0, has very small values during the motion of human limbs after injury. Hence it follows that the product of Deborah $\mathrm{De} \equiv \beta\omega/\eta_0$ and Strouhal $\mathrm{Str} \equiv 1/\omega t_0$ numbers i.e. DeStr and product of Reynolds number, relative radial clearance, and Strouhal number, i.e. Re ψ Str, have the order of the same magnitude. Moreover we assume $\mathrm{DeStr} \gg \alpha\omega/\eta_0$, $\mathrm{Str} > 1$, with ω denoting angular velocity of bone head. We assume rotational motion of human bone head with peripheral velocity $U = \omega R$, unsymmetrical synovial unsteady flow in the gap, viscoelastic and unsteady properties of synovial fluid, constant value of density ρ of synovial fluid, characteristic value of the gap height ε_0 of hip joint, no slip at the bone surfaces, R – radius of bone head. To estimate the governing equations we introduce relative radial clearance $\psi \equiv \varepsilon_0/R$. We neglect the terms multiplied by relative radial clearance because they are about thousand times smaller than the remaining terms. Thus taking into account the above mentioned assumptions, the system of equations of motion in spherical coordinates: $\phi, r\vartheta$ has the following form (Appendix A):

$$\text{(1.4)} \qquad \frac{\partial v_\varphi}{\partial t} = -\frac{1}{\rho R\sin\vartheta_1}\frac{\partial p}{\partial\varphi} + \frac{\eta_0}{\rho}\frac{\partial}{\partial r}\left(\frac{\partial v_\varphi}{\partial r}\right) + \frac{\beta}{\rho}\frac{\partial^3 v_\varphi}{\partial t\,\partial r^2},$$

(1.5) $$0 = \frac{\partial p}{\partial r},$$

(1.6) $$\frac{\partial v_\vartheta}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial \vartheta} + \frac{\eta_0}{\rho}\frac{\partial}{\partial r}\left(\frac{\partial v_\vartheta}{\partial r}\right) + \frac{\beta}{\rho}\frac{\partial^3 v_\vartheta}{\partial t\,\partial r^2},$$

(1.7) $$\frac{\partial v_\varphi}{\partial \varphi} + R\sin(\vartheta_1)\frac{\partial v_r}{\partial r} + \frac{\partial}{\partial \vartheta}\left[R v_\vartheta \sin(\vartheta_1)\right] = 0.$$

According to experimental research [1], the lubrication and pressure distribution region localizes in circumferential direction from angle $\varphi = 0$ to the half perimeter of spherical bone i.e. $\varphi = \pi$. In meridional direction pressure origin is in angle $\vartheta_1 = \pi/8$, (i.e. about 22 grade from upper pole of spherical bone) and contains the remaining part of upper hemisphere to the angle $\vartheta_1 = \pi/2$. Hence the lubrication region is defined as follows: $0 \le \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $\pi R/8 \le \vartheta \le \pi R/2$, $0 \le r \le \varepsilon$, $\vartheta_1 = \vartheta/R$, ε – gap height.

Symbols $v_\varphi, v_r, v_\vartheta$ denote synovial fluid velocity components in circumference, gap height and meridian directions of bone head, respectively. The terms multiplied by the coefficient β in right side of Eqs. (1.4), (1.6) denote influence of viscoelastic synovial fluid properties variable in the time on the hip joint lubrication. The terms in left side describe influences of accelerations occurring in the motion on the lubrication parameters.

In both classes for impulsive and periodic motions it is not possible to obtain similar solutions, a series expansion with respect to a non-similarity parameter will be given.

2. Impulsive lubrication

2.1. Method of solutions

Impulsive perturbations are started at the origin of the time interval. If time increases, then impulsive perturbations of lubrication parameters decrease. If time t tends to infinity, then perturbations tend to zero and we have classical lubrication with Newtonian properties of synovial fluid for human hip joint. Lubrication and flow parameters varying in the time for impulsive motion are presented in Fig. 2. In order to solve the system of equations (1.4)–(1.7) one introduces a solution as a power functional series expansion. Assuming successive powers of function $\beta/(\eta_0 t)$, we obtain finally:

(2.1) $$v_\varphi = U\left[v_{\varphi 0}(\chi,\varphi,\vartheta_1) + \frac{\beta}{\eta_0 t}v_{\varphi 1}(\chi,\varphi,\vartheta_1) + \left(\frac{\beta}{\eta_0 t}\right)^2 v_{\varphi 2}(\chi,\varphi,\vartheta_1) + \dots\right],$$

$$v_\vartheta = U\left[v_{\vartheta 0}(\chi,\varphi,\vartheta_1) + \frac{\beta}{\eta_0 t} v_{\vartheta 1}(\chi,\varphi,\vartheta_1) + \left(\frac{\beta}{\eta_0 t}\right)^2 v_{\vartheta 2}(\chi,\varphi,\vartheta_1) + \dots \right],$$

(2.1) [cont.]

$$v_r = U\Psi\left[v_{r0}(\chi,\varphi,\vartheta_1) + \frac{\beta}{\eta_0 t} v_{r1}(\chi,\varphi,\vartheta_1) + \left(\frac{\beta}{\eta_0 t}\right)^2 v_{r2}(\chi,\varphi,\vartheta_1) + \dots \right]$$

$$p = \frac{UR\eta_0}{\varepsilon_0^2}\left[p_{10}(\varphi,\vartheta_1,t_1) + \frac{\beta}{\eta_0 t} p_{11}(\varphi,\vartheta_1,t_1) + \left(\frac{\beta}{\eta_0 t}\right)^2 p_{12}(\varphi,\vartheta_1,t_1) + \dots \right]$$

with

(2.2)

$$\chi \equiv \frac{r}{2\sqrt{\nu t}} = r_1 N, \qquad N \equiv \frac{\varepsilon_0}{2\sqrt{\nu t}}, \qquad \nu \equiv \frac{\eta_0}{\rho},$$

$$t > 0, \qquad 0 < \frac{\beta}{\eta_0 t} < 1, \qquad 0 < r_1 < \varepsilon_1, \qquad r = \varepsilon_0 r_1.$$

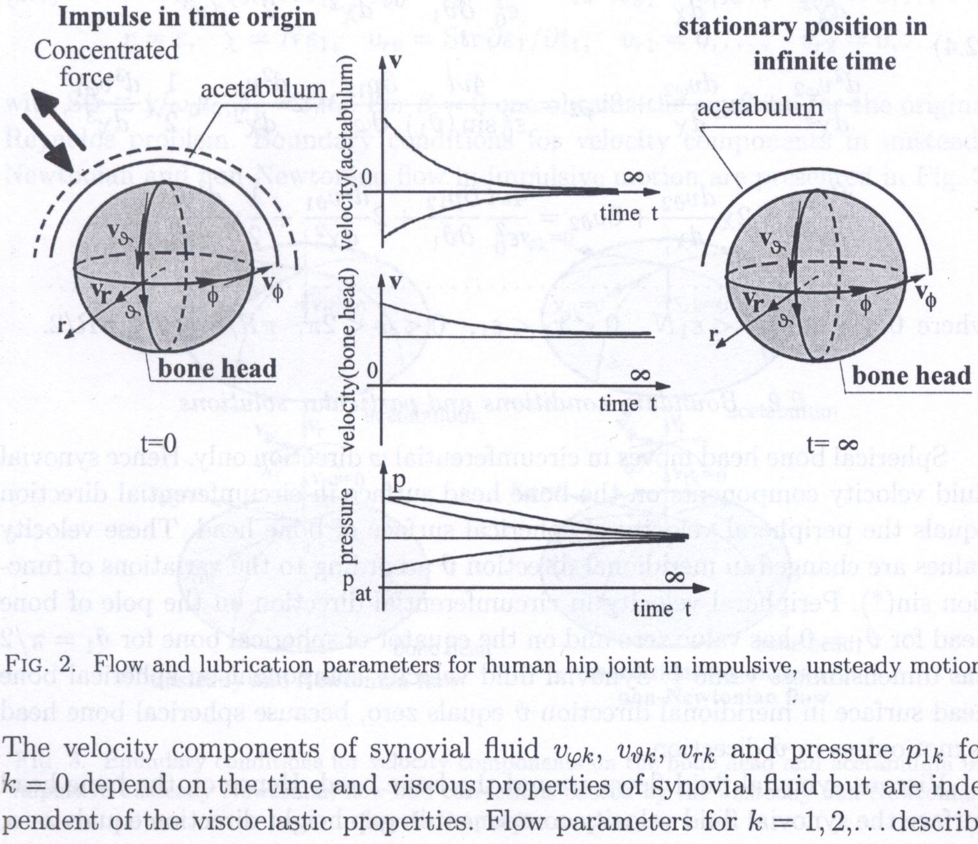


FIG. 2. Flow and lubrication parameters for human hip joint in impulsive, unsteady motion.

The velocity components of synovial fluid $v_{\varphi k}$, $v_{\vartheta k}$, v_{rk} and pressure p_{1k} for $k = 0$ depend on the time and viscous properties of synovial fluid but are independent of the viscoelastic properties. Flow parameters for $k = 1, 2, \dots$ describe corrections of synovial fluid velocity components and pressure caused by the time dependent viscoelastic properties of synovial fluid. The functions: $v_{\varphi k}$, $v_{\vartheta k}$, p_{1k} and symbols χ, N are dimensionless. By substituting the expressions (2.1)–(2.2) into system of equations (1.4), (1.6), one gets for the first six unknown functions $v_{\varphi 0}$, $v_{\vartheta 0}$, $v_{\varphi 1}$, $v_{\vartheta 1}$, $v_{\varphi 2}$, $v_{\vartheta 2}$ the following ordinary differential equations (Appendix B):

$$\frac{d^2 v_{\varphi 0}}{d\chi^2} + 2\chi \frac{d v_{\varphi 0}}{d\chi} = \frac{4\nu t}{\varepsilon_0^2 \sin(\vartheta_1)} \tag{2.3}$$

$$\frac{d^2 v_{\vartheta 0}}{d\chi^2} + 2\chi \frac{d v_{\vartheta 0}}{d\chi} = \frac{4\nu t}{\varepsilon_0^2} \frac{\partial p_{10}}{\partial \vartheta_1},$$

$$\frac{d^2 v_{\varphi 1}}{d\chi^2} + 2\chi \frac{d v_{\varphi 1}}{d\chi} + 4 v_{\varphi 1} = \frac{4\nu t}{\varepsilon_0^2 \sin(\vartheta_1)} \frac{\partial p_{11}}{\partial \varphi} + \frac{d^2 v_{\varphi 0}}{d\chi^2} + \frac{1}{2}\chi \frac{d^3 v_{\varphi 0}}{d\chi^3},$$

$$\frac{d^2 v_{\vartheta 1}}{d\chi^2} + 2\chi \frac{d v_{\vartheta 1}}{d\chi} + 4 v_{\vartheta 1} = \frac{4\nu t}{\varepsilon_0^2} \frac{\partial p_{11}}{\partial \vartheta_1} + \frac{d^2 v_{\vartheta 0}}{d\chi^2} + \frac{1}{2}\chi \frac{d^3 v_{\vartheta 0}}{d\chi^3}, \tag{2.4}$$

$$\frac{d^2 v_{\varphi 2}}{d\chi^2} + 2\chi \frac{d v_{\varphi 2}}{d\chi} + 8 v_{\varphi 2} = \frac{4\nu t}{\varepsilon_0^2 \sin(\vartheta_1)} \frac{\partial p_{12}}{\partial \varphi} + 2\frac{d^2 v_{\varphi 1}}{d\chi^2} + \frac{1}{2}\chi \frac{d^3 v_{\varphi 1}}{d\chi^3},$$

$$\frac{d^2 v_{\vartheta 2}}{d\chi^2} + 2\chi \frac{d v_{\vartheta 2}}{d\chi} + 8 v_{\vartheta 2} = \frac{4\nu t}{\varepsilon_0^2} \frac{\partial p_{12}}{\partial \vartheta_1} + 2\frac{d^2 v_{\vartheta 1}}{d\chi^2} + \frac{1}{2}\chi \frac{d^3 v_{\vartheta 1}}{d\chi^3},$$

....

where $0 \le \chi \equiv r_1 N < \varepsilon_1 N$, $0 < r_1 < \varepsilon_1$, $0 < \varphi < 2\pi$, $\pi R/8 \le \vartheta \le \pi R/2$.

2.2. Boundary conditions and particular solutions

Spherical bone head moves in circumferential φ direction only. Hence synovial fluid velocity components on the bone head surface in circumferential direction equals the peripheral velocity of spherical surface of bone head. These velocity values are changed in meridional direction ϑ according to the variations of function sin(*). Peripheral velocity in circumferential direction on the pole of bone head for $\vartheta_1 = 0$ has value zero and on the equator of spherical bone for $\vartheta_1 = \pi/2$ has dimensionless value 1. Synovial fluid velocity component on spherical bone head surface in meridional direction ϑ equals zero, because spherical bone head is motionless in ϑ direction.

Viscous synovial fluid flows around the bone head. Hence on the bone head surface the synovial fluid velocity component in gap height direction equals zero.

Spherical acetabulum surface is motionless in circumference and meridional direction. But spherical bonehead has any vibrations in gap height direction. Hence gap height changes in the time. Thus synovial fluid velocity components on the acetabulum surface are equal zero in circumference and meridional directions. Synovial fluid velocity component in gap height direction r equals the first derivative of the gap height with respect to the time.

The corrections of synovial fluid velocity components cannot change the above presented boundary conditions which are assumed on the bone head and acetabulum surface in circumference and meridional and gap height directions. Therefore for fluid velocity components of synovial fluid and its corrections we have following boundary conditions:

$$(2.5)\qquad \begin{array}{llll} r=0, & \chi=0, & v_{\varphi 0}=\sin\vartheta_1, & v_{\varphi 1}=0,\ldots,\quad v_{\varphi k}=0,\ldots \\ r=0, & \chi=0, & v_{\vartheta 0}=0, & v_{\vartheta 1}=0,\ldots,\quad v_{\vartheta k}=0,\ldots \\ r=0, & \chi=0, & v_{r0}=0, & v_{r1}=0,\ldots,\quad v_{rk}=0,\ldots \end{array}$$

$$(2.6)\qquad \begin{array}{llll} r=\varepsilon, & \chi=N\varepsilon_1, & v_{\varphi 0}=0, & v_{\varphi 1}=0,\ldots,\quad v_{\varphi k}=0,\ldots \\ r=\varepsilon, & \chi=N\varepsilon_1, & v_{\vartheta 0}=0, & v_{\vartheta 1}=0,\ldots,\quad v_{\vartheta k}=0,\ldots \\ r=\varepsilon, & \chi=N\varepsilon_1, & v_{r0}=\mathrm{Str}\,\partial\varepsilon_1/\partial t_1, & v_{r1}=0,\ldots,\quad v_{r2}=0,\ldots \end{array}$$

with $\mathrm{Str}\equiv 1/\omega_0 t_0$, $t_1=t/t_0$. For $\beta=0$ one obtains the equation for the original Reynolds problem. Boundary conditions for velocity components in unsteady Newtonian and non-Newtonian flow in impulsive motion are presented in Fig. 3.

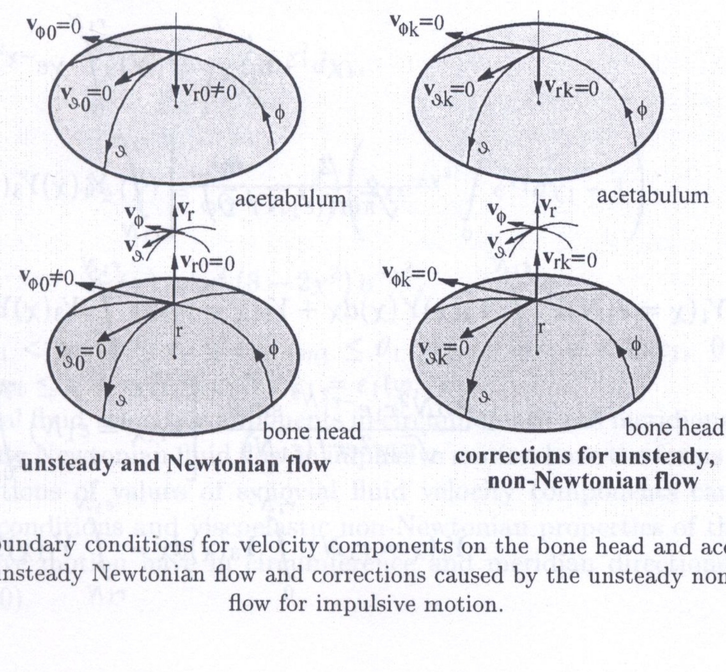


FIG. 3. Boundary conditions for velocity components on the bone head and acetabulum in impulsive unsteady Newtonian flow and corrections caused by the unsteady non-Newtonian flow for impulsive motion.

