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ALGORITHMS OF THE METHOD OF STATICALLY ADMISSIBLE DISCONTINUOUS STRESS FIELDS (SADSF) – PART III

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Summary of the whole paper: By now, the SADSF method is practically the only tool of shape design of complex machine elements that provides an effective solution even to the problems of 3D distribution of the material, and at the same time it is still enough user-friendly to be useful for engineers. This unique property of the method is due to the existence of its simple application version. When using it, a design engineer does not need to solve by oneself any statically admissible field – which could be very difficult – but obtains such a solution by assembling various ready-made particular solutions. The latter are in general obtained by means of individual and complex analyses and provided to a designer in a form of libraries.

The algorithms presented in this paper break up with the individual approach to a particular field. The algorithms are the first ones of general character, as they apply to the fundamental problems of the method. The algorithms enable solving practically any boundary problem that one encounters in constructing 2D statically admissible, discontinuous stress fields, first of all the limit fields. In the presented approach, one deals first with the fields arising around isolated nodes of stress discontinuity lines (Parts II and III), then integrates these fields into 2D complex fields (Part IV).

The software, created on the basis of the algorithms, among other things, allows one to find all the existing solutions of the discontinuity line systems and present them in a graphical form. It gives the possibility of analysing, updating and correcting these systems. In this way, it overcomes the greatest difficulty of the SADSF method following from the fact that the systems of discontinuity lines are not known a priori, and appropriate relationships are not known either, so that they could be found only in an arduous way by postulating the line systems and verifying them.

Application version of the SADSF method is not described in this paper; however, a reference is given to inform the reader where it can be found.

PART III

SOFTWARE IMPLEMENTATIONS AND EXAMPLE PROBLEMS FORMULATED FOR FIELDS AROUND NODES

Summary of Part III: Boundary problems, characteristic for the already-known fields around convex and concave corners, are used in this part of the paper as the examples to present juxtaposition of conditions, and to obtain a solution for general conditions of the

system – important for the fields that appear around nodes. The presented variants of these systems and the sets of unknowns, after minor completion, give the basis for deriving series of elementary problems, which are necessary to create the algorithm for solving arbitrary boundary problems, such as those encountered in the fields around nodes. The algorithm created on such a basis does not require formulating any particular relationships, and its implementation makes it possible to find any solution to the field around the node. The solution, presented in an illustrative graphical form, can then be easily edited. In effect, it becomes possible to test, almost instantaneously, admissibility of the structures, and verify the existence of solutions on the physical plane.

The paper also presents short description of properties of the fields around nodes that facilitates interpretation of the results. It is particularly useful in the cases when one obtains surprising results, for example when structural degenerations (collapses) appear.

It is worth mentioning that, with boundary conditions formulated for fields around both the above-mentioned types of corners, one obtains not only fields identical with the prototype, but also a whole variety of other fields that until now have been treated as different ones. Actually, these are the fields being solutions to the same boundary problem.

Key words: shape design, limit analysis, numerical methods.

12. Examples of problems

12.1. Example 1 (Field type A Fig. 14)

Let us consider a limit field around a node, which consists of three homogeneous regions 1, 2, 3 (Fig. 14a). Let us omit determination stress parameters from boundary conditions, and now assume that $\overset{(1)}{\omega} = 60$, $\overset{(3)}{\omega} = 60$ are given in the regions 1, 3, and the angle between principal directions of stresses in these regions (Fig. 14a) equals $\chi = -60°$.

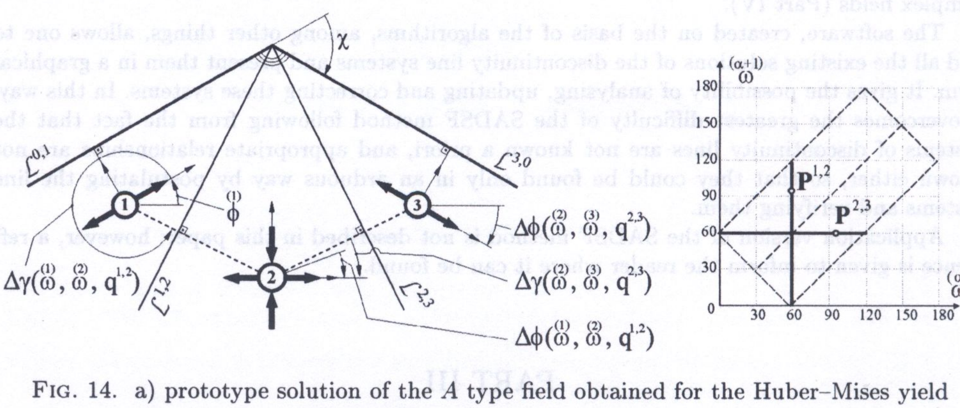

Fig. 14. a) prototype solution of the A type field obtained for the Huber–Mises yield condition with the data: $\overset{(1)}{\omega} = 60$, $\overset{(2)}{\omega} = 92.4364$, $\overset{(3)}{\omega} = 60$, $\chi = -60°$, $q^{1,2} = 4$, $q^{2,3} = 1$; b) area of existence Λ with plotted subareas $\Lambda^{1,2}_{1,3}$, $\Lambda^{2,3}_{1,3}$, (heavy lines) and images $\mathbf{P}^{1,2}$, $\mathbf{P}^{2,3}$ of the lines $\mathcal{L}^{1,2}$, $L^{2,3}$.

Although it is not necessary, let us also assume that the angle $\overset{(1)}{\phi}$ is equal to 30°. Assuming the Huber–Misses yield condition, we have to find the admissible state of stress in the intermediate region 2, and determine all parameters of the limit field around the node.

In literature ([1]), the field around a convex corner is called the field type *A*. In this example, however, it is treated as a fundamental solution, used for illustrative presentation of the problem on the basis of a commonly known case. As it turns out, if one starts from the boundary conditions characteristic for such a type of field, a whole series of solutions is obtained, in which the prototype of field *A* is just one of the elements.

12.1.1. Solution. According to (7.1), we formulate first the equality conditions of existence for both lines of the field $\mathcal{L}^{1,2}$ and $\mathcal{L}^{2,3}$ ($\alpha = 1$, $\alpha = 2$), then we get the functions:

(a) $$\Delta\phi\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, Q^{1,2}\right) = \overset{(2)}{\phi} - \overset{(1)}{\phi},$$

(b) $$\Delta\phi\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, Q^{2,3}\right) = \overset{(3)}{\phi} - \overset{(2)}{\phi},$$

which, after substitution into (11.1), lead to the condition:

(c) $$\Delta\phi\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, Q^{1,2}\right) + \Delta\phi\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, Q^{2,3}\right) = \chi.$$

The data in this equation are: $\overset{(1)}{\omega} = 60$, $\overset{(3)}{\omega} = 60$, $\chi = -60°$ ($\chi_1 = \chi = -60°$, $\chi_2 = \chi + 180° = 120°$) and the parameters $Q^{1,2}$ $Q^{2,3}$ can take values 1, 2. Then, for all combinations of indices $Q^{1,2}$ $Q^{2,3}$ we have four separate equations of type (c), each of them with one unknown $\overset{(2)}{\omega}$. The equations must be solved with the conditions:

$$(\overset{(1)}{\omega}, \overset{(2)}{\omega}) \in \Lambda^{1,2}_{1,3}, \quad (\overset{(2)}{\omega}, \overset{(3)}{\omega}) \in \Lambda^{2,3}_{1,3} \qquad \text{(see } \Lambda^{\alpha,\alpha+1}_{1,3} \text{ in Fig. 14b).}$$

Four roots are obtained here:

$$\overset{(2)}{\omega} = 113.1370, \quad 92.4364, \quad 20.7006, \quad 0.0000.$$

Substituting the roots into (7.1) and (7.3), we now can calculate:

$$\Delta\phi\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2}\right), \quad \Delta\phi\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, q^{2,3}\right),$$

$$\Delta\gamma\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2}\right), \quad \Delta\gamma\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, q^{2,3}\right),$$

for all parameters of subfamilies $q^{1,2} = 1..4$, $q^{2,3}$=1..4, and then test the structural conditions (10.1) for the existence of a physical place for each homogeneous region.

In this way we obtain 16 solutions related to the roots $\overset{(2)}{\omega}$ and to different paths determined by the values of the two parameters of subfamilies $\{q^{1,2}, q^{2,3}\}$. Eight of those are different solutions, for example:

$$\begin{array}{llll}
 & -\overset{(2)}{\omega} = 92.4364, & \{q^{1,2} = 2, q^{2,3} = 1\}, & \delta = 232.952°; \\
 & \overset{(2)}{\omega} = 92.4364, & \{q^{1,2} = 4, q^{2,3} = 1\}, & \delta = 52.952°; \quad (\leftarrow \text{see Fig. 14}) \\
 & -\overset{(2)}{\omega} = 20.7006, & \{q^{1,2} = l, q^{2,3} = 2\}, & \delta = 96.053°; \\
\text{(d)} & -\overset{(2)}{\omega} = 20.7006, & \{q^{1,2} = l, q^{2,3} = 4\}, & \delta = 276.053°; \\
 & -\overset{(2)}{\omega} = 0.0000, & \{q^{1,2} = l, q^{2,3} = 3\}, & \delta = 120.000°. \\
 & -\overset{(2)}{\omega} = 0.0000^{o}, & \{q^{1,2} = l, q^{2,3} = 1\}, & \delta = 300.000°; \\
 & -\overset{(2)}{\omega} = 113.1370, & \{q^{1,2} = 1, q^{2,3} = 2\}, & \delta = 150.992°; \\
 & -\overset{(2)}{\omega} = 113.1370, & \{q^{1,2} = 1, q^{2,3} = 4\}, & \delta = 330.992°;
\end{array}$$

Sketches of these solutions are presented in Fig. 15.