Solutions of equations (2.3), (2.4) have been found in closed form. Imposing boundary conditions $(2.5)_1$, $(2.5)_2$, $(2.6)_1$, $(2.6)_2$ on the general solutions of differential equations (2.3), (2.4),we obtain finally the following particular solutions (Appendix C):

$$(2.7)\quad v_{\varphi 0}(\varphi, r_1, \vartheta_1, t_1) = +\sin\vartheta_1 - \left\{\sin\vartheta_1 - \frac{\sqrt{\pi}}{2N^2\sin\vartheta_1}\frac{\partial p_{10}}{\partial\varphi}Y(\chi = N\varepsilon_1)\right\} \times \frac{\mathrm{erf(r_1N)}}{\mathrm{erf}(\varepsilon_1\mathrm{N})} \frac{\sqrt{\pi}}{2N^2\sin\vartheta_1}\frac{\partial p_{10}}{\partial\varphi}Y(\chi = Nr_1),$$

$$(2.8)\quad v_{\vartheta 0}(\varphi, r_1, \vartheta_1, t_1) = \frac{\sqrt{\pi}}{2N^2}\frac{\partial p_{10}}{\partial\vartheta_1}Y(\chi = N\varepsilon_1) \times \frac{\mathrm{erf(r_1N)}}{\mathrm{erf}(\varepsilon_1\mathrm{N})}\frac{\sqrt{\pi}}{2N^2}\frac{\partial p_{10}}{\partial\vartheta_1}Y(\chi = Nr_1),$$

$$(2.9)\quad v_{\varphi 1}(\phi, \vartheta_1, r_1, t_1)$$

$$= \frac{4\eta_0 t}{\rho\varepsilon^2}\frac{Nr_1e^{-r_1^2N^2}}{\sin\vartheta_1}\left\{\frac{\partial p_{11}}{\partial\varphi}\left[\int\limits_{r_1N}^{\varepsilon_1N}\chi Y_1(\chi)d\chi + \frac{N^2}{2}(\varepsilon_1^2 - r_1^2)Y_1(\chi = \varepsilon_1 N)\right]\right.$$

$$+\frac{\partial p_{10}}{\partial\varphi}\left[\int\limits_{r_1N}^{\varepsilon_1N}Y_2(\chi)d\chi - Y_1(\chi = \varepsilon_1N)\int\limits_0^{\varepsilon_1N}\chi e^{-\chi^2}Y_2(\chi)d\chi\right.$$

$$\left. + Y_1(\chi = r_1N)\int\limits_0^{r_1N}\chi e^{-\chi^2}Y_2(\chi)d\chi\right]$$

$$-\frac{2}{\sqrt{\pi}\mathrm{erf}(\varepsilon_1N)}\frac{\partial p_{10}}{\partial\phi}\left[\int\limits_{r_1N}^{\varepsilon_1N}Y_1(\chi)Y_3(\chi)Y(\chi)d\chi\right.$$

$$\left.\left. - Y_1(\chi = \varepsilon_1N)Y\int\limits_0^{\varepsilon_1N}Y_3(\chi)Y(\chi)d\chi + Y_1(\chi = r_1N)\int\limits_0^{r_1N}Y_3(\chi)Y(\chi)d\chi\right]\right\}$$

$$-\frac{8\beta N^2r_1e^{-r_1^2N^2}\sin\vartheta_1}{\sqrt{\pi}\rho\varepsilon^2erf(\varepsilon_1N)}\left[Y_1(\chi = \varepsilon_1N)\int\limits_0^{\varepsilon_1N}Y_3(\chi)d\chi\right.$$

$$\left. - Y_1(\chi = r_1N)\int\limits_0^{r_1N}Y_3(\chi)d\chi - \int\limits_{r_1N}^{\varepsilon_1N}Y_1(\chi)Y_3(\chi)d\chi\right],$$

(2.10) $v_{\vartheta 1}(\varphi,\vartheta_1,r_1,t_1)$

$$= \frac{4\eta_o t}{\rho\varepsilon^2} N r_1 e^{-r_1^2 N^2} \left\{ \frac{\partial p_{11}}{\partial \vartheta_1} \left[\int\limits_{r_1 N}^{\varepsilon_1 N} \chi Y_1(\chi) d\chi + \frac{N^2}{2}(\varepsilon_1^2 - r_1^2) Y_1(\chi = \varepsilon_1 N) \right] \right.$$

$$+ \frac{\partial p_{10}}{\partial \vartheta_1} \left[\int\limits_{r_1 N}^{\varepsilon_1 N} Y_2(\chi) d\chi - Y_1(\chi = \varepsilon_1 N) \int\limits_0^{\varepsilon_1 N} \chi e^{-\chi} Y_2(\chi) d\chi + Y_1(\chi = r_1 N) \right.$$

$$\left. \times \int\limits_0^{r_1 N} \chi e^{-\chi^2} Y_2(\chi) d\chi \right] - \frac{2}{\sqrt{\pi} erf(\varepsilon_1 N)} \frac{\partial p_{10}}{\partial \vartheta_1} \left[\int\limits_{r_1 N}^{\varepsilon_1 N} Y_1(\chi) Y_3(\chi) Y(\chi) d\chi \right.$$

$$\left. \left. - Y_1(\chi = \varepsilon_1 N) \int\limits_0^{\varepsilon_1 N} Y_3(\chi) Y(\chi) d\chi + Y_1(\chi = r_1 N) \int\limits_0^{r_1 N} Y_3(\chi) Y(\chi) d\chi \right] \right\},$$

with

$$Y(\chi) \equiv \int\limits_0^{\chi} e^{\chi_1^2} \mathrm{erf}\chi_1 d\chi_1 - \mathrm{erf}\chi \int\limits_0^{\chi} e^{\chi_1^2} d\chi_1,$$

$$\mathrm{erf}\chi_1 \equiv \frac{2}{\sqrt{\pi}} \int\limits_0^{\chi_1} \exp\left(-\chi_2^2\right) d\chi_2,$$

(2.11)
$$Y_1(\chi) \equiv \int\limits_{\delta}^{\chi} \frac{1}{\chi_1^2} e^{\chi_1^2} d\chi_1,$$

$$Y_2(\chi) \equiv \left(\frac{3}{2}\chi - \chi^3\right)\left(2\chi e^{-\chi^2} \int\limits_0^{\chi} e^{\chi_1^2} d\chi_1 - 1\right),$$

$$Y_3(\chi) \equiv \chi^2 (3 - 2\chi^2) e^{-\chi^2},$$

and $0 \le t_1 < \infty,\ 0 \le r_1 \le \varepsilon_1,\ b_{m1} \le \vartheta_1 \le b_{s1},\ 0 < \varphi < 2\pi c_1,\ 0 \le c_1 < \infty,$ $0 \le \chi_2 \le \chi_1 \le \chi \equiv r_1 N \le \varepsilon_1 N,\ \varepsilon_1 = \varepsilon_1(\varphi,\ \vartheta_1).$

Synovial fluid velocity components in circumference and meridional directions for unsteady Newtonian fluid flow in impulsive motion have the forms (2.7), (2.8).

Corrections of values of synovial fluid velocity components caused by the unsteady conditions and viscoelastic non-Newtonian properties of the fluid flow in impulsive motion have in circumference and meridian directions the forms: (2.9), (2.10).

We put series $(2.1)_1$–$(2.1)_3$ into continuity equation (1.7) and we equate terms multiplied by the same powers of small coefficient $\beta/\eta_0 t$. Hence we obtain following equations:

$$(2.12)\quad \begin{aligned} &\frac{\partial v_{\varphi 0}}{\partial \varphi}+\sin(\vartheta/R)\frac{\partial v_{r0}}{\partial r_1}+\frac{\partial}{\partial \vartheta_1}\left[v_{\vartheta 0}\sin(\vartheta/R)\right]=0,\\ &\frac{\partial v_{\varphi 1}}{\partial \varphi}+\sin(\vartheta/R)\frac{\partial v_{r1}}{\partial r_1}+\frac{\partial}{\partial \vartheta_1}\left[v_{\vartheta 1}\sin(\vartheta/R)\right]=0.\end{aligned}$$

Integrating equations $(2.12)_1$, $(2.12)_2$ with respect to the r_1 and imposing boundary conditions $(2.5)_3$ on the synovial fluid velocity component and its corrections in gap height direction we obtain:

$$v_{r0}(\varphi,\vartheta_1,r_1,t_1)=-\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\int\limits_0^{r_1} v_{\varphi 0}dr_1-\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\vartheta_1}\int\limits_0^{r_1}(\sin\vartheta_1)v_{\vartheta 0}dr_1,$$

$$(2.13)\quad v_{r1}(\varphi,\vartheta_1,r_1,t_1)=-\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left(\int\limits_0^{r_1} v_{\varphi 1}(\varphi,\vartheta_1,r_1,t_1)dr_1\right)$$

$$-\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\vartheta_1}\left(\int\limits_0^{r_1}(\sin\vartheta_1)\,v_{\vartheta 1}(\varphi,\vartheta_1,r_1,t_1)dr_1\right).$$

Velocity component of the synovial fluid in gap height direction for unsteady but Newtonian fluid flow in impulsive motion has the form $(2.13)_1$.Corrections of velocity component of the synovial fluid in gap height direction caused by the unsteady and viscoelastic non-Newtonian fluid flow properties in impulsive motion, have the form $(2.13)_2$. Imposing boundary conditions $(2.6)_3$ on the velocity components (2.13) and substituting into equations (2.13) the solutions (2.7)–(2.10), thus we obtain following modified Reynolds Equations:

$$(2.14)\quad \frac{\sqrt{\pi}}{2N^2}\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left\{\left[\frac{\int\limits_0^{\varepsilon_1}\operatorname{erf}(r_1N)\,dr_1}{\operatorname{erf}(\varepsilon_1 N)}Y(\chi=N\varepsilon_1)\right.\right.$$

$$\left.\left.-\int\limits_0^{\varepsilon_1}Y(\chi=Nr_1)\,dr_1\right]\frac{\partial p_{10}}{\partial\varphi}\right\}$$

$$
\text{(2.14) [cont.]} \quad + \frac{\sqrt{\pi}}{2N^2}\frac{\partial}{\partial \vartheta_1}\left\{\left[\frac{\int\limits_0^{\varepsilon_1} \operatorname{erf}(r_1 N)\,dr_1}{\operatorname{erf}(\varepsilon_1 N)} Y(\chi = N\varepsilon_1) - \int\limits_0^{\varepsilon_1} Y(\chi = N r_1)\,dr_1\right]\frac{\partial p_{10}}{\partial \vartheta_1}\sin\vartheta_1\right\}
$$

$$
= -(\sin\vartheta_1)\frac{\partial}{\partial\varphi}\left(\int\limits_0^{\varepsilon_1}\left[1 - \frac{\operatorname{erf}(r_1 N)}{\operatorname{erf}(\varepsilon_1 N)}\right] dr_1\right) - \mathrm{Str}\frac{\partial \varepsilon_1}{\partial t_1}\sin\vartheta_1,
$$

$$
\text{(2.15)} \quad \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left\{\frac{\partial p_{11}}{\partial\varphi}\left[\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_1(r_1)dr_1 + X_2(\varepsilon_1)\right]\right\} + \frac{\partial}{\partial\vartheta_1}\left\{\frac{\partial p_{11}}{\partial\vartheta_1}\sin\vartheta_1\left[\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_1(r_1)dr_1 + X_2(\varepsilon_1)\right]\right\}
$$

$$
= + \frac{2N\sin\vartheta_1}{\operatorname{erf}(h_1 N)}\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_3(r_1)dr_1
$$

$$
+ \frac{2}{\sqrt{\pi}\operatorname{erf}(h_1 N)\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left\{\frac{\partial p_{10}}{\partial\varphi}\left[\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_4(r_1)dr_1 - X_5(\varepsilon_1)\right]\right\}
$$

$$
+ \frac{2}{\sqrt{\pi}\operatorname{erf}(h_1 N)}\frac{\partial}{\partial\vartheta_1}\left\{\frac{\partial p_{10}}{\partial\vartheta_1}\sin\vartheta_1\left[\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_4(r_1)dr_1 - X_5(\varepsilon_1)\right]\right\}
$$

$$
- \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left\{\frac{\partial p_{10}}{\partial\varphi}\left[\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_6(r_1)\,dr_1 + X_7(\varepsilon_1)\right]\right\}
$$

$$
- \frac{\partial}{\partial\vartheta_1}\left\{\frac{\partial p_{10}}{\partial\vartheta_1}\sin\vartheta_1\left[\int\limits_0^{\varepsilon_1} r_1 e^{-r_1^2 N^2} X_6(r_1)\,dr_1 + X_7(\varepsilon_1)\right]\right\},
$$

with

$$
\text{(2.16)} \quad X_1(r_1) \equiv \int\limits_{r_1 N}^{\varepsilon_1 N} \chi Y_1(\chi)d\chi - r_1^2 Y_1(\chi = \varepsilon_1 N),
$$

(2.16) [cont.]
$$X_2(\varepsilon_1) = \frac{1}{4}\left(1 - e^{-\varepsilon_1^2 N^2}\right)\varepsilon_1^2 Y_1(\chi = N\varepsilon_1),$$

$$X_3(r_1) \equiv Y_1(\chi_1 = r_1 N)\int_0^{\varepsilon_1 N} Y_3(\chi)d\chi - Y_1(\chi = r_1 N)\int_0^{r_1 N} Y_3(\chi)d\chi$$
$$- \int_{r_1 N}^{\varepsilon_1 N} Y_1(\chi)Y_3(\chi)d\chi,$$

$$X_4(r_1) = \int_{r_1 N}^{\varepsilon_1 N} Y_1(\chi)Y_3(\chi)Y(\chi)d\chi + Y_1(\chi = r_1 N)\int_0^{r_1 N} Y_1(\chi)Y_3(\chi)Y(\chi)d\chi,$$

$$X_5(\varepsilon_1) \equiv \frac{1 - e^{-\varepsilon_1^2 N^2}}{2N^2} Y_1(\chi = \varepsilon_1 N)\int_0^{\varepsilon_1 N} Y_1(\chi)Y_3(\chi)Y(\chi)d\chi,$$

$$X_6(r_1) \equiv \int_{r_1 N}^{\varepsilon_1 N} Y_2(\chi)\,d\chi + Y_1(\chi = r_1 N)\int_0^{\varepsilon_1 N} Y_2(\chi)\,\chi e^{-\chi^2}d\chi,$$

$$X_7(\varepsilon_1) \equiv \frac{1 - e^{-\varepsilon_1^2 N^2}}{2N^2} Y_1(\chi = \varepsilon_1 N)\int_0^{\varepsilon_1 N} Y_2(\chi)\chi e^{-\chi^2}d\chi,$$

$0 \le r_2 \le r_1 \le \varepsilon_1$, $0 \le \varphi < 2\pi c_1$, $0 \le c_1 < 1$, $0 \le \vartheta_1 < \pi/2$, $0 \le t_1 < \infty$, $0 \le \chi_2 \le \chi \le \varepsilon_1 N$.

Modified Reynolds Equation (2.14) determines unknown function $p_{10}(\varphi, \vartheta_1 t_1)$. If t_1 tends to infinity i.e. $N \to 0$, $\mathrm{Str} \to 0$, then equation (2.14) tends to classical Reynolds equation (Appendix D):

(2.17)
$$\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left(\varepsilon_1^3\frac{\partial p_{10}}{\partial\varphi}\right) + \frac{\partial}{\partial\vartheta_1}\left(\varepsilon_1^3\frac{\partial p_{10}}{\partial\vartheta_1}\sin\vartheta_1\right) = 6\frac{\partial\varepsilon_1}{\partial\varphi}\sin\vartheta_1.$$

Equation (2.15) determines unknown function $p_{11}(\varphi, \vartheta_1, t_1)$ of pressure corrections caused by the viscoelastic properties of synovial fluid under unsteady conditions.

Time depended gap height with perturbations has the following form (see Fig. 4):

(2.18)
$$\varepsilon = \varepsilon_0\varepsilon_1 = \varepsilon^{(0)}\left[1 + s\cdot\exp(-t_0 t_1 \omega_0)\right],$$

$$(2.19)\quad \varepsilon^{(0)}(\varphi,\vartheta_1) \equiv \Delta\varepsilon_x \cos\varphi \sin\vartheta_1 + \Delta\varepsilon_y \sin\varphi \sin\vartheta_1 - \Delta\varepsilon_z \cos\vartheta_1 - R + [(\Delta\varepsilon_x \cos\varphi \sin\vartheta_1 + \Delta\varepsilon_y \sin\varphi \sin\vartheta_1 - \Delta\varepsilon_z \cos\vartheta_1)^2 + (R+\varepsilon_{\min})(R+2D+\varepsilon_{\min})]^{0.5}.$$

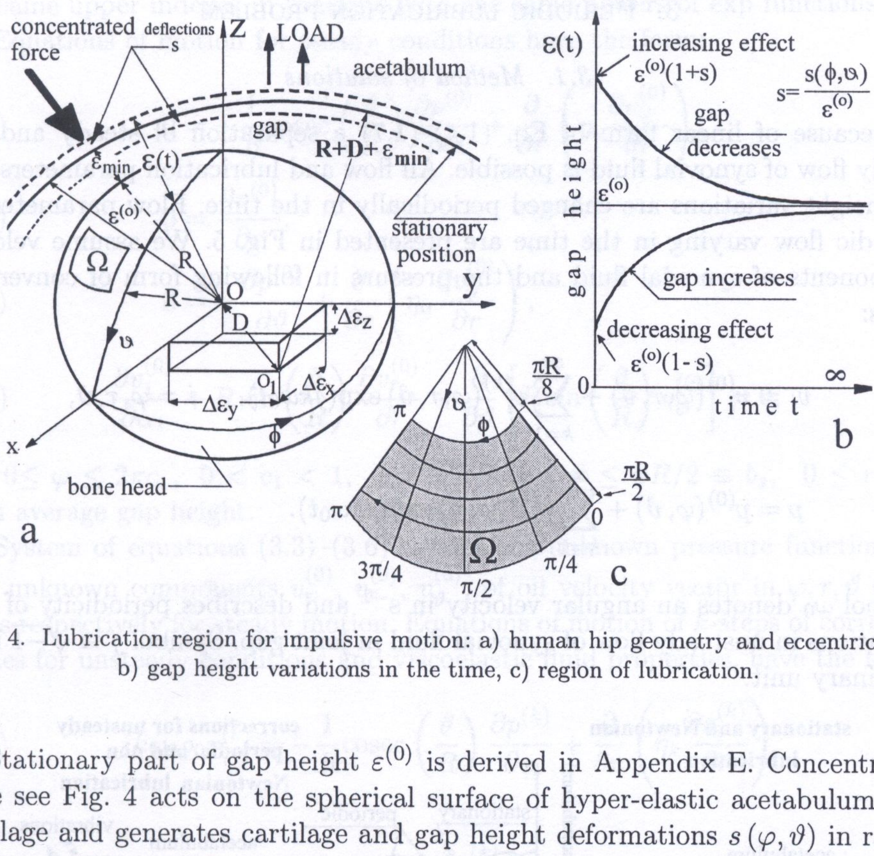


FIG. 4. Lubrication region for impulsive motion: a) human hip geometry and eccentricities, b) gap height variations in the time, c) region of lubrication.

Stationary part of gap height $\varepsilon^{(0)}$ is derived in Appendix E. Concentrated force see Fig. 4 acts on the spherical surface of hyper-elastic acetabulum and cartilage and generates cartilage and gap height deformations $s(\varphi,\vartheta)$ in radial direction. If longer the time up to the impulse is, then we have the smaller deformations multiplied by s according to the exponential function. Coefficient s describes the changes of gap height caused by the impulsive load during the motion. Gap height increases, if $s > 0$. Gap height decreases, if $s < 0$. If the concentrated force of impulse is greater, then absolute value of coefficient s is greater. Symbol ω_0 denotes an angular velocity in s^{-1} and describes changes of time perturbations in unsteady flow of synovial fluid in joint gap in height direction. If t_1 tends to infinity, then Eq. (2.17) tends to the classical Reynolds equation for stationary flow and gap height (2.18) tends to the time independent gap height for stationary flow. We assume a centre of spherical bone head in point $O(0,0,0)$ and centre of spherical cartilage in point $O_1(x-\Delta\varepsilon_x,\, y-\Delta\varepsilon_y,\, z+\Delta\varepsilon_z)$. Eccentricity has value D. Lubrication region $(\Omega : 0 \le \varphi \le \pi,\, \pi R/8 \le \vartheta \le \pi R/2)$

for impulsive motion and gap height changes in the time for impulsive motion is presented in Fig. 4c. It is a section of the bowl of the sphere. Pressure distribution p_{10} and its corrections p_{11}, p_{12}, ... occur in lubrication region and on the boundary of the region have values of atmospheric pressure p_{at}.