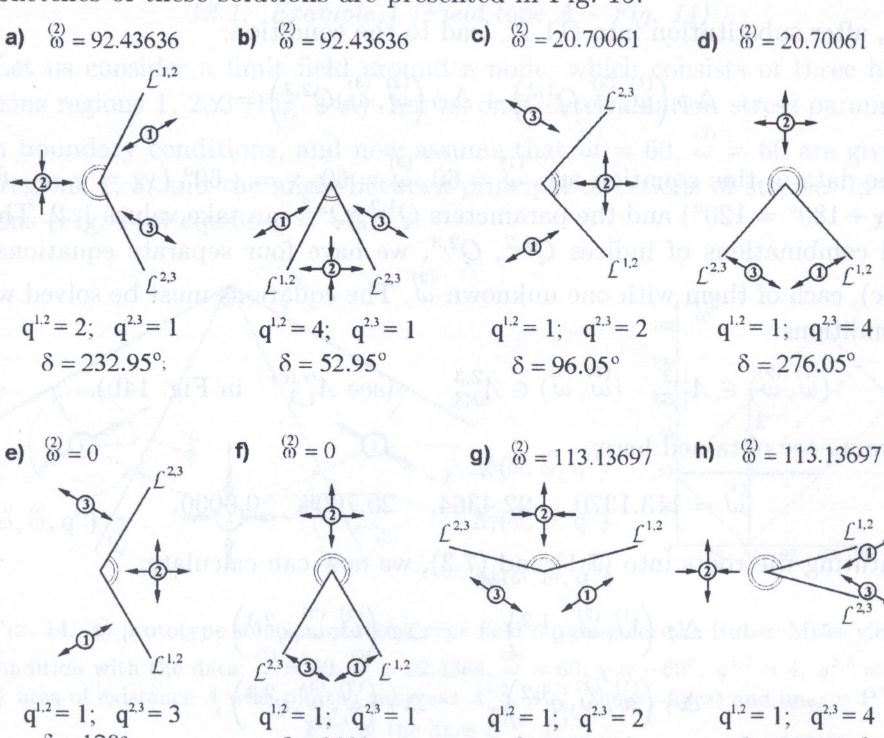

FIG. 15. A fragment of the set of fields around a node, generated from the prototype field type A (solutions for different roots $\overset{(2)}{\omega}$ and different paths $(q^{1,2}, q^{2,3})$).

12.1.2. Discussion of results. The multitude of the obtained solutions results from the fact that we have not imposed here many limitations that usually appear in practical problems. It has been done on purpose, in order to find all the solutions that can be generated from the same prototype field type A. In fact, we have obtained as much as eight qualitatively different varieties of the field.

Among the neglected conditions, we can mention first of all those pertaining to the field within the 2π complement of angle δ. By discarding these conditions, we assume *a priori* that there exists a statically admissible extension of the analysed field in this area. However, we do not specify this field precisely, and do not verify the possibility of its realisation on the physical plane. If one assumes, for example, that the area within the 2π complement of angle δ is free of stress (as it is usually assumed for fields of type A), then there must exist two more lines of discontinuity $\mathcal{L}^{0,1}$ and $\mathcal{L}^{3,0}$, one before region 1, and a second one behind region 3 (see Fig. 14). At the same time, the solutions shown in Figs. 15 d,f would be excluded as inadmissible. However, these solutions could be admissible with different assumptions and for different configurations of stress discontinuity lines systems. Then, at the stage of analysis, these solutions must not be neglected.

Neither half-lines δ_1, δ_2, nor angles $\nu^{0,1}$, $\nu^{N,0}$, between whom the field we sought for should exist, were not specified in the problem. For that reason there appear all the solutions encompassed by the angle $\delta \leq 2\pi$. Among them, one can also find identical ones, which are simply rotated by the angles $\Delta\gamma$ related to consecutive values of the parameter of subfamily $q^{1,2} = 1..4$ in the homogeneous region 1. It means that the solutions are rotated with respect to the configuration which is used as a reference for the configurations in next regions, and gives the basis for measuring angular parameters of lines $\mathcal{L}^{\alpha,\alpha+1}$. The multitude of solutions results from the possibility of choosing as much as 4 positions ($q^{1,2} = 1..4$) of the line $\mathcal{L}^{1,2}$ that was determined as the first one (see also Fig. 4).

Let us notice that one of the calculated roots: $\overset{(2)}{\omega}=113.1370$ would not be included into the above-presented set of solutions (d) if – besides of $\chi_1 = -60°$ – we would not take into account $\chi_2 = 120°$ (see (11.2)). The latter value also complies with the conditions of the problem, because the system $\{\xi\}^{(3)}$, associated with principal directions of the stress in region 3, remains associated with the same state of stress, even if we rotate it by the angle π.

The fields obtained for $\chi_2 = 120°$ are shown in Fig. 15 g,h. It is visible that they do not fulfil the condition of admissibility if, as in the previous case, one assumes the state of zero stress within the 2π complement of δ. However, we obtain a total of 4 varieties of field type A which satisfy such conditions (Fig. 15 a,b,c,e).

We give up juxtaposing the full set of solutions whose prototypes are boundary conditions characteristic for field type A.

Finally, it is worth mentioning that, instead of solving four equations of type (c), one can as well find the roots $\overset{(2)}{\omega}$ of only one equation, which can be derived from (c) by elimination of $Q^{1,2}$, $Q^{2,3}$:

$$(12.1)\quad \left\{\chi^2 - \left[\Delta\hat{\phi}(\overset{(1)}{\omega},\overset{(2)}{\omega})\right]^2 - \left[\Delta\hat{\phi}(\overset{(2)}{\omega},\overset{(3)}{\omega})\right]^2\right\}^2 - 4\left[\Delta\hat{\phi}(\overset{(1)}{\omega},\overset{(2)}{\omega})\ \Delta\hat{\phi}(\overset{(2)}{\omega},\overset{(3)}{\omega})\right]^2 = 0,$$

or the equation created in the form of a product of conditions (c) for all combinations of signs associated with $Q^{1,2}$, $Q^{2,3}$. Despite formal identity of both equations, the product form seems inconvenient for numerical methods.

12.2. Example 2 (Field type B – Fig. 16)

The prototype of field around concave corner, called field type B [1], also consists of three homogeneous region, 1, 2, 3 (Fig. 16a). However, in the external regions there are given the values $\overset{(1)}{\omega} = 60$, $\overset{(3)}{\omega} = 120$, and – for instance – $\overset{(1)}{\phi} = 90°$.

Let us assume that the angle between the directions of the greatest principal stresses in these regions equals $\chi = 0°$ $(\chi_1 = 0°, \chi_2 = 180°)$, and let us determine the admissible state of stress in the region 2 and the remaining field parameters.

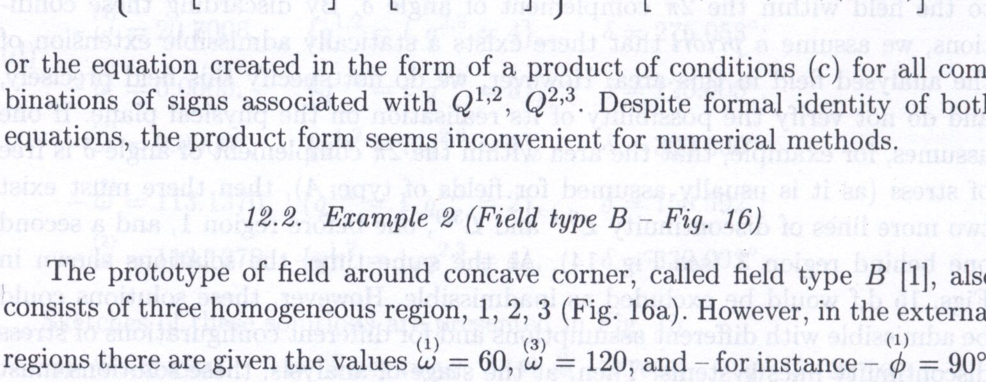

FIG. 16. a) prototype solution of the B-type field obtained for the Huber–Mises yield condition with the data: $\{\overset{(1)}{\omega} = 60, \overset{(2)}{\omega} = 90, \overset{(3)}{\omega} = 120,\ \chi = 0,\ q^{1,2} = 4,\ q^{2,3} = 1\}$, b) area of existence Λ with plotted subareas $\Lambda^{1,2}_{1,3}$, $\Lambda^{2,3}_{1,3}$ (heavy lines) and images $\mathbf{P}^{1,2}$, $\mathbf{P}^{2,3}$ of the lines $\mathcal{L}^{1,2}$, $\mathcal{L}^{2,3}$.

12.2.1. Solution. The present problem differs from the previous one only by the values of data. Let us then formulate an equation similar to type (c) (to the one from the previous example), to which we substitute the new data: $\overset{(1)}{\omega} = 60, \overset{(3)}{\omega} = 120, \chi = 0^\circ$. The equation is to be solved for all parameters of the families $Q^{\alpha,\alpha+1}(1,2)$ with the conditions:

$$\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}\right) \in \Lambda^{1,2}_{1,3}, \quad \left(\overset{(2)}{\omega}, \overset{(3)}{\omega}\right) \in \Lambda^{1,2}_{1,3}, \qquad \text{(see Fig. 16b).}$$

We now obtain one root: $\overset{(2)}{\omega} = 90$, and by substituting it to (7.1) and (7.3) we can calculate:

$$\Delta\phi\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2}\right), \quad \Delta\phi\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, q^{2,3}\right),$$

$$\Delta\gamma\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2}\right), \quad \Delta\gamma\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, q^{2,3}\right)$$

for all parameters of the subfamilies $q^{1,2} = 1..4$, $q^{2,3} = 1..4$. Next, we test the fulfilment of structural conditions (10.1). Eight solutions related to different paths $\{q^{1,2}, q^{2,3}\}$ satisfy these conditions. The fact that the number of solutions is so great can be explained by the same reasons as those mentioned in the example of field type A. Only two of the solutions are different (Fig. 17a,b):

$$\overset{(2)}{\omega} = 90.0000, \quad \{q^{1,2} = 3, q^{2,3} = 4\}, \quad \delta = 90^\circ;$$

$$\overset{(2)}{\omega} = 90.0000, \quad \{q^{1,2} = 2, q^{2,3} = 1\}, \quad \delta = 270^\circ.$$