3. Periodic lubrication problem

3.1. *Method of solutions*

Because of linear form of Eq. (1.4)–(1.7) a separation of steady and unsteady flow of synovial fluid is possible. All flow and lubrication parameters and gap height variations are changed periodically in the time. Flow parameters in periodic flow varying in the time are presented in Fig. 5. We assume velocity components of synovial fluid and the pressure in following form of convergent series:

$$v_i = v_i^{(0)}(\varphi, r, \vartheta) + \sum_{k=1}^{\infty} v_i^{(k)}(\varphi, r, \vartheta) \exp(jk\omega_0 t), \qquad i = \varphi, r, \vartheta, \tag{3.1}$$

$$p = p^{(0)}(\varphi, \vartheta) + \sum_{k=1}^{\infty} p^{(k)}(\varphi, \vartheta) \exp(jk\omega_0 t). \tag{3.2}$$

Symbol ω_0 denotes an angular velocity in s^{-1} and describes periodicity of perturbations in unsteady flow of synovial fluid in joint gap. Symbol $j \equiv \sqrt{-1}$ is an imaginary unit.

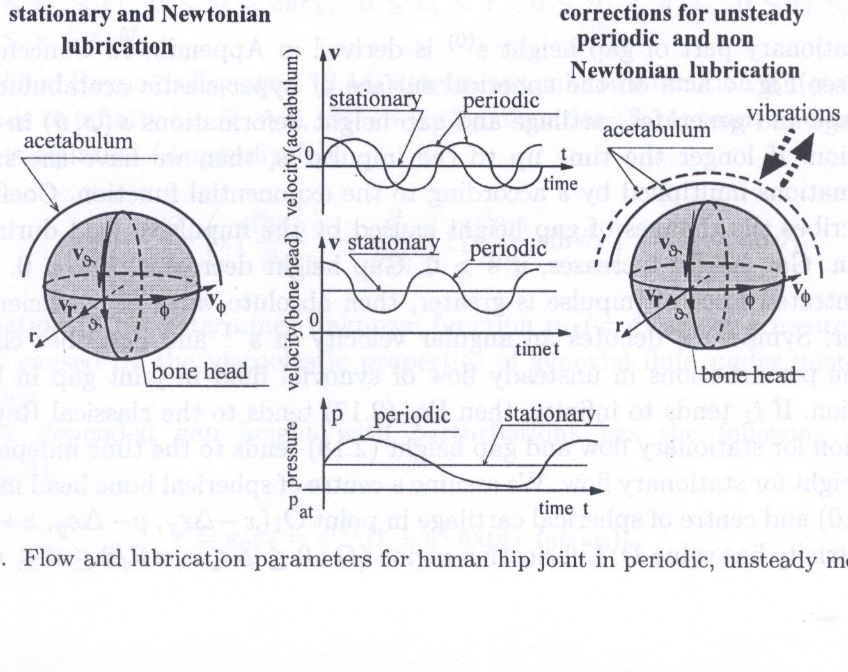


FIG. 5. Flow and lubrication parameters for human hip joint in periodic, unsteady motion.

Unknown functions with upper index (0) describe velocity vector components and pressure for stationary, non-viscoelastic fluid properties. Unknown functions with upper index (k), denote corrections of velocity vector components and pressure caused by the non-stationary viscoelastic properties of synovial fluid.

We put series (3.1)–(3.2) in set of Eqs. (1.4)–(1.7) and we compare terms of the same upper indexes in brackets with the same powers of exp functions.

Equations of motion for steady conditions have the form:

$$(3.3)\qquad 0 = -\frac{1}{R}\mathrm{cosec}\left(\frac{\vartheta}{R}\right)\frac{\partial p^{(0)}}{\partial \varphi} + \frac{\partial}{\partial r}\left(\eta_0 \frac{\partial v_\varphi^{(0)}}{\partial r}\right),$$

$$(3.4)\qquad 0 = \frac{\partial p^{(0)}}{\partial r},$$

$$(3.5)\qquad 0 = -\frac{\partial p^{(0)}}{\partial \vartheta} + \frac{\partial}{\partial r}\left(\eta_0 \frac{\partial v_\vartheta^{(0)}}{\partial r}\right),$$

$$(3.6)\qquad \frac{\partial v_1^{(0)}}{\partial \alpha_1} + R\sin\left(\frac{\vartheta}{R}\right)\frac{\partial v_r^{(0)}}{\partial r} + \frac{\partial}{\partial \vartheta}\left[R\sin\left(\frac{\vartheta}{R}\right)v_\vartheta^{(0)}\right] = 0,$$

for: $0 \leq \varphi \leq 2\pi c_1,\ \ 0 < c_1 < 1,\ \ b_m \equiv \pi R/8 \leq \vartheta \leq \pi R/2 \equiv b_s,\ \ 0 \leq r \leq \varepsilon$; with average gap height.

System of equations (3.3)–(3.6) determines unknown pressure function $p^{(0)}$ and unknown components $v_\varphi^{(0)}$, $v_r^{(0)}$, $v_\vartheta^{(0)}$, of oil velocity vector in φ, r, ϑ directions respectively for steady motion. Equations of motion of k-steps of correction values for unsteady conditions and viscoelastic fluid properties, have the form:

$$(3.7)\qquad jk\omega_0\rho_0 v_\varphi^{(k)} = -\frac{1}{R}\mathrm{cosec}\left(\frac{\vartheta}{R}\right)\frac{\partial p^{(k)}}{\partial \varphi} + \frac{\partial}{\partial r}\left(\eta_k \frac{\partial v_\varphi^{(k)}}{\partial r}\right)$$

$$(3.8)\qquad 0 = \frac{\partial p^{(k)}}{\partial r},$$

$$(3.9)\qquad jk\omega_0\rho_0 v_\vartheta^{(k)} = -\frac{\partial p^{(k)}}{\partial \vartheta} + \frac{\partial}{\partial r}\left(\eta_k \frac{\partial v_\vartheta^{(k)}}{\partial r}\right)$$

$$(3.10)\qquad \frac{\partial v_1^{(k)}}{\partial \alpha_1} + R\sin\left(\frac{\vartheta}{R}\right)\frac{\partial v_r^{(k)}}{\partial r} + \frac{\partial}{\partial \vartheta}\left[R\sin\left(\frac{\vartheta}{R}\right)v_\vartheta^{(k)}\right] = 0,$$

for: $k = 1, 2, 3, ...,\ \ 0 < \varphi \leq 2\pi c_1,\ \ 0 < c_1 < 1,\ \ \pi R/8 \leq \vartheta \leq \pi R/2,\ \ 0 \leq r \leq \varepsilon$; $\eta_k \equiv \eta_0 + jk\omega_0\beta$.

System of equations (3.7)–(3.10) determines unknown corrections $p^{(k)}$ of pressure function and unknown corrections $v_{\varphi}^{(k)}$, $v_r^{(k)}$, $v_{\vartheta}^{(k)}$ of components of oil velocity vector in φ, r, ϑ directions respectively. Symbol η_k – denotes apparent viscosity of synovial fluid and depends on pseudoviscosity coefficient β. If β tends to zero, then apparent viscosity tends to the classical dynamic viscosity η_0.

3.2. Boundary conditions and particular solutions

During the motion of human hip joint take place the periodically changes of gap height in the time and in circumferential and meridional directions. Unsteady part of gap height will be expressed by the trigonometric sin and cos(*) functions with variable period. We assume that the total gap height has following form:

$$\varepsilon_{\text{tot}} \equiv \varepsilon^{(0)}(\varphi,\vartheta)\left[1+\sum_{k=1}^{\infty}\frac{1}{k^2}\exp(jk\omega_0 t)\right], \tag{3.11}$$

where: symbol $\varepsilon^{(0)}$ – time independent primary gap height value defined by the equation (2.19). Time independent average gap height with periodic perturbations has following form:

$$\varepsilon \equiv re\frac{1}{t_s}\int_0^{t_s}\varepsilon_{\text{tot}}dt = \varepsilon^{(0)}(\varphi,\vartheta_1)\left[1+\sum_{k=1}^{\infty}\frac{\sin(\omega_0 t_s k)}{k^3\omega_0 t_s}\right], \tag{3.12}$$

where t_s – average time period of joint gap perturbations. Viscous synovial fluid flows around the bone head. Hence on the bone head surface the synovial fluid velocity component in gap height direction equals zero. On the acetabulum surface, synovial fluid velocity component in gap height direction r equals the first derivative of the gap height with respect to the time. Equating the terms of the same powers of exp functions in Eqs. (3.1) and the first derivative with respect to the time of formula (3.11) we have:

$$\sum_{k=1}^{\infty} v_r^{(k)}(\varphi, r=\varepsilon,\vartheta)\exp(jk\omega_0 t) = \frac{\partial\varepsilon_{\text{tot}}}{\partial t} = \sum_{k=1}^{\infty} j\varepsilon^{(0)}\frac{1}{k}\omega_0\exp(jk\omega_0 t), \tag{3.13}$$

hence

$$v_r^{(k)}(\varphi, r=\varepsilon,\vartheta) = j\frac{1}{k}\varepsilon^{(0)}\omega_0 \equiv V_{rk}. \tag{3.14}$$

Bonehead realizes angular velocity ω_1 in φ direction and ω_3 in ϑ direction. Acetabulum moves in circumference φ and meridian ϑ direction. Moreover we take into account tangential acceleration of bone head and acetabulum surface variation in the time. Time dependent total tangential velocities of bone surface and acetabulum surface in φ, ϑ directions have following forms of trigonometric sin(*) and cos(*) functions with variable period:

$$
\begin{aligned}
&U_{\varphi} = U_{\varphi 0} + \sum_{k=1}^{\infty} U_{\varphi k} \exp\left(jk\omega_0 t\right), \qquad U_{\varphi k} \equiv \frac{U^*_{\varphi 0}}{k^2},\\
&U_{\vartheta} = U_{\vartheta 0} + \sum_{k=1}^{\infty} U_{\vartheta k} \exp\left(jk\omega_0 t\right), \qquad U_{\vartheta k} \equiv \frac{U^*_{\vartheta 0}}{k^2},\\
&V_{\varphi} = \sum_{k=1}^{\infty} V_{\varphi k} \exp\left(jk\omega_0 t\right), \qquad V_{\varphi k} \equiv \frac{V_{\varphi 0}}{k^2},\\
&V_{\vartheta} = \sum_{k=1}^{\infty} V_{\vartheta k} \exp\left(jk\omega_0 t\right), \qquad V_{\vartheta k} \equiv \frac{V_{\vartheta 0}}{k^2},
\end{aligned} \tag{3.15}
$$

where: $U_{\varphi k}$, $U_{\vartheta k}$, $V_{\varphi k}$, $V_{\vartheta k}$ – time independent coefficients of tangential velocity changes of bone surface and acetabulum for $k = 1, 2, 3,$....

Synovial fluid velocity component on the bone head surface in circumference direction equals the peripheral circumference velocity of spherical surface of bone head. These velocity values are changed in meridional direction ϑ according to the variations of function sin(*). Peripheral velocity in circumferential direction on the pole of bone head for $\vartheta = 0$ has value zero and on the equator of spherical bone for $\vartheta = \pi R/2$ has value $\omega_1 R$.

Synovial fluid velocity component on the bone head surface in meridional direction equals the peripheral meridional velocity of spherical surface of bone head. These velocity values are changed in circumference direction φ according to the variations of function sin(*). Peripheral velocity in meridian direction on bone head for $\varphi = 0$ has value zero and on the prime meridian of spherical bone for $\varphi = \pi R/2$ has value $\omega_3 R$. Hence time independent circumference and meridional velocities for spherical bonehead in stationary motion have following forms:

$$
\begin{aligned}
&U_{\varphi 0} = \omega_1 R \sin(\vartheta/R), \qquad U_{\vartheta 0} = \omega_3 R \sin(\varphi),\\
&U^*_{\varphi 0} = \omega_{10} R \sin(\vartheta/R), \qquad U^*_{\vartheta 0} = \omega_{30} R \sin(\varphi).
\end{aligned} \tag{3.16}
$$

We denote: ω_1, ω_{10} – angular velocity and its perturbations of spherical bone head along circumference φ direction and ω_3, ω_{30} – angular velocity and its perturbations for spherical bone head along the meridional ϑ direction. Symbol R denotes radius of spherical bone head.

Equating the terms of the same powers of exp functions in series (3.1) and (3.15) and taking into account Eq. (3.14), therefore for fluid velocity components of synovial fluid and its corrections we have following boundary conditions:

$$
\begin{aligned}
&r = 0, \quad v_\varphi^{(0)} = U_{\varphi o}, \quad v_\varphi^{(1)} = U_{\varphi 1}, \quad v_\varphi^{(2)} = U_{\varphi 2}, \ldots\ldots \quad v_\varphi^{(k)} = U_{\varphi k}, \ldots\\
(3.17) \qquad &r = 0, \quad v_\vartheta^{(0)} = U_{\vartheta o}, \quad v_\vartheta^{(1)} = U_{\vartheta 1}, \quad v_\vartheta^{(2)} = U_{\vartheta 2}, \ldots\ldots \quad v_\vartheta^{(k)} = U_{\vartheta k}, \ldots\\
&r = 0, \quad v_r^{(0)} = 0, \quad v_r^{(1)} = 0, \quad v_r^{(2)} = 0, \ldots\ldots \quad v_r^{(k)} = 0, \ldots
\end{aligned}
$$

$$
\begin{aligned}
&r = \varepsilon, \quad v_\varphi^{(0)} = 0, \quad v_\varphi^{(1)} = V_{\varphi 1}, \quad v_\varphi^{(2)} = V_{\varphi 2}, \ldots\ldots \quad v_\varphi^{(k)} = V_{\varphi k}, \ldots\\
(3.18) \qquad &r = \varepsilon, \quad v_\vartheta^{(0)} = 0, \quad v_\vartheta^{(1)} = V_{\vartheta 1}, \quad v_\vartheta^{(2)} = V_{\vartheta 2}, \ldots\ldots \quad v_\vartheta^{(k)} = V_{\vartheta k}, \ldots\\
&r = \varepsilon, \quad v_r^{(0)} = 0, \quad v_r^{(1)} = V_{r1}, \quad v_r^{(2)} = V_{\vartheta 2}, \ldots\ldots \quad v_r^{(k)} = V_{rk}, \ldots
\end{aligned}
$$

Boundary conditions for velocity components for steady Newtonian and unsteady non-Newtonian flow in periodic motion are presented in Fig. 6.

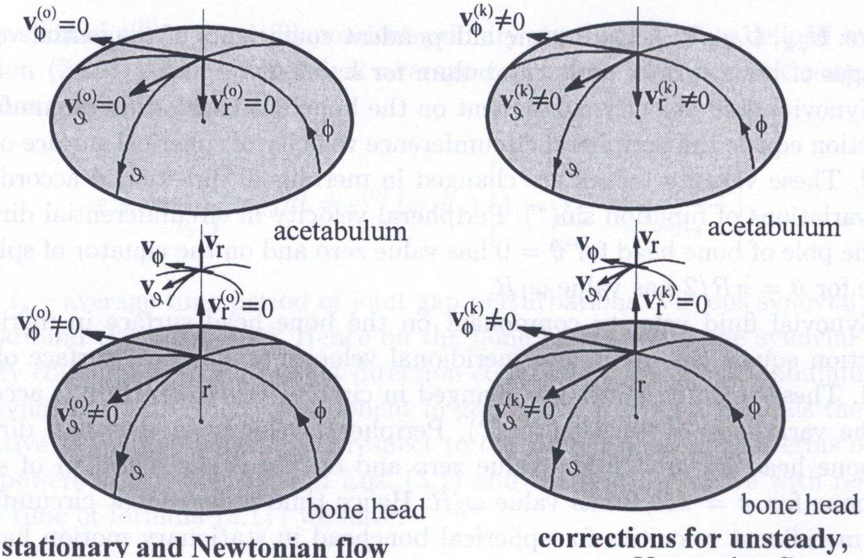


FIG. 6. Boundary conditions for velocity components on the bone head and acetabulum in stationary Newtonian flow and corrections caused by the unsteady and non-Newtonian flow for periodic motion.

Imposing boundary conditions $(3.17)_1$, $(3.17)_2$, $(3.18)_1$, $(3.18)_2$ on the system of equations (3.3)–(3.6) and on the system of Eqs. (3.7)–(3.10), we determine unknown functions of velocity components for stationary flow $v_\varphi^{(0)}$, $v_\vartheta^{(0)}$, and functions of corrections $v_\varphi^{(k)}$, $v_\vartheta^{(k)}$ for $k = 1, 2, 3, \ldots$ caused by unsteady fluid flow with viscoelastic properties. Results of mathematical calculations are presented in following solutions [2]:

$$v_\varphi^{(0)} = -\frac{1}{2\eta_0}\frac{1}{R}\operatorname{cosec}\left(\frac{\vartheta}{R}\right)\frac{\partial p^{(0)}}{\partial\varphi}\varepsilon^2 s(1-s) + U_{10}(1-s),$$

(3.19)

$$v_\vartheta^{(0)} = -\frac{1}{2\eta_0}\frac{\partial p^{(0)}}{\partial\vartheta}\varepsilon^2 s(1-s) + U_{30}(1-s).$$

$$v_\varphi^{(k)} = \Pi_{\varphi k}W_k + U_{\varphi k}\frac{\sinh\left[(\varepsilon - r)A_k\right]}{\sinh(\varepsilon A_k)} + V_{\varphi k}\frac{\sinh(rA_k)}{\sinh(\varepsilon A_k)},$$

(3.20)

$$v_\vartheta^{(k)} = \Pi_{\vartheta k}W_k + U_{\vartheta k}\frac{\sinh\left[(\varepsilon - r)A_k\right]}{\sinh(\varepsilon A_k)} + V_{\vartheta k}\frac{\sinh(rA_k)}{\sinh(\varepsilon A_k)},$$

with

$$W_k \equiv \left[1-\exp(rA_k)\right] - \left[1-\exp(\varepsilon A_k)\right]\frac{\sinh(rA_k)}{\sinh(\varepsilon A_k)},$$

$$\Pi_{\varphi k} \equiv \frac{j}{k\omega_0\rho_0 R\sin\vartheta_1}\frac{\partial p^{(k)}}{\partial\varphi}, \quad \Pi_{\vartheta k} \equiv \frac{j}{k\omega_0\rho_0}\frac{\partial p^{(k)}}{\partial\vartheta},$$

(3.21)

$$A_k \equiv \sqrt{\frac{jk\omega_0\rho_0}{\eta_0 + jk\omega_0\beta}}$$

for $k = 1, 2, 3, \ldots s \equiv r/\varepsilon$, $0 \leq \varphi \leq 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \leq \vartheta \leq \pi R/2 \equiv b_s$, $0 \leq r \leq \varepsilon$.