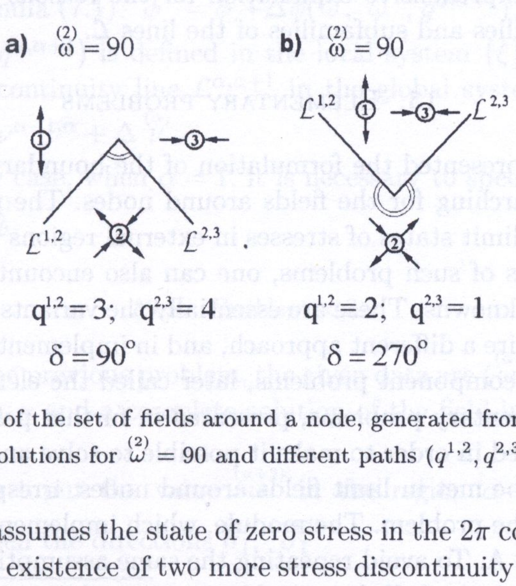

FIG. 17. A fragment of the set of fields around a node, generated from the prototype field type B (solutions for $\overset{(2)}{\omega} = 90$ and different paths $(q^{1,2}, q^{2,3}) = 90$)).

If one additionally assumes the state of zero stress in the 2π complement of angle δ, which implies the existence of two more stress discontinuity lines $\mathcal{L}^{0,1}$ and $\mathcal{L}^{3,0}$, then only one among the mentioned solutions will remain the admissible one – and this is shown in Fig. 17a.

12.2.2. Practical remarks. Before the analysis of both examples is completed, it is worth to emphasise the general sense of operating on both the parameters of families Q and subfamilies q of the lines $\mathcal{L}$, as it was demonstrated in the solutions of the problems. It turns out that it does not make a difference whether one uses the families, or alternatively the subfamilies of lines, in determining the function $\Delta\phi$. The differences arise only during the analysis of the function $\Delta\gamma$, where the choice of the parameter value $q = 1..4$ means that we choose one out of four possible positions of the line $\mathcal{L}$ with respect to the principal directions of stress in the existing homogeneous system. In the case of the analysed nodes, the position of the complete field on the physical plane will be determined already by the choice of the value $q^{1,2}$ in the region 1. It explains the repetition of identical solutions that differ only by the angle of rotation. However, it becomes important to have these solutions at our disposal when the conditions for the contents of the field in the half-plane limited by half-lines δ_1, δ_2 (or $\nu^{0,1}$, $\nu^{N,0}$) become active. Here, determining the function $\Delta\gamma$ on the basis of parameters Q instead of q would lead to the rejection of two solutions that actually exist, and can be found in this sector of the plane.

In practical problems, one often encounters active geometrical conditions, especially in the cases when a part of the field around a node has already been solved, and the problem consists in finding a fragment of the field "inscribed" into the field. The considerations on different configurations of the regions and lines $\mathcal{L}$ in relation to the values of Q and q, were presented in this section in order to give a fairly comprehensive explanation for the reasons of selecting specific parameters of families and subfamilies of the lines $\mathcal{L}$.

13. Elementary problems

In Sec. 11, we presented the formulation of the boundary problem most frequently met in searching for the fields around nodes. The problem pertains to the case when the limit states of stresses in external regions 1 and N are known. In practice, besides of such problems, one can also encounter those with other sets of data and unknowns. These are essentially the variants of the previous one, however, they require a different approach, and in implementations they must be treated as specific component problems, later called the elementary problems.

The set of elementary problems, presented in further part of the paper, was purposefully selected in order to make it possible to solve an arbitrary boundary problem that can be met in limit fields around nodes, irrespective of particular characteristics of the problem. The module, which implements these problems is denoted by symbol $\mathbb{A}$. To avoid repeating the same assumption in the future, we state now that solutions of all these problems are conditional and are determined only when for each $\alpha = 1..N$ there is $\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\} \in \Lambda$.

13.1. Problem <1>[1)]

There are given $(\overset{(\alpha)}{\omega}, \overset{(\alpha)}{\phi})$ in the current homogeneous region α and a complete solution of the field in the regions up to α, inclusively.

For the mentioned value of stress parameter $\overset{(\alpha+1)}{\omega}$ in the region $\alpha + 1$ we calculate:

$$\Delta\hat{\phi}\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\right) \quad \text{and} \quad \Delta\hat{\gamma}\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\right),$$

and next, for each parameter of the subfamily $q^{\alpha,\alpha+1} = 1..4$ and for the calculated values

$$\Delta\phi\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}\right), \quad \Delta\gamma\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha+\alpha+1}\right)$$

we verify the structural conditions (10.1), and also the possible geometrical conditions of solution existence for the whole considered field. The admissible values of $q^{\alpha,\alpha+1}$ are proposed for choice. Determining the value of $q^{\alpha,\alpha+1}$ can be done by inspection of graphs of all admissible solutions presented on the physical plane. Obviously, the greater the number of constrains imposed by the solution on the region up to α, inclusively, the lower the possibilities of choice of $q^{\alpha,\alpha+1}$. The existence of the preceding homogeneous regions leaves less and less freedom for creating new ones.

The state of stress in the region $\alpha + 1$ in the co-ordinate system of the total field {a} is determined by $\overset{(\alpha+1)}{\omega}$, $\overset{(\alpha+1)}{\phi}$, while the angle of principal stresses can be calculated from formula (7.1) $\overset{(\alpha+1)}{\phi} = \overset{(\alpha)}{\phi} + \Delta\phi(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1})$, because the increment $\Delta\phi(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1})$ is defined in the local system $\{\xi\}^{(\alpha)}$. Consequently, the direction of discontinuity line $\mathcal{L}^{\alpha,\alpha+1}$ in the global system {a} determines the angle $\nu^{\alpha,\alpha+1} = \nu^{\alpha-1,\alpha} + \Delta\overset{(\alpha)}{\nu}$.

In the particular case, when $\alpha = 1$, it is necessary to specify not only $\overset{(1)}{\omega}$, $\overset{(2)}{\omega}$, but also the angle $\overset{(1)}{\phi}$,

13.2. Problem <2>

Similarly as in the previous problem, the given data are $(\overset{(\alpha)}{\omega}, \overset{(\alpha)}{\phi})$ in the current homogeneous region α and a complete solution of the field in the regions up to this one, inclusively.

Additionally, we input the value $\overset{(\alpha+2)}{\omega}$ in the region $\alpha + 2$ and the angle $\chi = \overset{(\alpha+2)}{\phi} - \overset{(\alpha)}{\phi}$ between the directions $\overset{(\alpha)}{\sigma_1}$, $\overset{(\alpha+2)}{\sigma_1}$.

[1)] The denotations of elementary problems used here are consistent with those assumed in the implementation.

The quantities to be found are $\overset{(\alpha+1)}{\omega}$, $\overset{(\alpha+1)}{\phi}$ and the parameters of the lines $\mathcal{L}^{\alpha,\alpha+1}$, $\mathcal{L}^{\alpha+1,\alpha+2}$, which separate the intermediate region $\alpha+1$.

The problem can be reduced to finding a conditional solution to equation of the following form:

$$\Delta\phi\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, Q^{\alpha,\alpha+1}\right) + \Delta\phi\left(\overset{(\alpha+1)}{\omega}, \overset{(\alpha+2)}{\omega}, Q^{\alpha+1,\alpha+2}\right) = \chi,$$

whose unknowns are: $\overset{(\alpha+1)}{\omega}$, $Q^{\alpha,\alpha+1}$, $Q^{\alpha+1,\alpha+2}$.

Similarly as in the previous examples, the technique of solving such equations consists, first of all, in eliminating $Q^{\alpha,\alpha+1}$, $Q^{\alpha+1,\alpha+2}$, which essentially means reduction of the equation to a set of four equations defined on the functions $\Delta\hat{\phi}$, and formulated for all four combinations of the signs:

$$\chi = \pm\Delta\hat{\phi}\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\right) \pm \Delta\hat{\phi}\left(\overset{(\alpha+1)}{\omega}, \overset{(\alpha+2)}{\omega}\right).$$

Alternatively, we can create one fourth-order equation of type (12.1), from which we determine single and multiple roots $\overset{(\alpha+1)}{\omega}$.

Next, for all roots $\overset{(\alpha+1)}{\omega}$ (not more than four roots) we test the values of parameters $q^{\alpha,\alpha+1} = 1..4$, $q^{\alpha+1,\alpha+2} = 1..4$ to find the roots for which the conditions of existence (structural (10.1), and possibly also geometrical ones) are satisfied for the field on the physical plane. In doing so, we have to take into account the whole of the previously solved field (up to the region α, inclusively). The series of the admissible paths $\{q^{\alpha,\alpha+1}, q^{\alpha+1,\alpha+2}\}$ (for which conditions (10.1) are satisfied) are put together in the object created by the algorithm, and presented in a graphical form that enables us to make a choice.

13.3. Problem <3>

Also in this case, there is a solution of the field given for the region up to α, inclusively. We have to find $\overset{(\alpha+1)}{\omega}$ and the line $\mathcal{L}^{\alpha,\alpha+1}$ when the increment of the principle stress angle $\Delta\phi(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1})$ is given, so that $\overset{(\alpha+1)}{\phi}$ in the next region $\alpha+1$ is also given.

In order to solve this problem, we formulate the equation [(7.1)]:

$$\overset{(\alpha+1)}{\phi} - \overset{(\alpha)}{\phi} = \Delta\phi\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, Q^{\alpha,\alpha+1}\right),$$

whose unknowns are $\overset{(\alpha+1)}{\omega}$, $Q^{\alpha,\alpha+1}$, and which must be solved with $\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\} \in \Lambda$.