Synovial fluid velocity components in circumference and meridional directions for stationary flow in periodic motion have the forms $(3.19)_1$, $(3.19)_2$.

Corrections of values of synovial fluid velocity components caused by the unsteady conditions and viscoelastic properties of the fluid in periodic motion have in circumference and meridional directions the forms: $(3.20)_1$, $(3.20)_2$.

Integrating Eqs. (3.6), (3.10) with respect to the r_1 and imposing boundary conditions $(3.17)_3$ on the synovial fluid velocity component and its corrections in gap height direction, we obtain:

$$v_r^{(0)} = -\frac{1}{R}\operatorname{cosec}\left(\frac{\vartheta}{R}\right)\int_0^r \frac{\partial v_\varphi^{(0)}}{\partial\varphi}dr - \operatorname{cosec}\left(\frac{\vartheta}{R}\right)\int_0^r \frac{\partial}{\partial\vartheta}\left[v_\vartheta^{(0)}\sin\left(\frac{\vartheta}{R}\right)\right]dr,$$

(3.22)

$$v_r^{(k)} = -\frac{1}{R}\operatorname{cosec}\left(\frac{\vartheta}{R}\right)\int_0^r \frac{\partial v_\varphi^{(k)}}{\partial\varphi}dr - \operatorname{cosec}\left(\frac{\vartheta}{R}\right)\int_0^r \frac{\partial}{\partial\vartheta}\left[v_\vartheta^{(k)}\sin\left(\frac{\vartheta}{R}\right)\right]dr.$$

Velocity component of the synovial fluid in gap height direction for stationary Newtonian fluid flow in periodic motion has the form $(3.22)_1$.

Corrections of velocity component of the synovial fluid in gap height direction caused by the unsteady and viscoelastic non-Newtonian fluid flow properties in periodic motion, have the form $(3.22)_2$.

Imposing boundary conditions $(3.18)_3$ on the velocity components (3.22) and using the law of differentiation of integrals with variable limits of integration and boundary conditions (3.17), (3.18), thus from Eq. (3.22) yields:

$$
\begin{aligned}
&\frac{\partial}{\partial\varphi}\int_0^{\varepsilon} v_{\varphi}^{(0)}\,dr + R\frac{\partial}{\partial\vartheta}\int_0^{\varepsilon} v_{\vartheta}^{(0)}\sin\left(\frac{\vartheta}{R}\right)dr = 0,\\
(3.23)\qquad &\frac{\partial}{\partial\varphi}\int_0^{\varepsilon} v_{\varphi}^{(k)}\,dr + R\frac{\partial}{\partial\vartheta}\int_0^{\varepsilon} v_{\vartheta}^{(k)}\sin\left(\frac{\vartheta}{R}\right)dr\\
&\qquad = -j\omega_0\varepsilon^{(0)}\frac{R}{k}\sin\left(\frac{\vartheta}{R}\right) + V_{\varphi k}\frac{\partial\varepsilon}{\partial\varphi} + RV_{\vartheta k}\sin\left(\frac{\vartheta}{R}\right)\frac{\partial\varepsilon}{\partial\vartheta},
\end{aligned}
$$

Substituting into Eqs. (3.23) the solutions (3.19), (3.20), thus we obtain following modified Reynolds Equations (Appendix F):

$$
\begin{aligned}
(3.24)\qquad &\frac{1}{R}\operatorname{cosec}\left(\frac{\vartheta}{R}\right)\frac{\partial}{\partial\varphi}\left\{\frac{\varepsilon^3}{\eta_0}\frac{\partial p^{(0)}}{\partial\varphi}\right\} + R\frac{\partial}{\partial\vartheta}\left\{\frac{\varepsilon^3}{\eta_0}\frac{\partial p^{(0)}}{\partial\vartheta}\sin\frac{\vartheta}{R}\right\}\\
&\qquad = 6\omega_1 R\sin\left(\frac{\vartheta}{R}\right)\frac{\partial\varepsilon}{\partial\varphi} + 6\omega_3 R^2\sin(\varphi)\frac{\partial}{\partial\vartheta}\left[\varepsilon\sin\left(\frac{\vartheta}{R}\right)\right],
\end{aligned}
$$

for $\Omega : 0 \le \varphi \le \pi$, $\pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2$,

$$
\begin{aligned}
(3.25)\qquad &\frac{1}{R}\operatorname{cosec}\left(\frac{\vartheta}{R}\right)\frac{\partial}{\partial\varphi}\left[\frac{\varepsilon^3}{\eta_k}\frac{\partial p^{(k)}}{\partial\varphi}\right] + R\frac{\partial}{\partial\vartheta}\left[\frac{\varepsilon^3}{\eta_k}\sin\left(\frac{\vartheta}{R}\right)\frac{\partial p^{(k)}}{\partial\vartheta}\right]\\
&\qquad = 12j\,\omega_0\varepsilon^{(0)}\frac{R}{k}\sin\left(\frac{\vartheta}{R}\right) - 12\left[V_{\varphi k}\frac{\partial\varepsilon}{\partial\varphi} + RV_{\vartheta k}\sin\left(\frac{\vartheta}{R}\right)\frac{\partial\varepsilon}{\partial\vartheta}\right]\\
&\qquad + 6\left[U_{\varphi k}(\vartheta) + V_{\varphi k}\right]\left[\frac{\partial\varepsilon}{\partial\varphi} - \frac{1}{12}\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3}{\eta_k}\right)jk\omega_0\rho_0\right]\\
&\qquad + 6R\left[U_{\vartheta k}(\varphi) + V_{\vartheta k}\right]\left\{\frac{\partial}{\partial\vartheta}\left[\varepsilon\sin\left(\frac{\vartheta}{R}\right)\right] - \frac{1}{12}\frac{\partial}{\partial\vartheta}\left[\frac{\varepsilon^3}{\eta_k}\sin\left(\frac{\vartheta}{R}\right)\right]jk\omega_0\rho_0\right\}.
\end{aligned}
$$

for $k = 1, 2, 3, \ldots$, $0 \le \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \vartheta \le \pi R/2 \equiv b_s$, $0 \le r \le \varepsilon$.

Multiplying both hands of Eq. (3.25) by expression $\exp(jk\omega_0 t)$ and equating terms of real parts of complex number in both hands of equation, we get following sequence of modified Reynolds equations:

$$(3.26)\quad \frac{1}{R}\operatorname{cosec}\left(\frac{\vartheta}{R}\right)\frac{\partial}{\partial\varphi}\left[\frac{\varepsilon^3}{\eta_k^*}\frac{\partial p^{(k)}}{\partial\varphi}\right]+R\frac{\partial}{\partial\vartheta}\left[\frac{\varepsilon^3}{\eta_k^*}\frac{\partial p^{(k)}}{\partial\vartheta}\sin\left(\frac{\vartheta}{R}\right)\right]$$

$$=-12\,\omega_0\varepsilon^{(0)}\frac{R}{k}\sin\left(\frac{\vartheta}{R}\right)\sin(k\omega_0 t)+6\left[U_{1k}(\vartheta)+V_{1k}\right]\left[\frac{\partial\varepsilon}{\partial\varphi}\cos(k\omega_0 t)\right.$$

$$\left.+\frac{1}{12}\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3\eta_0}{\eta_0^2+\omega_0^2k^2\beta^2}\right)k\omega_0\rho_o\sin(k\omega_0 t)\right]$$

$$-12\left[V_{1k}\frac{\partial\varepsilon}{\partial\varphi}+RV_{3k}\frac{\partial}{\partial\vartheta}\left(\varepsilon\sin\frac{\vartheta}{R}\right)\right]\cos(k\omega_0 t)$$

$$+6\left[U_{3k}(\varphi)+V_{3k}\right]\left\{R\frac{\partial}{\partial\vartheta}\left[\varepsilon\sin\left(\frac{\vartheta}{R}\right)\right]\cos(k\omega_0 t)\right.$$

$$\left.-\frac{1}{12}\frac{\partial}{\partial\vartheta}\left(\frac{\varepsilon^3\,\eta_0 h_1}{\eta_0^2+\omega_0^2k^2\beta^2}\right)k\omega_0\rho_0\sin(k\omega_0 t)\right\}$$

with

$$(3.27)\quad \frac{1}{\eta_k^*}\equiv re\frac{\exp(jk\omega_0 t)}{\eta_k}=\frac{\eta_0}{\eta_0^2+\omega_0^2k^2\beta^2}\cos(k\omega_0 t)$$

$$+\frac{\omega_0 k\beta}{\eta_0^2+\omega_0^2k^2\beta^2}\sin(k\omega_0 t)\,.$$

for $k=1,2,3,\ldots,0<\varphi\leq 2\pi c_1$, $0<c_1<1$, $\pi R/8\leq\vartheta\leq\pi R/2$, $0\leq r\leq\varepsilon$ and Eq. (3.26) determines pressure corrections $p^{(k)}$ caused by the unsteady flow conditions of synovial fluid with viscoelastic properties in periodic motion.

Summation of left and right hands of equations (3.24) and (3.26) for $k=1,2,3,\ldots$, respectively is possible if and only if coefficient β tends to zero i.e. if viscoelastic properties of synovial fluid are neglected and retain only influences of unsteady motion [2, 3, 8]. After summation we obtain modified Reynolds equation in final form [3]:

$$(3.28)\quad \frac{1}{R\sin\frac{\vartheta}{R}}\frac{\partial}{\partial\varphi}\left\{\frac{\varepsilon^3}{\eta_0}\frac{\partial p}{\partial\varphi}\right\}+R\frac{\partial}{\partial\vartheta}\left\{\frac{\varepsilon^3}{\eta_0}\frac{\partial p}{\partial\vartheta}\sin\frac{\vartheta}{R}\right\}$$

$$\text{(3.28) [cont.]} \quad = 6\omega_1 R\left(\sin\frac{\vartheta}{R}\right)\frac{\partial\varepsilon}{\partial\varphi} + 6\omega_3 R^2(\sin\varphi)\frac{\partial}{\partial\vartheta}\left(\varepsilon\sin\frac{\vartheta}{R}\right)$$

$$- 12\,\omega_0 R\left(\sin\frac{\vartheta}{R}\right)\varepsilon^{(0)}\sum_{k=1}^{\infty}\frac{\sin(k\omega_0 t)}{k}$$

$$- 12\left[V_{10}\frac{\partial\varepsilon}{\partial\varphi} + RV_{30}\frac{\partial}{\partial\vartheta}\left(\varepsilon\sin\frac{\vartheta}{R}\right)\right]\sum_{k=1}^{\infty}\frac{\cos(k\omega_0 t)}{k^2}$$

$$+6\left[\omega_{10}R\left(\sin\frac{\vartheta}{R}\right) + V_{10}\right]\left[\frac{\partial\varepsilon}{\partial\varphi}\sum_{k=1}^{\infty}\frac{\cos(k\omega_0 t)}{k^2}\right.$$

$$\left.+\frac{1}{12}\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3}{\eta_0}\right)\omega_0\rho_0\sum_{k=1}^{\infty}\frac{\sin(k\omega_0 t)}{k}\right]$$

$$+ 6R\left[\omega_{30}R(\sin\varphi) + V_{30}\right]\left[\frac{\partial}{\partial\vartheta}\left(\varepsilon\sin\frac{\vartheta}{R}\right)\sum_{k=1}^{\infty}\frac{\cos(k\omega_0 t)}{k^2}\right.$$

$$\left.+\frac{1}{12}\frac{\partial}{\partial\vartheta}\left(\frac{\varepsilon^3}{\eta_0}\sin\frac{\vartheta}{R}\right)\omega_0\rho_0\sum_{k=1}^{\infty}\frac{\sin(k\omega_0 t)}{k}\right],$$

$...0<\varphi \le 2\pi c_1,\ 0 < c_1 < 1,\ b_m \equiv \pi R/8 \le \vartheta \le \pi R/2 \equiv b_s,\ 0 \le r \le \varepsilon.$

Modified Reynolds equation (3.28) determines total pressure function p for unsteady periodic flow. The sums presented in right hand side of equation (3.28) are convergent for arbitrary time. Presented model describes the hip joint periodic motion without vibration damping and without changes of frequencies in the time. Thus if the time tends to infinity, then the pressure distribution has periodic values too.

Center of spherical bonehead and acetabulum and gap height variations change in the time for periodic motion are presented in Fig. 7a, b, c, where $\varepsilon(t) = re(\varepsilon_{\text{tot}})$.

Pressure distribution $p^{(0)}$ and its corrections $p^{(1)}$, $p^{(2)}$, $p^{(3)}$,... occur in lubrication region Ω and on the boundary of region Ω (bowl of the sphere) have values of atmospheric pressure p_{at}. Region Ω was indicated in Fig. 7 and for periodic motion is identically defined by the following inequalities: $\Omega : 0 \le \varphi \le \pi$, $\pi R/8 \le \vartheta \le \pi R/2$.

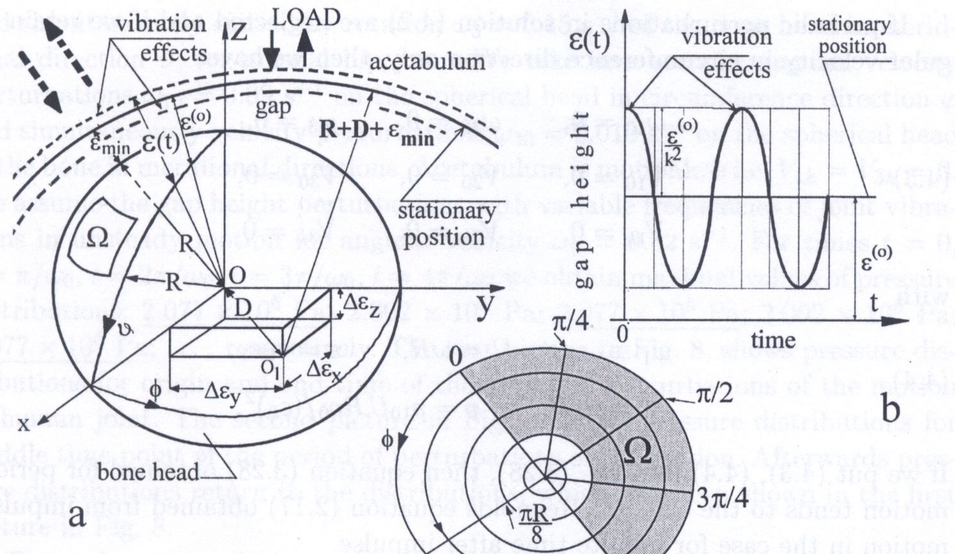


FIG. 7. Lubrication region for impulsive motion,: a) human hip geometry and eccentricities, b) gap height variations in the time, c) region of lubrication.

4. COMPARISONS

Final dimensional form of pressure distribution for unsteady impulsive motion and viscoelastic properties of synovial fluid has the form:

$$p = \frac{\omega R^2 \eta_0}{\varepsilon_o^2}\left[p_{10}(\varphi,\vartheta_1,t_1) + \frac{\beta}{\eta_o t}p_{11}(\varphi,\vartheta_1,t_1) + O\left(\frac{\beta}{\eta_0 t}\right)^2\right]. \tag{4.1}$$

where dimensionless functions p_{10}, p_{11},... are determined by the Eqs. (2.14), (2.15).

Final dimensional form of pressure distribution for unsteady periodic motion and viscoelastic properties of synovial fluid has the form:

$$p = p^{(0)}(\varphi,\vartheta_1,t_1) + p^{(1)}(\varphi,\vartheta_1,t_1)\cos(\omega_0 t) + \ldots + p^{(k)}(\varphi,\vartheta_1,\ t_1)\cos(k\omega_0 t)+ \tag{4.2}$$

where dimensional functions $p^{(0)}$, $p^{(1)}$, ... are determined by the Eqs. (3.24), (3.26) for $k = 1,2,3,\ldots$.

If coefficient β tends to zero i.e. if viscoelastic properties of synovial fluid are neglected for periodic motion, then we can find the total sum p of infinite terms of series (4.2) in numerical form. In this case function of pressure p determines Eq. (3.28).

If periodic perturbations in solution (4.2) are neglected and if we retain angular velocity in circumference direction only, then we have:

$$(4.3)\qquad \begin{aligned} &\omega_{10} = 0, && \omega_{30} = 0, && \omega_3 = 0,\\ &V_{10} = 0, && V_{20} = 0, && V_{30} = 0,\\ &V_{1k} = 0, && V_{2k} = 0, && V_{3k} = 0 \end{aligned}$$

with

$$(4.4)\qquad \begin{aligned} &\omega_1 \equiv \omega, \qquad U = \omega R, \qquad \varepsilon = \varepsilon_0 \varepsilon_1,\\ &\vartheta = \vartheta_1 R, \qquad p = p_{10} U R \eta_0 / (\varepsilon_0)^2. \end{aligned}$$

If we put (4.3), (4.4) into Eq. (3.28), then equation (3.28) obtained for periodic motion tends to the classical Reynolds equation (2.17) obtained from impulsive motion in the case for infinite time after impulse.

For periodic motion the components of velocity of viscoelastic, synovial fluid (3.1) and pressure (3.2) are periodic functions with respect to the time (see Fig. 5).

For impulsive motion the components of velocity of viscoelastic, synovial fluid and pressure (2.1) are mostly monotone decreasing or increasing functions in the interval from time origin to infinite time values. If time tends to infinity, then pressure function and velocity components tend to the pressure and velocity components for Newtonian properties of synovial fluid in stationary motion and without impulsive effects (see Fig. 2).