For $\overset{(\alpha+1)}{\omega}$ determined in this way (not more than four roots) we test the conditions of existence of the solution on the physical plane for all values of $q^{\alpha,\alpha+1} = 1..4$. Only admissible values of the subfamilies, i.e. those satisfying conditions (10.1), are available for the choice.

13.4. Problem <0>

This problem concerns the boundary line $\mathcal{L}^{0,1}$ externally loaded by the stress vector $\mathbf{p}^{0,1}$ (see Figs. 4 and 6).

The given quantities are the angle parameter $\nu^{0,1}$, which determines the system of co-ordinates $\{\xi\}^{(0)}$ associated with the line $\mathcal{L}^{0,1}$, and the components of stress vector $\mathbf{p}^{0,1}$ applied on this line. The quantity we seek for is the limit state of stress $\{\overset{(1)}{\omega}\overset{(1)}{\phi}\}$ in the region 1 adjacent to this line.

In order to solve the problem, we assume that the axis ξ_2 of the system $\{\xi\}^{(0)}$ (perpendicular to $\mathcal{L}^{0,1}$) is drawn outside the field in the region 1 (Figs. 4 and 6) in such a way that the direction parameter $\nu^{0,1}$ of the line $\mathcal{L}^{0,1}$ is associated with $\overset{(0)}{\phi}$ through the following dependence: $\overset{(0)}{\phi} = \nu^{0,1} - \pi$. Therefore, it is sufficient to specify $\nu^{0,1}$ in order to define the line $\mathcal{L}^{0,1}$ and the system $\{\xi\}^{(0)}$. In so defined system $\{\xi\}^{(0)}$, the vector $\mathbf{e}^{1,0}$ normal to $\mathcal{L}^{0,1}$ has the co-ordinates (0,1), which enables us to use formulae (8.3).

Next, we input the components of loads $\mathbf{p}^{0,1}$ into the system $\{\xi\}^{(0)}$, which must satisfy the condition $(8.3)_1$ $(\Delta(\mathbf{p},\mathbf{e}) \geq 0)$. Consequently, from $(8.3)_2$ $(\overset{(1)}{\omega} = \bar{\omega}(\mathbf{p},\mathbf{n},Q^{0,1}))$ we calculate two roots $\overset{(1)}{\omega}$ $(Q^{0,1} = 1, 2)$. The choice of the index $Q^{0,1}$ of root $\overset{(1)}{\omega}$ means that we have established one of the roots of the parameter $\Delta\phi^{(0)}$ determined from $(8.3)_3$ $(\Delta\phi^{(0)} = \bar{\phi}(\mathbf{p},\mathbf{n}, \overset{(1)}{\omega}, Q^{0,1}))$.

The angle $\Delta\phi^{(0)}$ is measured with respect to the axis $\{\xi_1\}^{(0)}$, and, similarly as in the previous problems, the condition $\overset{(1)}{\phi} = \overset{(0)}{\phi} + \Delta\phi^{(0)}$ holds here.

The state of stress outside of the field is not defined, however, in the algorithm one assumes $\overset{(0)}{\boldsymbol{\sigma}} = \mathbf{0}$, and the boolean variables $\sigma B^{(0)}$, $\omega B^{(0)}$, defined there, are given values TRUE or FALSE, respectively. It means that the stresses are defined, although they have no relation with the parameter ω.

The case of loaded external line $\mathcal{L}^{N,0}$ of the field is solved in the next subsection, as elementary problem <d>.

13.5. Problem <d>

This problem is included into the set of elementary problems in order to facilitate solving fields where the load $\mathbf{p}^{\alpha,\alpha+1}$ is applied along internal lines of discontinuity $\mathcal{L}^{\alpha,\alpha+1}$. It turns out that it is convenient to extend this problem to include also the case of an external line of the field $\mathcal{L}^{N,0}$, which might be loaded or not.

Then, two cases will be discerned within Problem <d>:

1. The considered line type $\mathcal{L}^{N,0}$ might be externally loaded or not, but behind it exists the state of zero stress $(\overset{(N+1)}{\boldsymbol{\sigma}} = \mathbf{0})$;

2. Some arbitrary limit state of stress is created ($\overset{(\alpha+1)}{\sigma} \neq \mathbf{0}$) behind the line of type $\mathcal{L}^{\alpha,\alpha+1}$

In both cases, the given data create $\overset{(\alpha)}{\omega}$, $\overset{(\alpha)}{\phi}$ in the start region α, and the solution of the field in the region up to this one, inclusively.

In order to define the position of the created discontinuity line $\mathcal{L}^{\alpha,\alpha+1}$ or $\mathcal{L}^{N,0}$, one must specify its parameter $\nu^{\alpha,\alpha+1}$ ($\nu^{N,0}$), which makes it possible to calculate $\Delta\gamma^{\alpha} = \nu^{\alpha,\alpha+1} - \overset{(\alpha)}{\phi} + \frac{1}{2}\pi$, and to place this line in the local system $\{\xi\}^{\alpha}$.