5. Numerical calculations

5.1. *Periodic motion*

Numerical calculations of variable in the time and periodic distributions of total pressure p are performed by virtue of equation (3.28) inside the region Ω in the three regular intervals of time namely for $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$. Results are presented in Fig. 8. Time period of joint vibration equals $t_s = 2\pi/\omega_0$. On the boundary of the region Ω pressure has atmospheric values p_{at}. Region Ω is indicated in Fig. 7 and is defined by the following inequalities: $\Omega : 0 \le \varphi \le \pi$, $\pi R/8 \le \vartheta \le \pi R/2$. It is a section of the bowl of the sphere. Numerical calculations are performed for gap height (3.12), (2.19) and for $R = 0.0265$ [m], $\omega_1 = 0.8$ [s^{-1}], $\omega_3 = 0.150$ [s^{-1}], $\omega_0 = 0.02$ [s^{-1}], $\omega_{10} = 0.09$ [s], $\omega_{30} = 0.01$ [s], $\Delta\varepsilon_x = 1$ [μm], $\Delta\varepsilon_y = 0.5$ [μm], $\Delta\varepsilon_z = 3$ [μm], $\eta_0 = 0.20$ [Pas], $\rho_0 = 800$ [kg/m^3]. Minimal value of gap height equals $\varepsilon_{\min} = 3.0$ [μm], maximal value of gap height equals $\varepsilon_{\max} = 7.12$ [μm]. Fig. 8 shows variable in time pressure distribution caused by rotation $\omega_1 = 0.8$ s^{-1} of the bone head in circumference direction φ

and simultaneously caused by rotation $\omega_3 = 0.15\ \mathrm{s}^{-1}$ of the bone head in meridional direction ϑ, for normal hip joint. We take into account angular velocity perturbations $\omega_{10} = 0.09\ \mathrm{s}^{-1}$ on the spherical head in circumference direction φ and simultaneously velocity perturbations $\omega_{30} = 0.010\ \mathrm{s}^{-1}$ on the spherical head of the bone in meridional directions. Acetabulum is motionless i.e. $V_{1k} = V_{3k} = 0$. We assume the gap height perturbations with variable frequencies of joint vibrations in unsteady motion for angular velocity $\omega_0 = 0.02\ \mathrm{s}^{-1}$. For times $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, $t = 3\pi/\omega_0$, $t = 4\pi/\omega_0$ we obtain maximal values of pressure distributions: $\underline{2.077 \times 10^6\ \mathrm{Pa}}$; 2.002×10^6 Pa; $\underline{2.077 \times 10^6\ \mathrm{Pa}}$; 2.002×10^6 Pa; $\underline{2.077 \times 10^6\ \mathrm{Pa}}$; ... , respectively. The first picture in Fig. 8, shows pressure distributions for origin and end time of the period of perturbations of the motion of human joint. The second picture in Fig. 8, shows pressure distributions for middle time point of the period of perturbations of the motion. Afterwards pressure distributions return to the distributions, which had been shown in the first picture in Fig. 8.

Figure 9 presents three curves of capacity distributions versus time inside the time period of joint vibration for three assumptions corresponding to the three cases namely for motion of bone head in meridional direction (◇), circumference direction (x), and simultaneously in circumference and meridian directions (?) respectively. For substantial case in periodic motion we calculate capacity values for following times: $t = 0$ [s], $t = \pi/3\omega_0$ [s], $t = 2\pi/3\omega_0$ [s], $t = \pi/\omega_0$ [s], $t = 4\pi/3\omega_0$ [s], $t = 5\pi/3\omega_0$ [s], $t = 2\pi/\omega_0$ [s],....... For simultaneously periodic motion of bone head in circumference and meridional directions (see symbol □ in Fig. 9),we obtain capacity values: <u>1696 N</u>; 1950 N, 1800 N, 1653 N; 1510 N; 1470 N; <u>1696 N</u>, ... Hence follows that capacity changes in the time period attain 22 percent. It is easy to see that the pressure distributions and capacities in following times $t = 0$ [s], $t = 2\pi/\omega_0$ [s], $t = 4\pi/\omega_0$ [s],... have the same values. Pressure distributions and capacities in times $t = (m-1)\pi/\omega_0$ [s] for $m = 2, 4,$... have the same values too.

5.2. Impulsive motion

In impulsive motion the dimensionless pressure p_{10} and its dimensionless corrections: $p_{11}, p_{12},\ldots$ are determined in lubrication region Ω by virtue of modified Reynolds Equations (2.14), (2.15), taking into account gap height (2.18), (2.19). Region Ω is defined in the same manner as for periodic motion. Numerical calculations are performed for: radius of spherical bone head $R = 0.0265$ m, angular velocity of the impulsive perturbations of acetabulum $\omega_0 = 0.2\ \mathrm{s}^{-1}$, characteristic dimensional time $t_0 = 0.0001$ s. The gap height (2.18), (2.19) is taken into account, where we assume following eccentricities of bone head: $\Delta\varepsilon_x = 4.0$ μm, $\Delta\varepsilon_y = 0.5$ μm, $\Delta\varepsilon_z = 3$ μm. After D. Dowson experiment [1] follows that dy-

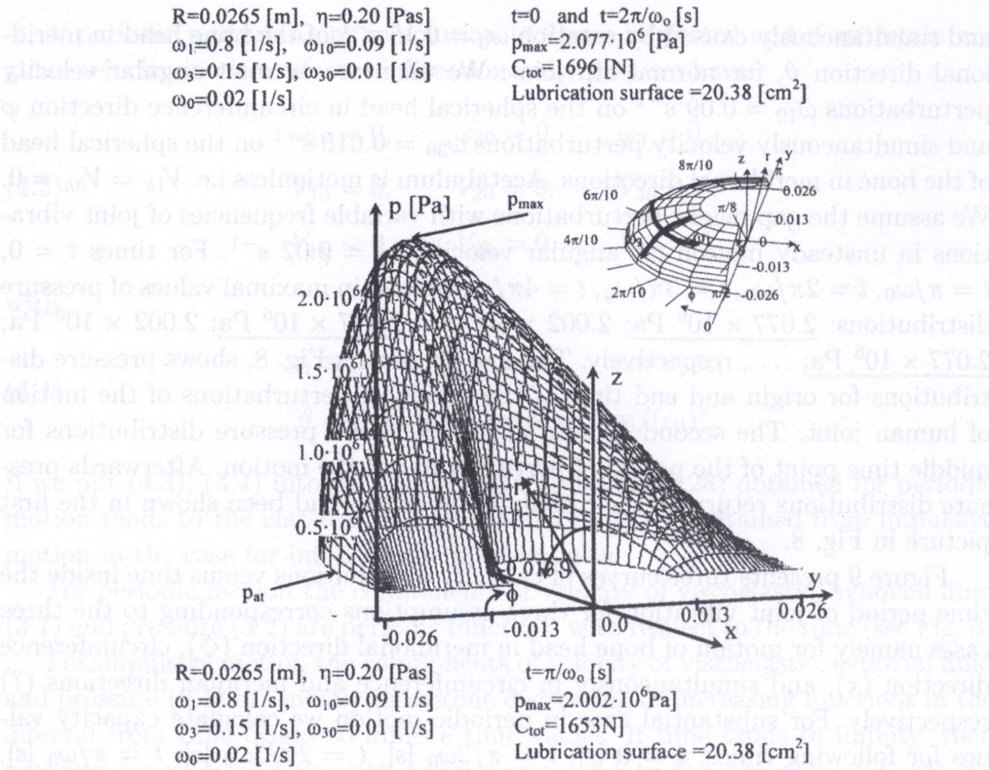


FIG. 8. Pressure distributions in periodic motion caused by rotation of bone head in circumference ϕ direction and simultaneously in meridian direction in ϑ direction where nonzero values of angular velocity $\omega_{10}.8\,\mathrm{s}^{-1}$, $\omega_{30}.15\,\mathrm{s}^{-1}$ and nonzero angular velocity perturbations ω_{10}, ω_{30}, in unsteady flow and nonzero angular velocity perturbations ω_0 of gap height perturbations are taken into account.

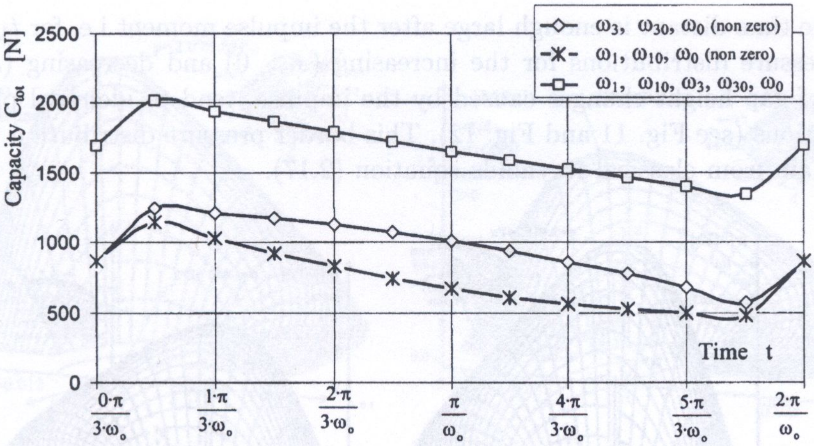


FIG. 9. Capacity distributions versus time in the range of time period for three assumptions corresponding to the three cases namely for motion of bone head in meridian direction (◇), circumference direction (×), and simultaneously in circumference and meridian directions (□) respectively.

namic viscosity of synovial fluid has value $\eta_0 = 0.02$ Pas and according to the theory of viscoelastic fluids [1, 18] we deduce that pseudo viscosity coefficient equals $\beta = 0.000001$ Pas^2 (Appendix G). Moreover we assume: density of synovial fluid $\rho = 800$ kg/m^3, angular velocity of spherical bone head $\omega = 0.8$ s^{-1}, average gap height minimum $\varepsilon_{\min}$ changes in the time interval 0.0001 s $\le t \le 100$ s and attains values from 2 μm to 3.5 μm. Average relative radial clearance has value $\psi \equiv \varepsilon/R = 2 \cdot 10^{-3}$ Strouhal number Str $= 12500$, Re $\cdot\, \psi$Str $= 0.112$, De $\cdot$ Str $= 0.50$. In this case we have $0 \le \beta/\eta_0 t < 1$. For dimensionless times: $t_1 = 1$, $t_1 = 100$, $t_1 = 1000$, $t_1 = 10\,000$, $t_1 = 100\,000$, $t_1 = 1000\,000$ i.e. for dimensional times: $t = 0.0001$ s; $t = 0.01$ s; $t = 0.1$ s; $t = 1.0$ s; $t = 10.0$ s; $t = 100.0$ s respectively and for $s = \pm 0.25$ we obtain pressure distributions in Fig. 10, Figs. 11, 12. To obtain real values of time, we must multiply dimensionless values t_1 by the characteristic time value $t_0 = 0.0001$ s. For example $t_1 = 10000$ denotes 1 s after impulse. Presented time scale enables determination of important pressure changes in some microseconds after injury.

The pressure distributions on the right side in Fig. 10, Fig. 11 are obtained for the increasing of gap height caused by impulsive effects. In this case if the time after the impulse increases, then gap height decreases (see Fig. 4b) hence pressure increases.

The pressure distributions on the left side in Fig. 10, Fig. 11 are obtained for the decreasing of gap height caused by impulsive effects. In this case if the time after the impulse increases, then gap height increases (see Fig. 4b) thus pressure decreases.

If the time distant is enough large after the impulse moment i.e. for $t_1 \to \infty$, then pressure distributions for the increasing ($s > 0$) and decreasing ($s < 0$) effects of gap height changes caused by the impulse, tend to identical pressure distributions (see Fig. 11 and Fig. 12). This border pressure distribution we can also obtain from classical Reynolds equation (2.17).

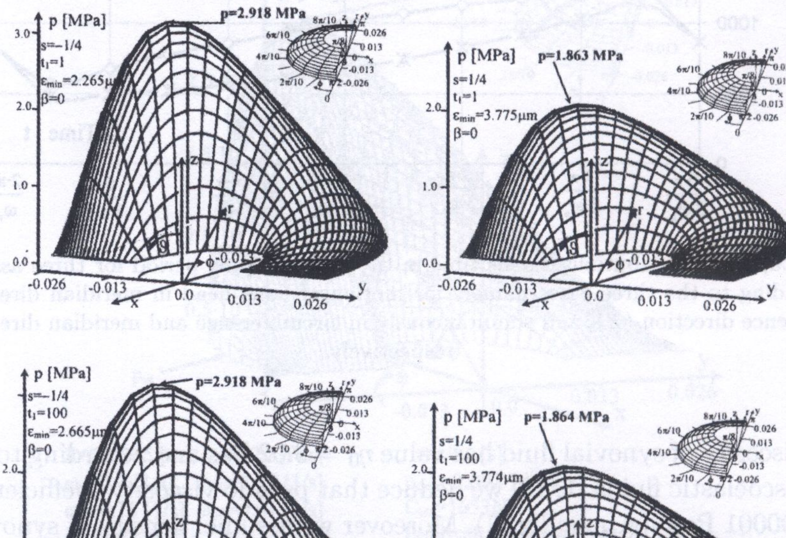


FIG. 10. Dimensional hydrodynamic pressure distributions inside the gap of human spherical hip joint on the region $\Omega : 0 \le \varphi \le \pi$, $\pi R/8 \le \vartheta \le \pi R/2$ in dimensionless times: $t_1 = 1$ (i.e. $t = 0.0001\,\mathrm{s}$), $t_1 = 100$ (i.e. $t = 0.01\,\mathrm{s}$), $t_1 = 1000$ (i.e. $t = 0.1\,\mathrm{s}$), after the impulse moment for the increasing(decreasing) effects of gap height changes see the right (left) column of figures respectively. Results are obtained for the following data: $R = 0.0265\,\mathrm{m}$; $\eta_0 = 0.02\,\mathrm{Pas}$; $\rho = 800\ \mathrm{kg/m^3}$; $\Delta\varepsilon_1 = 4\ \mu\mathrm{m}$; $\Delta\varepsilon_2 = 0.5\ \mu\mathrm{m}$; $\Delta\varepsilon_3 = 3\ \mu\mathrm{m}$; $\psi \equiv \varepsilon/R \approx 2 \cdot 10^{-3}$; $\omega = 0.8\,\mathrm{s}^{-1}$; $\omega_0 = 0.2\,\mathrm{s}^{-1}$; Str = 12500; Re·Str = 0.112; De · Str = 0.50.

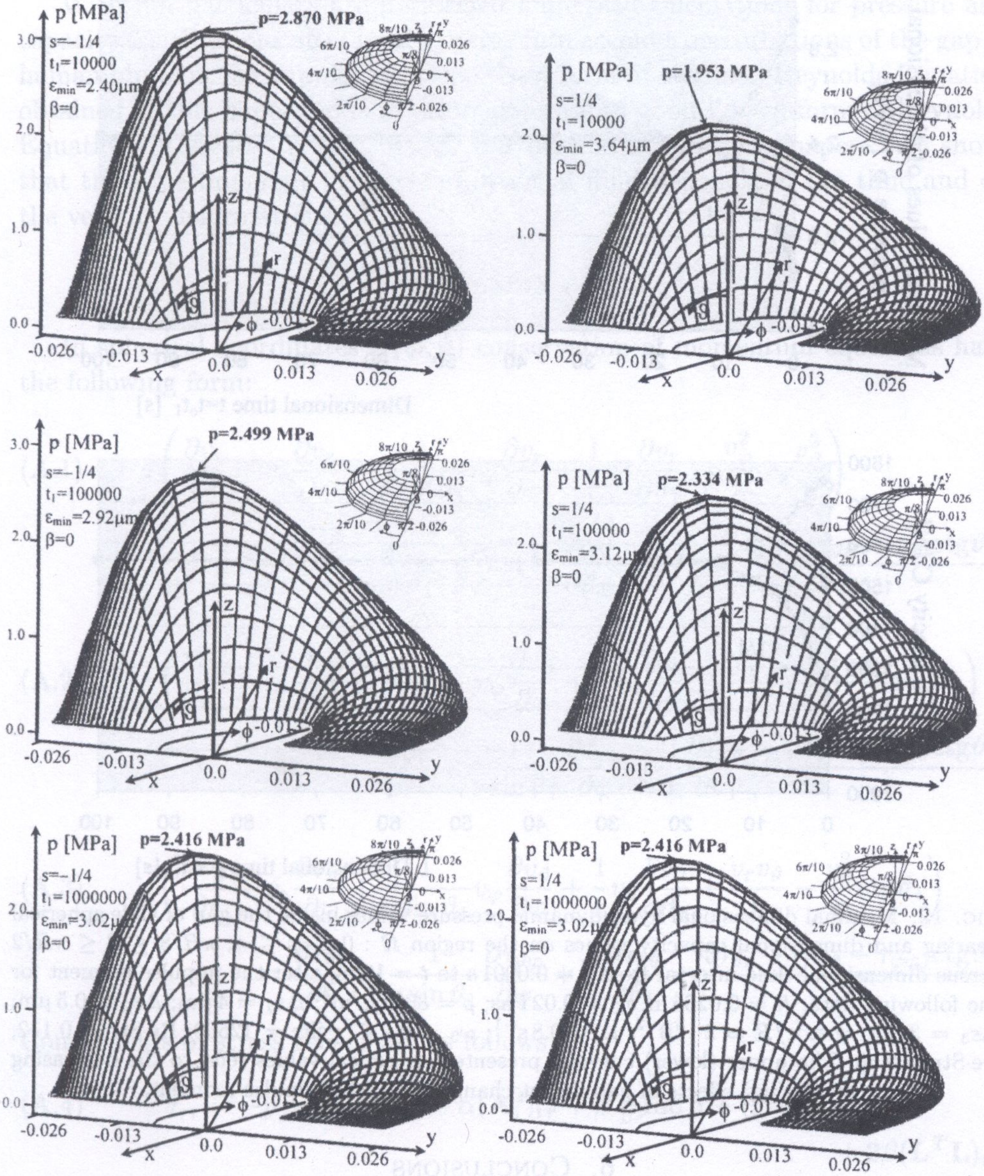


FIG. 11. Dimensional hydrodynamic pressure distributions inside the gap of human spherical hip joint on the region $\Omega : 0 \leq \varphi \leq \pi$, $\pi R/8 \leq \vartheta \leq \pi R/2$ in dimensionless times: $t_1 = 10\,000$, (i.e. $t = 1$ s), $t_1 = 100\,000$ (i.e. $t = 10$ s), $t_1 = 1\,000\,000$ (i.e. $t = 100$ s), after the impulse moment for the increasing(decreasing) effects of gap height changes see the right (left) column of figures respectively. Results are obtained for the following data: $R = 0.0265$ m; $\eta_0 = 0.02$ Pas; $\rho = 800$ kg/m^3; $\Delta\varepsilon_1 = 4\,\mu$m; $\Delta\varepsilon_2 = 0.5\,\mu$m; $\Delta\varepsilon_3 = 3\,\mu$m; $\omega = 0.8\,\text{s}^{-1}$; $\omega_0 = 0.2\,\text{s}^{-1}$; $\Psi \equiv \varepsilon/R \approx 2 \cdot 10^{-3}$; Str = 12500; Re · Str = 0.112; De · Str = 0.50.

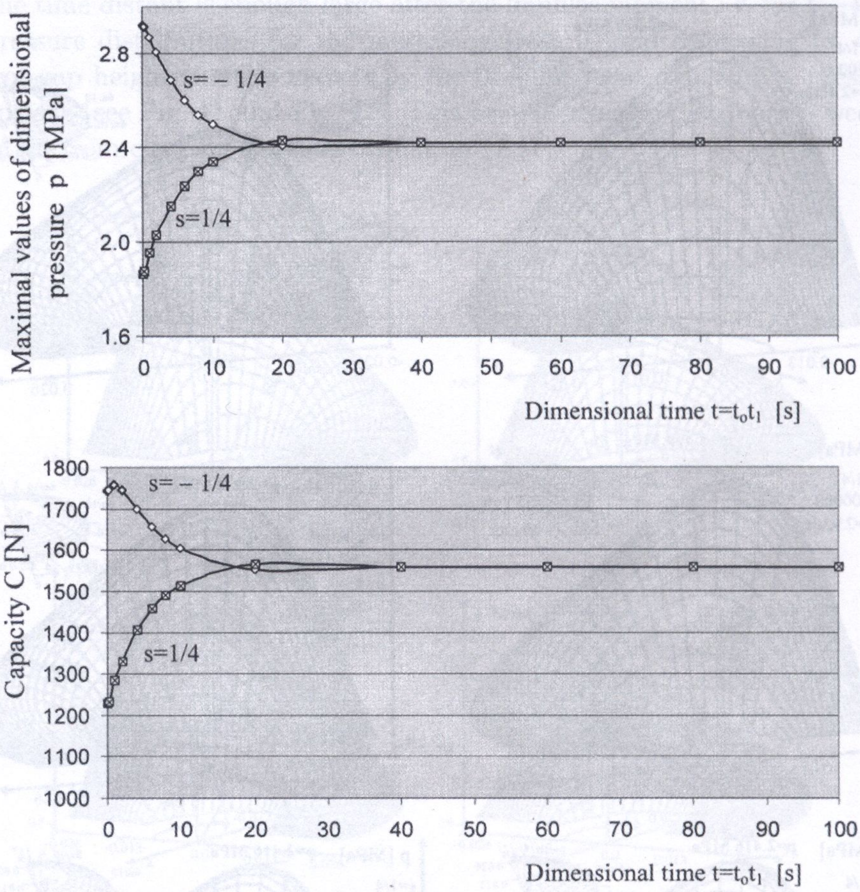


FIG. 12. Maximal dimensional hydrodynamic pressure values inside the gap of slide spherical bearing and dimensional capacity values on the region $\Omega : 0 \leq \varphi \leq \pi$, $\pi R/8 \leq \vartheta \leq \pi R/2$ versus dimensional time interval from $t = 0.0001$ s to $t = 100$ s, after the impulse moment for the following data: $R = 0.0265$ m; $\eta_0 = 0.02$ Pas; $\rho = 800\,\mathrm{kg/m^3}$; $\Delta\varepsilon_1 = 4\,\mu\mathrm{m}$; $\Delta\varepsilon_2 = 0.5\,\mu\mathrm{m}$; $\Delta\varepsilon_3 = 3\,\mu\mathrm{m}$; $\psi \equiv \varepsilon/R \approx 2 \cdot 10^{-3}$; $\omega = 0.8\,\mathrm{s^{-1}}$; $\omega_0 = 0.2\,\mathrm{s^{-1}}$; Str = 12500; Re·Str = 0.112; De·Str = 0.50. The upper (lower) curve of presented numerical values refer to the decreasing (increasing) effects of gap height changes after the impulse moment.

6. CONCLUSIONS

Present paper shows analytical and numerical comparisons of velocities of synovial fluid and pressure between impulsive and periodic motion occurring in gap of spherical human hip joint. Periodic perturbations and impulsive motions during the unsteady lubrication and simultaneously viscoelastic properties of the fluid are taken into account. It is proved, that principle of superposition is not valid for the pressure and capacity values, for simultaneous motion of bone head of human hip joint in two directions.

With full particulars are performed numerical calculations for pressure and capacity distributions after injury taking into account perturbations of the gap of human hip joint for impulsive motion. New form of modified Reynolds Equation obtained in this paper tends in particular case to good known form of Reynolds Equation for steady motion, which was derived in foregoing papers. We show, that the total apparent viscosity of synovial fluid depends on the time and on the velocity deformations.

Appendix A.

In spherical coordinates (r, φ, ϑ) conservation of momentum equations have the following form:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_r}{\partial\varphi}+\frac{1}{r}v_\vartheta\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\varphi^2}{r}-\frac{v_\vartheta^2}{r}\right)$$
$$=\frac{\partial\tau_{rr}}{\partial r}+\frac{1}{r\sin\vartheta_1}\frac{\partial\tau_{r\varphi}}{\partial\varphi}+\frac{1}{r}\frac{\partial\tau_{r\vartheta}}{\partial\vartheta_1}+\frac{2\tau_{rr}-\tau_{\varphi\varphi}-\tau_{\vartheta\vartheta}-\tau_{r\vartheta}\mathrm{ctg}\vartheta_1}{r},\tag{A.1}$$

$$\rho\left(\frac{\partial v_\varphi}{\partial t}+v_r\frac{\partial v_\varphi}{\partial r}+\frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\varphi}{\partial\varphi}+\frac{1}{r}v_\vartheta\frac{\partial v_\varphi}{\partial\vartheta_1}+\frac{v_rv_\varphi}{r}+\frac{v_\vartheta v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)$$
$$=\frac{\partial\tau_{\varphi r}}{\partial r}+\frac{1}{r\sin\vartheta_1}\frac{\partial\tau_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}\frac{\partial\tau_{\varphi\vartheta}}{\partial\vartheta_1}+\frac{3\tau_{\varphi r}+2\tau_{\varphi\vartheta}\mathrm{ctg}\vartheta_1}{r},\tag{A.2}$$

$$\rho\left(\frac{\partial v_\vartheta}{\partial t}+v_r\frac{\partial v_\vartheta}{\partial r}+\frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\vartheta}{\partial\varphi}+\frac{1}{r}v_\vartheta\frac{\partial v_\vartheta}{\partial\vartheta_1}+\frac{v_rv_\vartheta}{r}-\frac{v_\vartheta^2}{r}\mathrm{ctg}\vartheta_1\right)$$
$$=\frac{\partial\tau_{\vartheta r}}{\partial r}+\frac{1}{r\sin\vartheta_1}\frac{\partial\tau_{\vartheta\varphi}}{\partial\varphi}+\frac{1}{r}\frac{\partial\tau_{\vartheta\vartheta}}{\partial\vartheta_1}+\frac{3\tau_{\vartheta r}+(\tau_{\vartheta\vartheta}-\tau_{\varphi\varphi})\mathrm{ctg}\vartheta_1}{r}.\tag{A.3}$$

Components of stress tensor are as follows:

$$\tau_{rr}=-p+\eta_0(\mathbf{A}_1)_{rr}+\alpha(\mathbf{A}_1^2)_{rr}+\beta\left[\mathrm{grad}\,\mathbf{a}+(\mathrm{grad}\,\mathbf{a})^T\right]_{rr}+2\beta(\mathbf{L}^T\mathbf{L})_{rr},\tag{A.4}$$

$$\tau_{\varphi\varphi}=-p+\eta_0(\mathbf{A}_1)_{\varphi\varphi}+\alpha(\mathbf{A}_1^2)_{\varphi\varphi}+\beta\left[\mathrm{grad}\,\mathbf{a}+(\mathrm{grad}\,\mathbf{a})^T\right]_{\varphi\varphi}+2\beta(\mathbf{L}^T\mathbf{L})_{\varphi\varphi},\tag{A.5}$$

$$\tau_{\vartheta\vartheta}=-p+\eta_0(\mathbf{A}_1)_{\vartheta\vartheta}+\alpha(\mathbf{A}_1^2)_{\vartheta\vartheta}+\beta\left[\mathrm{grad}\,\mathbf{a}+(\mathrm{grad}\,\mathbf{a})^T\right]_{\vartheta\vartheta}+2\beta(\mathbf{L}^T\mathbf{L})_{\vartheta\vartheta},\tag{A.6}$$

$$\text{(A.7)}\quad \tau_{r\varphi} = +\eta_0(\mathbf{A}_1)_{r\varphi} + \alpha(\mathbf{A}_1^2)_{r\varphi} + \beta\left[\operatorname{grad}\mathbf{a} + (\operatorname{grad}\mathbf{a})^T\right]_{r\varphi} + 2\beta(\mathbf{L}^T\mathbf{L})_{r\varphi},$$

$$\text{(A.8)}\quad \tau_{\varphi\vartheta} = +\eta_0(\mathbf{A}_1)_{\varphi\vartheta} + \alpha(\mathbf{A}_1^2)_{\varphi\vartheta} + \beta\left[\operatorname{grad}\mathbf{a} + (\operatorname{grad}\mathbf{a})^T\right]_{\varphi\vartheta} + 2\beta(\mathbf{L}^T\mathbf{L})_{\varphi\vartheta},$$

$$\text{(A.9)}\quad \tau_{r\vartheta} = +\eta_0(\mathbf{A}_1)_{r\vartheta} + \alpha(\mathbf{A}_1^2)_{r\vartheta} + \beta\left[\operatorname{grad}\mathbf{a} + (\operatorname{grad}\mathbf{a})^T\right]_{r\vartheta} + 2\beta(\mathbf{L}^T\mathbf{L})_{r\vartheta}.$$

Tensor $\mathbf{G} = \operatorname{grad}\mathbf{a}$ of rank two has nine coordinates presented in following matrix:

$$\text{(A.10)}\quad \begin{Vmatrix} G_{rr} & G_{r\varphi} & G_{rz} \\ G_{\varphi r} & G_{\varphi\varphi} & G_{\varphi z} \\ G_{zr} & G_{z\varphi} & G_{zz} \end{Vmatrix} = \begin{Vmatrix} \dfrac{\partial a_r}{\partial r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_r}{\partial\varphi} - \dfrac{a_\varphi}{r} & \dfrac{1}{r}\dfrac{\partial a_r}{\partial\vartheta_1} - \dfrac{a_\vartheta}{r} \\ \dfrac{\partial a_\varphi}{\partial r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_\varphi}{\partial\varphi} + \dfrac{a_r}{r} + \dfrac{a_\vartheta}{r}\operatorname{ctg}\vartheta_1 & \dfrac{1}{r}\dfrac{\partial a_\varphi}{\partial\vartheta_1} \\ \dfrac{\partial a_\vartheta}{\partial r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_\vartheta}{\partial\varphi} - \dfrac{a_\varphi}{r}\operatorname{ctg}\vartheta_1 & \dfrac{1}{r}\dfrac{\partial a_\vartheta}{\partial r} + \dfrac{a_r}{r} \end{Vmatrix}$$

Coordinates of the tensor:

$$\text{(A.11)}\quad \mathbf{U} \equiv \operatorname{grad}\mathbf{a} + (\operatorname{grad}\mathbf{a})^T$$

are presented in the following symmetrical matrix:

$$\text{(A.12)}\quad \begin{Vmatrix} U_{rr} & U_{r\varphi} & U_{rz} \\ U_{\varphi r} & U_{\varphi\varphi} & U_{\varphi z} \\ U_{zr} & U_{z\varphi} & U_{zz} \end{Vmatrix} = \begin{Vmatrix} 2\dfrac{\partial a_r}{\partial r} & \dfrac{\partial a_\varphi}{\partial r} + \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_r}{\partial\varphi} - \dfrac{a_\varphi}{r} & \dfrac{\partial a_\vartheta}{\partial r} + \dfrac{1}{r}\dfrac{\partial a_r}{\partial\vartheta_1} - \dfrac{a_\vartheta}{r} \\ \dfrac{\partial a_\varphi}{\partial r} + \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_r}{\partial\varphi} - \dfrac{a_\varphi}{r} & 2\left(\dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_\varphi}{\partial\varphi} + \dfrac{a_r}{r} + \dfrac{a_\vartheta}{r}\operatorname{ctg}\vartheta_1\right) & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_\vartheta}{\partial\varphi} + \dfrac{1}{r}\dfrac{\partial a_\varphi}{\partial\vartheta_1} - \dfrac{a_\varphi}{r}\operatorname{ctg}\vartheta_1 \\ \dfrac{\partial a_\vartheta}{\partial r} + \dfrac{1}{r}\dfrac{\partial a_r}{\partial\vartheta_1} - \dfrac{a_\vartheta}{r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial a_\vartheta}{\partial\varphi} + \dfrac{1}{r}\dfrac{\partial a_\varphi}{\partial\vartheta_1} - \dfrac{a_\varphi}{r}\operatorname{ctg}\vartheta_1 & 2\left(\dfrac{1}{r}\dfrac{\partial a_\vartheta}{\partial r} + \dfrac{a_r}{r}\right) \end{Vmatrix}$$

where:

$$\text{(A.13)}\quad a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi^2}{r} + \frac{v_\vartheta}{r}\frac{\partial v_r}{\partial\vartheta_1} - \frac{v_\vartheta^2}{r},$$

$$\text{(A.14)}\quad a_\varphi = \frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r v_\varphi}{r} + \frac{v_\vartheta}{r}\frac{\partial v_\varphi}{\partial\vartheta_1} + \frac{v_\varphi v_\vartheta}{r}\operatorname{ctg}\vartheta_1,$$

$$\text{(A.15)}\quad a_\vartheta = \frac{\partial v_\vartheta}{\partial t} + v_r\frac{\partial v_\vartheta}{\partial r} + \frac{v_\varphi}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi} + \frac{v_r v_\vartheta}{r} + \frac{v_\vartheta}{r}\frac{\partial v_\vartheta}{\partial\vartheta_1} - \frac{v_\varphi^2}{r}\operatorname{ctg}\vartheta_1.$$

Tensor $\mathbf{L} = \operatorname{grad} \mathbf{v}$ of rank two has nine coordinates presented in following matrix:

$$\text{(A.16)} \quad \begin{Vmatrix} L_{rr} & L_{r\varphi} & L_{rz} \\ L_{\varphi r} & L_{\varphi\varphi} & L_{\varphi z} \\ L_{zr} & L_{z\varphi} & L_{zz} \end{Vmatrix} = \begin{Vmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial v_r}{\partial \varphi} - \dfrac{v_\varphi}{r} & \dfrac{1}{r}\dfrac{\partial v_r}{\partial \vartheta_1} - \dfrac{v_\vartheta}{r} \\ \dfrac{\partial v_\varphi}{\partial r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial v_\varphi}{\partial \varphi} + \dfrac{v_r}{r} + \dfrac{v_\vartheta}{r}\operatorname{ctg}\vartheta_1 & \dfrac{1}{r}\dfrac{\partial v_\varphi}{\partial \vartheta_1} \\ \dfrac{\partial v_\vartheta}{\partial r} & \dfrac{1}{r\sin\vartheta_1}\dfrac{\partial v_\vartheta}{\partial \varphi} - \dfrac{v_\varphi}{r}\operatorname{ctg}\vartheta_1 & \dfrac{1}{r}\dfrac{\partial v_\vartheta}{\partial r} + \dfrac{v_r}{r} \end{Vmatrix}$$

Tensor $\mathbf{L}^* = \mathbf{L}^T\mathbf{L}$ is symmetrical and has following components:

$$L^*_{rr} = \left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{\partial v_\varphi}{\partial r}\right)^2 + \left(\frac{\partial v_\vartheta}{\partial r}\right)^2,$$

$$L^*_{\varphi\varphi} = \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)^2 + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\vartheta}{r}\operatorname{ctg}\vartheta_1\right)^2 + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial \varphi} - \frac{v_\varphi}{r}\operatorname{ctg}\vartheta_1\right)^2,$$

$$L^*_{\vartheta\vartheta} = \left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\varphi}{\partial \vartheta_1}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta_1} - \frac{v_r}{r}\right)^2,$$

$$\text{(A.17)} \quad L^*_{r\varphi} = \frac{\partial v_r}{\partial r}\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right) + \frac{\partial v_\varphi}{\partial r}\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r} + \frac{v_\vartheta}{r}\operatorname{ctg}\vartheta_1\right) + \frac{\partial v_\vartheta}{\partial r}\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial \varphi} - \frac{v_\varphi}{r}\operatorname{ctg}\vartheta_1\right),$$

$$L^*_{\vartheta\vartheta} = \frac{\partial v_r}{\partial r}\left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r}\right) + \frac{\partial v_\varphi}{\partial r}\left(\frac{1}{r}\frac{\partial v_\varphi}{\partial \vartheta_1}\right) + \frac{\partial v_\vartheta}{\partial r}\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta_1} - \frac{v_r}{r}\right),$$

$$L^*_{\varphi\vartheta} = \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)\left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r}\right) + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\vartheta}{r\operatorname{ctg}\vartheta_1}\right)\left(\frac{1}{r}\frac{\partial v_\varphi}{\partial \vartheta_1}\right) + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial \varphi} - \frac{v_\varphi}{r}\operatorname{ctg}\vartheta_1\right)\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta_1} - \frac{v_r}{r}\right).$$

We put results (A.10)–(A.17) into Eqs. (A.4)–(A.9). Hence we obtain following form of stress components:

$$\text{(A.18)}\quad \tau_{rr} = -p + \eta_0\left(2\frac{\partial v_r}{\partial r}\right) + \alpha\left[4\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)^2 + \left(\frac{\partial v_\vartheta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \vartheta_1} - \frac{v_\vartheta}{r}\right)^2\right]$$
$$+ 2\beta\left[\frac{\partial^2 v_r}{\partial r\partial t} + \frac{\partial}{\partial r}\left(v_r\frac{\partial v_r}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_r}{\partial \varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_r}{\partial \vartheta_1} - \frac{v_\varphi^2}{r} - \frac{v_\vartheta^2}{r}\right)\right]$$
$$+ 2\beta\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{\partial v_\varphi}{\partial r}\right)^2 + \left(\frac{\partial v_\vartheta}{\partial r}\right)^2\right],$$

$$\text{(A.19)}\quad \tau_{\varphi\varphi} = -p + \eta_0\left[\frac{2}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} + \frac{2v_\vartheta}{r}\text{ctg}\vartheta_1\right]$$
$$+ \alpha\left[\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)^2 + \left(\frac{2}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} - \frac{2v_\vartheta}{r}\text{ctg}\vartheta_1\right)^2\right.$$
$$\left. + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial \varphi} + \frac{\partial v_r}{r\partial \vartheta_1} - \frac{v_\vartheta}{r}\text{ctg}\vartheta_1\right)^2\right]$$
$$+ \beta\left[\left(\frac{2}{r\sin\vartheta_1}\frac{\partial^2 v_\varphi}{\partial \varphi\partial t} + \frac{2}{r}\frac{\partial v_1}{\partial t} + \frac{2\text{ctg}\vartheta_1}{r}\frac{\partial v_\vartheta}{\partial t}\right)\right.$$
$$+ \frac{2}{r\sin\vartheta_1}\frac{\partial}{\partial \varphi}\left(v_r\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_\varphi}{\partial \vartheta_1} + \frac{v_r v_\varphi}{r} + \frac{v_\vartheta v_\varphi}{r}\text{ctg}\vartheta_1\right)$$
$$+ \frac{2}{r}\left(v_r\frac{\partial v_r}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_r}{\partial \varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_r}{\partial \vartheta_1} - \frac{v_\varphi^2}{r} - \frac{v_\vartheta^2}{r}\right)$$
$$\left. + \frac{2\text{ctg}\vartheta_1}{r}\left(v_r\frac{\partial v_\vartheta}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\vartheta}{\partial \varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_\vartheta}{\partial \vartheta_1} + \frac{v_r v_\vartheta}{r} - \frac{v_\vartheta^2}{r}\text{ctg}\vartheta_1\right)\right]$$
$$+ 2\beta\left[\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)^2 + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\vartheta}{r}\text{ctg}\vartheta_1\right)^2\right.$$
$$\left. + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial \varphi} - \frac{v_\varphi}{r}\text{ctg}\vartheta_1\right)^2\right],$$

$$\text{(A.20)}\quad \tau_{\vartheta\vartheta} = -p + 2\eta_0\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta_1} + \frac{v_r}{r}\right) + \alpha\left[\left(\frac{\partial v_\vartheta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1} - \frac{v_\vartheta}{r}\right)^2 + \left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi} + \frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta} - \frac{v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)^2 + \left(\frac{2}{r}\frac{\partial v_\vartheta}{\partial\vartheta_1} + \frac{2v_r}{r}\right)^2\right]$$

$$+ \beta\left[\frac{2}{r}\left(\frac{\partial^2 v_\vartheta}{\partial\vartheta_1\partial t} + \frac{\partial v_r}{\partial t}\right) + \frac{2}{r}\frac{\partial}{\partial\vartheta_1}\left(v_r\frac{\partial v_\vartheta}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\vartheta}{\partial\varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_\vartheta}{\partial\vartheta_1} + \frac{v_r v_\vartheta}{r} - \frac{v_\varphi^2}{r}\mathrm{ctg}\vartheta_1\right) + \frac{2}{r}\left(v_r\frac{\partial v_r}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_r}{\partial\varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_r}{\partial\vartheta_1} + \frac{v_\varphi^2}{r} - \frac{v_\vartheta^2}{r}\right)\right]$$

$$+ 2\beta\left[\left(\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1} - \frac{v_\vartheta}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta_1}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta_1} + \frac{v_r}{r}\right)^2\right],$$

$$\text{(A.21)}\quad \tau_{r\varphi} = \eta_0\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi}{r}\right)$$

$$+ \alpha\left[2\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi}{r}\right)\left(\frac{\partial v_r}{\partial r} + \frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r} + \frac{v_\vartheta}{r}\mathrm{ctg}\vartheta_1\right) + \left(\frac{\partial v_\vartheta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1} - \frac{v_\vartheta}{r}\right)\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi} + \frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta_1} - \frac{v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)\right]$$

$$+ \beta\left[\frac{\partial}{\partial t}\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi}{r}\right) + \left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\left(v_r\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\varphi}{\partial\varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_\varphi}{\partial\vartheta_1} + \frac{v_\varphi v_r}{r} + \frac{v_\varphi v_\vartheta}{r}\mathrm{ctg}\vartheta_1\right) + \frac{1}{r\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left(v_r\frac{\partial v_r}{\partial r} + \frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_r}{\partial\varphi} + \frac{1}{r}v_\vartheta\frac{\partial v_r}{\partial\vartheta_1} + \frac{v_\varphi^2}{r} + \frac{v_\vartheta^2}{r}\right)\right]$$

$$+ 2\beta\left[\frac{\partial v_r}{\partial r}\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi}{r}\right) + \frac{\partial v_\varphi}{\partial r}\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r} + \frac{v_\vartheta}{r}\mathrm{ctg}\vartheta_1\right) + \frac{\partial v_\vartheta}{\partial r}\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi} - \frac{v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)\right],$$

$$(\text{A.22})\quad \tau_{r\vartheta}=\eta_0\left[\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\vartheta}{r}+\frac{\partial v_\vartheta}{\partial r}\right]$$

$$+\alpha\left[2\left(\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\vartheta}{r}+\frac{\partial v_\vartheta}{\partial r}\right)\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta_1}+\frac{v_r}{r}\right)\right.$$

$$\left.+\left(\frac{\partial v_\varphi}{\partial r}+\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi}-\frac{v_\vartheta}{r}\right)\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi}+\frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta_1}-\frac{v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)\right]$$

$$+\beta\left[\frac{\partial}{\partial t}\left(\frac{\partial v_\vartheta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\vartheta}{r}\right)\right.$$

$$+\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\left(v_r\frac{\partial v_\vartheta}{\partial r}+\frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_\vartheta}{\partial\varphi}+\frac{1}{r}v_\vartheta\frac{\partial v_\varphi}{\partial\vartheta_1}+\frac{v_\vartheta v_r}{r}-\frac{v_\varphi^2}{r}\mathrm{ctg}\vartheta_1\right)$$

$$\left.+\frac{1}{r}\frac{\partial}{\partial\vartheta_1}\left(v_r\frac{\partial v_r}{\partial r}+\frac{1}{r\sin\vartheta_1}v_\varphi\frac{\partial v_r}{\partial\varphi}+\frac{1}{r}v_\vartheta\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\varphi^2}{r}-\frac{v_\vartheta^2}{r}\right)\right]$$

$$+2\beta\left[\frac{\partial v_r}{\partial r}\left(\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\vartheta}{r}\right)\frac{\partial v_\varphi}{\partial r}\left(\frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta_1}\right)+\frac{\partial v_\vartheta}{\partial r}\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta_1}-\frac{v_r}{r}\right)\right],$$

$$(\text{A.23})\quad \tau_{\varphi\vartheta}=\eta_0\left[\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi}+\frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta_1}-\frac{v_\varphi}{r}\mathrm{ctg}\vartheta_1\right]$$

$$+\alpha\left[\left(\frac{\partial v_\varphi}{\partial r}+\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi}-\frac{v_\varphi}{r}\right)\left(\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\vartheta}{r}+\frac{\partial v_\vartheta}{\partial r}\right)\right.$$

$$+2\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi}+\frac{1}{r}\frac{\partial v_\varphi}{\partial\vartheta_1}-\frac{v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)$$

$$\left.\times\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_\varphi}{\partial\varphi}+\frac{2v_r}{r}+\frac{v_\vartheta}{r}\mathrm{ctg}\vartheta_1+\frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta}\right)\right]$$

$$+\beta\left[\frac{1}{r}\left(\frac{1}{\sin\vartheta_1}\frac{\partial^2 v_\vartheta}{\partial\varphi\partial t}+\frac{\partial^2 v_\varphi}{\partial\vartheta_1\partial t}-\mathrm{ctg}\vartheta_1\frac{\partial v_\varphi}{\partial t}\right)\right.$$

$$+\frac{1}{r\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left(v_r\frac{\partial v_\vartheta}{\partial r}+\frac{v_\varphi}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi}+\frac{v_\vartheta}{r}\frac{\partial v_\vartheta}{\partial\vartheta}+\frac{v_r v_\vartheta}{r}-\frac{v_\varphi^2}{r}\mathrm{ctg}\vartheta_1\right)$$

$$\left.+\left(\frac{1}{r}\frac{\partial}{\partial\vartheta_1}-\frac{\mathrm{ctg}\vartheta_1}{r}\right)\left(v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r\sin\vartheta_1}\frac{\partial v_\vartheta}{\partial\varphi}+\frac{v_\vartheta}{r}\frac{\partial v_\varphi}{\partial\vartheta_1}+\frac{v_r v_\varphi}{r}+\frac{v_\vartheta v_\varphi}{r}\mathrm{ctg}\vartheta_1\right)\right]$$

$$+2\beta\left[\left(\frac{1}{r\sin\vartheta_1}\frac{\partial v_r}{\partial\varphi}-\frac{v_\varphi}{r}\right)\left(\frac{1}{r}\frac{\partial v_r}{\partial\vartheta_1}-\frac{v_\vartheta}{r}\right)\right.$$

$$\text{(A.23) [cont.]} \quad + \left(\frac{1}{r \sin \vartheta_1} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\vartheta}{r} \operatorname{ctg} \vartheta_1 \right) \frac{1}{r} \frac{\partial v_\varphi}{\partial \vartheta_1} + \left(\frac{1}{r \sin \vartheta_1} \frac{\partial v_\vartheta}{\partial \varphi} - \frac{v_\varphi}{r} \operatorname{ctg} \vartheta_1 \right) \left(\frac{1}{r} \frac{\partial v_\vartheta}{\partial \vartheta_1} - \frac{v_r}{r} \right) \Bigg],$$

and $\tau_{\varphi\vartheta} = \tau_{\vartheta\varphi}$, $\tau_{\varphi r} = \tau_{r\varphi}$, $\tau_{r\vartheta} = \tau_{\vartheta r}$.

Continuity equation has the following form:

$$\text{(A.24)} \qquad \frac{1}{r^2 \sin \vartheta_1} \left[\frac{\partial}{\partial r_1} (v_r r^2 \sin \vartheta_1) + \frac{\partial}{\partial \varphi} (r v_\varphi) + \frac{\partial}{\partial \vartheta_1} (v_\vartheta r \sin \vartheta_1) \right] = 0.$$

We assume following dependencies between dimensionless synovial fluid velocity components $v_{\varphi\Sigma}$, $v_{r\Sigma}$, $v_{\vartheta\Sigma}$, pressure $p_{1\Sigma}$ and adequate dimensional values in following form:

$$\text{(A.25)} \quad v_\varphi = U v_{\varphi\Sigma}, \qquad v_r = U \Psi v_{r\Sigma}, \qquad v_\vartheta = U v_{\vartheta\Sigma}, \qquad p = (U R \eta_0 / \varepsilon_0^2) p_{1\Sigma}$$

Dependence between dimensional and dimensionless radial coordinate and time has the form:

$$\text{(A.26)} \qquad r = R + \varepsilon r_1 = R(1 + \Psi r_1), \qquad \frac{\partial}{\partial r} = \frac{1}{\varepsilon_0} \frac{\partial}{\partial r_1}, \qquad t = t_o t_1.$$

Deborah Numbers, Strouhal and Reynolds Numbers are as follows:

$$\text{(A.27)} \qquad \mathrm{De}_\beta \equiv \frac{\beta\omega}{\eta_0}, \qquad \mathrm{De}_\alpha \equiv \frac{\alpha\omega}{\eta_0}, \qquad \mathrm{Str} \equiv \frac{1}{\omega t_0}, \qquad \mathrm{Re} \equiv \frac{\rho\omega R \varepsilon_0}{\eta_0}.$$

In the system of equations (A.1)–(A.3) we put formulae (A.18)–(A.23) and next we put dependencies (A.25), (A.26), (A.27). Hence we obtain following system of equations of conservation of momentum in following dimensionless form:

$$\text{(A.28)} \qquad \mathrm{Re}\,\Psi \mathrm{Str} \frac{\partial v_{\varphi\Sigma}}{\partial t_1} + O(\mathrm{Re}\,\Psi) = \frac{1}{\sin \vartheta_1} \frac{\partial p_{1\Sigma}}{\partial \varphi} + \frac{\partial}{\partial r_1} \left(\frac{\partial v_{\varphi\Sigma}}{\partial r_1} \right) + O(\mathrm{De}_\alpha) + O(\Psi) + \mathrm{De}_\beta \, \mathrm{Str} \frac{\partial^3 v_{\varphi\Sigma}}{\partial t_1 \partial r_1^2},$$

$$\text{(A.29)} \qquad O(\mathrm{De}_\alpha) = \frac{\partial p_{1\Sigma}}{\partial r_1},$$

$$\text{(A.30)} \qquad \mathrm{Re}\,\Psi \mathrm{Str} \frac{\partial v_{\vartheta\Sigma}}{\partial t_1} + O(\mathrm{Re}\,\Psi) = \frac{\partial p_{1\Sigma}}{\partial \vartheta_1} + \frac{\partial}{\partial r_1} \left(\frac{\partial v_{\vartheta\Sigma}}{\partial r_1} \right) + O(\mathrm{De}_\alpha) + O(\Psi) + \mathrm{De}_\beta \mathrm{Str} \frac{\partial^3 v_{\vartheta\Sigma}}{\partial t_1 \partial r_1^2}.$$

In continuity equation (A.24) we put dependencies (A.25), (A.26). Thus continuity equation has the form:

$$\text{(A.31)} \qquad (\sin\vartheta_1)\frac{\partial v_{r\Sigma}}{\partial r_1} + \frac{\partial v_{\varphi\Sigma}}{\partial\varphi} + \frac{\partial}{\partial\vartheta_1}(v_{\vartheta\Sigma}\sin\vartheta_1) + O(\Psi) = 0.$$

Neglecting the terms of order O(De_α), O(Ψ), O($\mathrm{Re}\,\Psi$), then dimensionless system of equations of conservation of momentum and continuity equation (A.28)–(A.31) tend to the dimensional form (1.4)–(1.7).

Appendix B.

In Eqs. (1.4)–(1.7) the derivatives with respect of the variable t_1, r_1 can be replaced by the derivatives with respect of the one variable χ only, by using the following relationships:

$$\text{(B.1)} \qquad \frac{\partial}{\partial t_1} = \frac{\partial}{\partial\chi}\frac{\partial\chi}{\partial t_1} = -\frac{1}{4}\sqrt{\mathrm{Res}}\,\frac{r_1}{t_1\sqrt{t_1}}\frac{\partial}{\partial\chi} = -\frac{\chi}{2t_1}\frac{\partial}{\partial\chi},$$

$$\text{(B.2)} \qquad \frac{\partial^2}{\partial r_1^2} = \frac{\partial}{\partial r_1}\left(\frac{\partial}{\partial r_1}\right) = \frac{\partial}{\partial\chi}\left(\frac{\partial}{\partial\chi}\frac{\partial\chi}{\partial r_1}\right)\frac{\partial\chi}{\partial r_1} = \frac{\mathrm{Res}}{4t_1}\frac{\partial^2}{\partial\chi^2},$$

$$\text{(B.3)} \qquad \frac{\partial^3}{\partial t_1\partial r_1^2} = \frac{\partial}{\partial t_1}\left(\frac{\mathrm{Res}}{4t_1}\frac{\partial^2}{\partial\chi^2}\right)$$

$$= -\frac{\mathrm{Res}}{4t_1^2}\frac{\partial^2}{\partial\chi^2} + \frac{\mathrm{Res}}{4t_1}\frac{\partial}{\partial\chi}\left(\frac{\partial^2}{\partial\chi^2}\right)\frac{\partial\chi}{\partial t_1} = -\frac{\mathrm{Res}}{4t_1^2}\left(\frac{\partial^2}{\partial\chi^2} + \frac{\chi}{2}\frac{\partial^3}{\partial\chi^3}\right).$$

Next, the series (2.1) were put into the changed system (1.4)–(1.7), where the variables t_1, r_1 were replaced by the variable χ, and $\mathrm{Res} \equiv \mathrm{Re}\Psi\mathrm{Str}$.

Appendix C.

General solutions of ordinary differential equations:

$$\text{(C.1)} \qquad \frac{d^2 v_{i0}}{d\chi^2} + 2\chi\frac{dv_{i0}}{d\chi} = \frac{1}{N_i^2}\frac{\partial p_{10}}{\partial\alpha_i},$$

for $i = \varphi$, ϑ, $\alpha_\varphi \equiv \varphi$, $\alpha_\vartheta \equiv \vartheta_1$ and $(N_\varphi)^2 \equiv N^2\sin(\vartheta_1)$, $N_\vartheta \equiv N$, has the following form:

$$\text{(C.2)} \qquad v_{i0}(\chi) = C_{i1}v_{01}(\chi) + v_{02}C_{i2} + v_{i03}(\chi),$$

where: C_{i1}, C_{i2} are integral constants and particular solutions of homogeneous and non homogeneous differential equations are as follows:

$$\text{(C.3)} \qquad v_{01}(\chi) = \int_0^{\chi} e^{-\chi_1^2} d\chi_1, \qquad v_{02}(\chi) = 1,$$

$$\text{(C.4)} \qquad v_{i03}(\chi) = -\frac{1}{N_i^2}\frac{\partial p_{10}}{\partial \alpha_i}\left[\int_0^{\chi} e^{\chi_1^2} v_{i01}(\chi_1) d\chi_1 - v_{i01}(\chi)\int_0^{\chi} e^{\chi_1^2} d\chi_1\right],$$

where $0 \leq \chi_1 \leq \chi \equiv r_1 N$. For $t_1 \to 0$, then $N \to \infty$, thus $\chi \to \infty$. For $t_1 \to \infty$, thus $N \to 0$. Hence for $r_1 > 0$ we have $\chi \to 0$. For $t_1 > 0$ and $r_1 = 0$ we have $\chi = 0$. Following limits are true:

$$\begin{aligned}
&v_{01}(\chi) \to \pi^{0.5}/2, && \text{for } \chi \to \infty,\ t_1 \to 0,\ N \to \infty;\\
&v_{01}(\chi) \to 0, && \text{for } \chi \to 0,\ r_1 = 0,\ 0 < t_1 < t_2 < \infty,\ N > 0;\\
&v_{i03}(\chi) \to 0, && \text{for } \chi \to 0,\ r_1 = 0,\ 0 < t_1 < t_2 < \infty,\ N > 0;\\
&&& i = \varphi, \vartheta,\\
\text{(C.5)} \quad &v_{01}(\chi) \to 0, && \text{for } \chi \to 0,\ r_1 > 0,\ t_1 \to \infty,\ N \to 0;\\
&v_{\varphi 03}(\chi) \to -\frac{r_1^2}{2\sin\vartheta_1}\frac{\partial p_{10}}{\partial \varphi}, && \text{for } \chi \to 0,\ r_1 > 0,\ t_1 \to \infty,\ N \to 0,\\
&v_{\vartheta 03}(\chi) \to -\frac{r_1^2}{2}\frac{\partial p_{10}}{\partial \vartheta_1}, && \text{for } \chi \to 0,\ r_1 > 0,\ t_1 \to \infty,\ N \to 0,
\end{aligned}$$

Imposing proper conditions (2.5), (2.6) on the general solution (A3.2) we obtain:

$$\text{(C.6)} \quad \begin{aligned}
&C_{\varphi 1} v_{01}(\chi = 0) + C_{\varphi 2} + v_{\varphi 03}(\chi = 0) = \sin\vartheta_1, && \text{for } r_1 = 0,\\
&C_{\varphi 1} v_{01}(\chi = M) + C_{\varphi 2} + v_{\varphi 03}(\chi = M) = 0, && \text{for } r_1 = \varepsilon_1,\\
&C_{\vartheta 1} v_{01}(\chi = 0) + C_{\vartheta 2} + v_{\vartheta 03}(\chi = 0) = 0, && \text{for } r_1 = 0,\\
&C_{\vartheta 1} v_{01}(\chi = M) + C_{\vartheta 2} + v_{\vartheta 03}(\chi = M) = 0, && \text{for } r_1 = \varepsilon_1.
\end{aligned}$$

where $M \equiv \varepsilon_1 N$. Taking into account limits (C.5), then system of equations (C.6) has the following solutions:

$$\text{(C.7)} \quad \begin{aligned}
&C_{\varphi 1} = -\frac{\sin\vartheta_1 + v_{\varphi 03}(M)}{v_{01}(M)}, \qquad C_{\vartheta 1} = -\frac{v_{\vartheta 03}(M)}{v_{01}},\\
&C_{\varphi 2} = \sin\vartheta_1, \qquad C_{\vartheta 2} = 0.
\end{aligned}$$

Now into the right hand of equations:

$$\text{(C.8)}\qquad \frac{d^2 v_{i1\Sigma}}{d\chi^2} + 2\chi\frac{dv_{i1\Sigma}}{d\chi} + 4(v_{i1\Sigma}) = \frac{1}{N_i^2}\frac{\partial p_{11}}{\partial \alpha_i} + \frac{d^2 v_{i0}}{d\chi^2}\left(\frac{3}{2} - \chi^2\right),$$

for $i = \varphi,\ \vartheta;\ \alpha_\varphi \equiv \varphi,\ \alpha_\vartheta \equiv \vartheta_1$ and $(N_\varphi)^2 \equiv N^2\sin(\vartheta_1),\ N_\vartheta \equiv N$, we put solution (C.2), (C.3), (C.4), (C.8). Thus general solution of ordinary differential equation (C.8) has the following form:

$$\text{(C.9)}\qquad v_{i1}(\chi) = C_{i3}v_{11}(\chi) + C_{i4}v_{12}(\chi) + v_{i13}(\chi) \quad \text{for} \quad i = \varphi, \vartheta.$$

where C_{i3}, C_{i4} are integral constants. Particular solutions are as follows:

$$\text{(C.10)}\qquad v_{11}(\chi) = \chi e^{-\chi^2}, \qquad v_{12}(\chi) = \chi e^{-\chi^2}\int_\delta^\chi \frac{1}{\chi_1^2}e^{\chi_1^2}d\chi_1,$$

$$\text{(C.11)}\qquad v_{i13}(\chi, C_{i1}) = v_{11}(\chi)\int_0^\chi \left\{ C_{i1}\,\chi_1(3-2\chi_1^2) - \left(\frac{3}{2}-\chi_1^2\right)e^{\chi_1^2}\frac{d^2}{d\chi_1^2}\Big[v_{i03}(\chi_1)\Big] - \frac{1}{N_i^2}e^{\chi_1^2}\frac{\partial p_{11}}{\partial\alpha_i}\right\} v_{12}(\chi_1)d\chi_1 + v_{12}(\chi)\int_0^\chi\left\{\left(\frac{3}{2}-\chi_1^2\right)e^{\chi_1^2}\frac{d^2}{d\chi_1^2}\Big[v_{i03}(\chi_1)\Big] + \frac{1}{N_i^2}e^{\chi_1^2}\frac{\partial p_{11}}{\partial\alpha_i} - C_{i1}\chi_1(3-2\chi_1^2)\right\} v_{11}(\chi_1)d\chi_1,$$

for $i = \varphi, \vartheta;\ 0 < \delta \le \chi_1 \le \chi$. Following limits are true:

$$\text{(C.12)}\qquad \begin{aligned} &v_{11}(\chi) \to 0, && \text{for} \quad \chi \to 0, \quad r_1 = 0, \quad 0 < t_1 < t_2 < \infty, \quad N > 0,\\ &v_{12}(\chi) \to -1 && \text{for} \quad \chi \to 0, \quad r_1 = 0, \quad 0 < t_1 < t_2 < \infty, \quad N > 0,\\ &v_{i13}(\chi) \to 0, && \text{for} \quad \chi \to 0, \quad r_1 = 0, \quad 0 < t_1 < t_2 < \infty, \quad N > 0;\\ & && i = \varphi, \vartheta. \end{aligned}$$

Imposing proper conditions (2.5),(2.6) on the general solution (C.9) we get:

$$\text{(C.13)}\qquad \begin{aligned} C_{\varphi3}v_{11}(\chi = 0) + C_{\varphi4}v_{21}(\chi = 0) + v_{\varphi13}(\chi = 0) &= 0, \quad \text{for} \quad r_1 = 0,\\ C_{\varphi3}v_{11}(\chi = M) + C_{\varphi4}v_{21}(\chi = M) + v_{\varphi13}(\chi = M) &= 0, \quad \text{for} \quad r_1 = \varepsilon_1,\\ C_{\vartheta3}v_{11}(\chi = 0) + C_{\vartheta4}v_{21}(\chi = 0) + v_{\vartheta13}(\chi = 0) &= 0, \quad \text{for} \quad r_1 = 0,\\ C_{\vartheta3}v_{11}(\chi = M) + C_{\vartheta4}v_{21}(\chi = M) + v_{\vartheta13}(\chi = M) &= 0, \quad \text{for} \quad r_1 = \varepsilon_1. \end{aligned}$$

Taking into account limits (C.12), then system of equations (C.13) has the following solutions:

$$\text{(C.14)}\qquad C_{i3} = -\frac{v_{i13}\,(\chi = M)}{v_{11}\,(\chi = M)}, \quad C_{i4} = 0, \quad \text{for} \quad i = \varphi, \vartheta.$$

Appendix D.

If t_1 tends to infinity i.e. N tends to zero, then Eq. (2.14) tends to classical Reynolds equation for steady motion. To explain this fact we calculate the following limits:

$$\text{(D.1)}\qquad \lim_{N\to 0}\frac{\sqrt{\pi}}{2N^2}Y\,(\chi = \varepsilon_1 N)$$

$$\equiv \lim_{N\to 0}\frac{\sqrt{\pi}}{2N^2}\left[\int_0^{\varepsilon_1 N}\exp(\chi^2)\mathrm{erf}(\chi)d\chi - \mathrm{erf}(\varepsilon_1 N)\int_0^{\varepsilon_1 N}\exp(\chi^2)d\chi\right]$$

$$= \lim_{N\to 0}\frac{1}{N^2}\left\{\int_0^{\varepsilon_1 N}\left[\exp(\chi^2)\int_0^{\chi}\exp(-\chi_1^2)d\chi_1\right]d\chi - \left(\int_0^{\varepsilon_1 N}\exp(-\chi^2)d\chi\right)\left(\int_0^{\varepsilon_1 N}\exp(\chi^2)d\chi\right)\right\}$$

$$\overset{H}{=} -\lim_{N\to 0}\frac{\varepsilon_1\int_0^{N\varepsilon_1}\exp(\chi_1^2)d\chi_1}{2N\exp(\varepsilon_1^2N^2)} \overset{H}{=} -\frac{\varepsilon_1^2}{2}\lim_{N\to 0}\frac{\exp(\varepsilon_1^2N^2)}{\exp(\varepsilon_1^2N^2)+2\varepsilon_1^2N^2\exp(\varepsilon_1^2N^2)} = -\frac{\varepsilon_1^2}{2}.$$

Analogously:

$$\text{(D.2)}\qquad \lim_{N\to 0}\frac{\sqrt{\pi}}{2N^2}Y\,(\chi_1 = N r_1) = -\frac{r_1^2}{2},$$

$$\text{(D.3)}\qquad \lim_{N\to 0}\frac{\mathrm{erf}(r_1 N)}{\mathrm{erf}(\varepsilon_1 N)} = \frac{r_1}{\varepsilon_1}.$$

Thus Eq. (D.1) for $N \to 0$ tends to the following form:

$$\text{(D.4)}\qquad \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\left\{\left[\left(-\frac{\varepsilon_1^2}{2}\right)\int_0^{\varepsilon_1}\frac{r_1}{\varepsilon_1}dr_1 - \int_0^{\varepsilon_1}\left(-\frac{r_1^2}{2}\right)dr_1\right]\frac{\partial p_{10}}{\partial\varphi}\right\}$$

(D.4) [cont.]
$$+\frac{\partial}{\partial\vartheta_1}\left\{\left[\left(-\frac{\varepsilon_1^2}{2}\right)\sin\vartheta_1\int_0^{\varepsilon_1}\frac{r_1}{\varepsilon_1}dr_1-\int_0^{\varepsilon_1}\left(-\frac{r_1^2}{2}\right)\sin\vartheta_1 dr_1\right]\frac{\partial p_{10}}{\partial\vartheta_1}\right\}$$
$$=-(\sin\vartheta_1)\frac{\partial}{\partial\varphi}\left[\int_0^{\varepsilon_1}\left(1-\frac{r_1}{\varepsilon_1}\right)dr_1\right]-\mathrm{Str}\frac{\partial\varepsilon_1}{\partial t_1}\sin\vartheta_1.$$

If Str tends to zero after final calculations, then we obtain following form of classical Reynolds Equations in spherical coordinates (2.17).

Appendix E.

Dependencies between rectangular (x, y, z) and spherical (φ, r, ϑ) co-ordinates (see Fig. 4) have a following form:

(E.1) $$x = r\sin(\vartheta_1)\cos\varphi, \quad y = r\sin(\vartheta_1)\sin\varphi, \quad z = r\cos(\vartheta_1), \quad 0 < r < R.$$

Graphical illustration of a centre of spherical bone head O(0,0,0) and centre of spherical acetabulum in point $O_1(x-\Delta\varepsilon_x,\ y-\Delta\varepsilon_y,\ z+\Delta\varepsilon_z)$ is presented in Fig. 4. Equation of spherical acetabulum surface at centre point $O_1(x-\Delta\varepsilon_x,\ y-\Delta\varepsilon_y,\ z+\Delta\varepsilon_z)$ we can write in following form:

(E.2)
$$(x-\Delta\varepsilon_x)^2+(y-\Delta\varepsilon_y)^2+(z+\Delta\varepsilon_z)^2=(R+D+\varepsilon_{\min})^2,$$
$$D=\left[(\Delta\varepsilon_x)^2+(\Delta\varepsilon_y)^2+(\Delta\varepsilon_z)^2\right]^{0.5}.$$

We put dependencies (E.1) in Eq. (E.2), hence we obtain:

(E.3)
$$(r\cos\varphi\sin\vartheta_1-\Delta\varepsilon_x)^2+(r\sin\varphi\sin\vartheta_1-\Delta\varepsilon_y)^2+(r\cos\vartheta_1+\Delta\varepsilon_z)^2 = (R+D+\varepsilon_{\min})^2.$$

Gap height has following form:

(E.4) $$\varepsilon^{(0)}(\varphi,\vartheta_1)\equiv r-R.$$

We find r from Eq. (E.3) and put in formula (E.4). Hence gap height has finally a following form (2.19).

APPENDIX F.

If we substitute solutions (3.20) + (3.21) in Eq. $(3.23)_2$, then we have:

$$\text{(F.1)}\quad \frac{\partial}{\partial\varphi}\left[\frac{1}{R\sin\left(\dfrac{\vartheta}{R}\right)}\frac{\partial p^{(k)}}{\partial\varphi}\int_0^{\varepsilon} W_k dr\right]+\frac{\partial}{\partial\vartheta}\left[R\sin\left(\frac{\vartheta}{R}\right)\frac{\partial p^{(k)}}{\partial\vartheta}\int_0^{\varepsilon} W_k dr\right]$$

$$+\frac{\partial}{\partial\varphi}\left\{\frac{k\omega_0\rho_0}{j}\left[U_{\varphi k}\int_0^{\varepsilon}\frac{\sinh\left[(\varepsilon-r)A_k\right]}{\sinh\left[\varepsilon A_k\right]}dr+V_{\varphi k}\int_0^{\varepsilon}\frac{\sinh\left(rA_k\right)}{\sinh\left(\varepsilon A_k\right)}dr\right]\right\}$$

$$+\frac{\partial}{\partial\vartheta}\left\{\frac{k\omega_0\rho_0}{j}R\sin\left(\frac{\vartheta}{R}\right)\left[U_{\vartheta k}\int_0^{\varepsilon}\frac{\sinh\left[(\varepsilon-r)A_k\right]}{\sinh\left[\varepsilon A_k\right]}dr+V_{\vartheta k}\int_0^{\varepsilon}\frac{\sinh\left(rA_k\right)}{\sinh\left[\varepsilon A_k\right]}dr\right]\right\}$$

$$=-k^2\omega_0^2\rho_0\varepsilon^{(k)}R\sin\left(\frac{\vartheta}{R}\right)+\frac{k\omega_0\rho_0}{j}\left[V_{\varphi k}\frac{\partial\varepsilon}{\partial\varphi}+V_{\vartheta k}R\sin\left(\frac{\vartheta}{R}\right)\frac{\partial\varepsilon}{\partial\vartheta}\right],$$

for $k=1,2,3,...,0\le\varphi\le 2\pi c_1,\ 0<c_1<1,\ \pi R/8\le\vartheta\le\pi R/2,\ 0\le r\le\varepsilon$.

For the further reduction of Eq. (F.1) it is necessary to calculate for $i=\varphi,\vartheta$ the following integrals:

$$\text{(F.2)}\quad \frac{k\omega_0\rho_0}{j}U_{ik}\int_0^{\varepsilon}\frac{\sinh\left[(\varepsilon-r)A_k\right]}{\sinh\left[\varepsilon A_k\right]}dr=\frac{k\omega_0\rho_0}{j}U_{ik}\int_0^{\varepsilon}\frac{e^{(\varepsilon-r)A_k}-e^{-(\varepsilon-r)A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}dr$$

$$=\frac{k\omega_0\rho_o}{jA_k}U_{ik}\frac{e^{\varepsilon A_k}-2+e^{-\varepsilon A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}=\frac{k\omega_0\rho_0}{jA_k}U_{ik}\tanh\left(\frac{\varepsilon A_k}{2}\right)$$

$$=-\frac{1}{2}j\,k\omega_0\rho_0U_{ik}\left[\varepsilon-\frac{1}{12}\varepsilon^3A_k^2+O\left(\varepsilon^4\right)\right],$$

$$\text{(F.3)}\quad \int_0^{\varepsilon}W_k dr_2=\int_0^{\varepsilon}\left[\left(1-e^{rA_k}\right)-\left(1-e^{\varepsilon A_k}\right)\frac{e^{rA_k}-e^{-rA_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}\right]dr$$

$$=\int_0^{\varepsilon}\left[1-\frac{e^{rA_k}-e^{(\varepsilon-r)A_k}}{1+e^{\varepsilon A_k}}\right]dr=\varepsilon+\frac{2}{A_k}\frac{1-e^{\varepsilon A_k}}{1+e^{\varepsilon A_k}}$$

$$=\varepsilon+\frac{2}{A_k}\tanh\left(\frac{\varepsilon A_k}{2}\right)=\frac{j}{12}\frac{\varepsilon^3}{\eta_k}k\omega_0\rho_0-O\left(\varepsilon^4\right),$$

$$\text{(F.4)}\quad V_{ik}\frac{k\omega_0\rho_0}{j}\int\limits_0^{\varepsilon}\frac{\sinh rA_k}{\sinh\varepsilon A_k}dr$$

$$=V_{ik}\frac{k\omega_0\rho_0}{j}\int\limits_0^{\varepsilon}\frac{e^{rA_k}-e^{-rA_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}dr=\frac{k\omega_0\rho_0}{jA_k}V_{ik}\frac{e^{\varepsilon A_k}-2+e^{-\varepsilon A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}$$

$$=\frac{k\omega_0\rho_0}{jA_k}V_{ik}\tanh\left(\frac{\varepsilon A_k}{2}\right)=-\frac{1}{2}j\,k\omega_0\rho_0V_{ik}\left[\varepsilon-\frac{1}{12}\varepsilon^3A_k^2+O\left(\varepsilon^4\right)\right].$$

Integrals (F.2), (F.3), (F.4) we put in equation (F.1). Thus we get Eq. (3.25).

Appendix G.

Classical stress-strain relation has the following form [1, 6]:

$$\text{(G.1)}\qquad \mathbf{S}=-p\mathbf{I}+\eta_p\mathbf{A}_1,$$

The majority of experiments performed on the biological synovial fluids indicate that dynamic viscosity decreases with shear rate increasing [1]. By virtue of Dowson's experimental values, the apparent viscosity we can show in following form [18]:

$$\text{(G.2)}\qquad \eta_p(A,B)=\eta_\infty+\frac{\eta_0-\eta_\infty}{1+A\,\mathrm{tr}\mathbf{A}_1+B\,\mathrm{tr}\mathbf{A}_1\,\mathrm{tr}\mathbf{A}_1+B\,\mathrm{tr}\mathbf{A}_2},$$

where coefficient A obtained by the experimental way has value from 1.200 s to 2.000 s and coefficient B most often attain values from $0.00300\,\mathrm{s}^2$ to $0.00600\,\mathrm{s}^2$. Symbol η_0 – denotes dynamic viscosity in Pas of motionless synovial fluid or for the very slow movement of synovial fluid, η_∞ – dynamic viscosity in Pas of synovial fluid in large motion. Viscoelastic properties of synovial fluids are described by means of Rivlin Ericksen constitutive relations (1.2). We assume following approximate form:

$$\text{(G.3)}\qquad \mathbf{S}\cong-p\mathbf{I}+\mathbf{A}_1\eta_p,\quad \eta_p\approx\eta_0+\alpha\,\mathrm{tr}\mathbf{A}_1+\beta\frac{\mathrm{tr}\mathbf{A}_2}{\mathrm{tr}\mathbf{A}_1}.$$

Apparent viscosity (F.2) as a function of two variables A and B we expand in Taylor series in neighborhood of point $A=0$, $B=0$ and obtain series we equate with the apparent viscosity presented in formula (G.3). Hence we obtain following approximation dependencies between unknown coefficients α,β and known experimental values A and B:

$$\text{(G.4)}\qquad \begin{aligned}\alpha&\cong-A(\eta_0-\eta_\infty)+\eta_0B/A+O(B^2/A^2),\\ \beta&\cong0.5\eta_0B/A+O(B^2/A^2).\end{aligned}$$

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