In the Case 1. ($\overset{(N+1)}{\sigma} = \mathbf{0}$), the system $\{\xi\}^{(N+1)}$ is associated with line $\mathcal{L}^{N,0}$, and its axis ξ_2 is directed outside the region N; it is assumed that $\overset{(N+1)}{\phi} = \nu^{N,0}$, which is different from that on line $\mathcal{L}^{0,1}$. The load determined in this system, $\mathbf{p}^{N,0}$, is applied to the line $\mathcal{L}^{N,0}$.

In the Case 2., the state of stress in the region $\alpha + 1$ does not depend on the states in preceding regions, so that it is necessary to input additionally the parameters $\overset{(\alpha+1)}{\omega}$, $\overset{(\alpha+1)}{\phi}$.

The calculated external load on the line $\mathbf{p}^{\alpha,\alpha+1}$ is also given in the system associated with the stress discontinuity line.

14. Properties of limit systems of stress discontinuity lines in the fields around nodes

14.1. General remarks

The solutions of boundary problems formulated for fields around nodes are sometimes surprising and their interpretation might be difficult. It results, first of all, from the properties of the problems themselves and these reasons can be shortly described as follows:

1. The problems met here often lead to the situation in which even the dimension of the system of conditions is not *a priori* known. Then, we can define *a priori* neither the number of regions (N), nor the number of stress discontinuity lines that separate them. At the same time, it is known that, when these numbers are arbitrarily (incorrectly) assumed, the solution to the field we seek for might not exist (and in general it does not exist).
2. We have at our disposal only one equation type (11.1) (nonlinear, with singularities and very complicated) for determining the parameters of field around a node. The equation is formulated for all possible combinations of parameter values $Q^{1,2}$, $Q^{2,3}$,.., $Q^{N-1,N}$ and the conditional solutions to this equation can be found for a fixed (assumed) number N. The number of unknowns, however, becomes equal to the number of equations only when the field we seek for contains $N = 3$ homogeneous regions.

3. The set of limitations comprises the conditions for the contents in the domain $(\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\} \in \Lambda_{1,N}^{\alpha,\alpha+1})$, which are the necessary conditions of existence. Moreover, it comprises structural conditions (10.1) and – possibly – geometrical ones, all of them nonlinear.

The properties of the equations (11.1) and the conditions formulated on lines $\mathcal{L}$ can be best characterised by the graphical forms of functions $\Delta\hat{\phi}$ and $\Delta\hat{\gamma}$ (7.2), (7.4), on which these equations are determined. The plots of the functions are sketched in Fig. 18a,b in the form of three-dimensional graphs spread over the admissible area Λ, which is the domain of the functions. The Huber–Mises yield condition was assumed when deriving these functions.

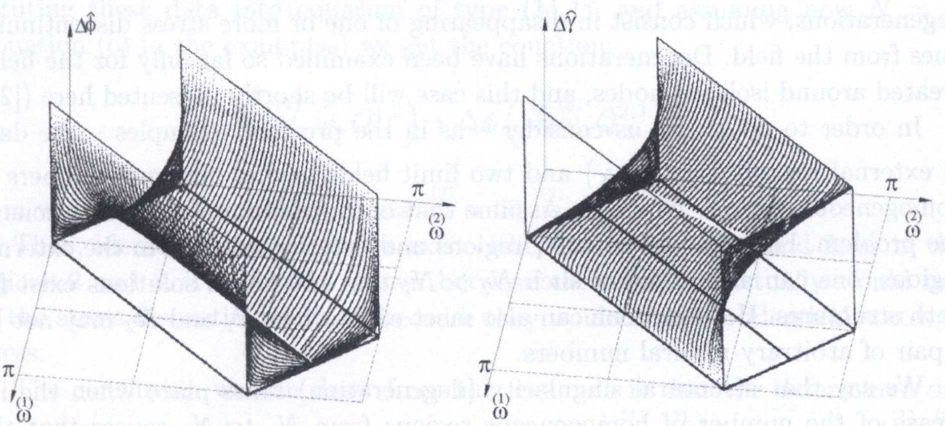

FIG. 18. **Plots of functions $\Delta\hat{\phi}\,(\overset{(1)}{\omega}, \overset{(2)}{\omega})$ and $\hat{\gamma}(\overset{(1)}{\omega}, \overset{(2)}{\omega})$ drawn over the area Λ for the Huber–Mises yield condition (similary as for arbitrary $\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}$ in adjacent regions).**

As it can be seen in Fig. 18, the function $|\Delta\,\phi\,(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega})|$ initially takes an approximately cylindrical shape when moving from its central cross-section $\overset{(\alpha)}{\omega} + \overset{(\alpha+1)}{\omega} = \pi$ towards the origin of the co-ordinate system. Then, one can see a growing and finally rapid change of the function shape near the section $\overset{(\alpha)}{\omega} + \overset{(\alpha+1)}{\omega} = \pi/3$, and finally the function takes – in the same section – the form of a U-shaped broken line. Similar changes of the function shape can be observed when moving in opposite direction, because the function is symmetrical about the plane $\overset{(\alpha)}{\omega} + \overset{(\alpha+1)}{\omega} = \pi$.

We omit the description of the surface $\Delta\hat{\gamma}(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega})$, which is clearly represented in Fig. 18b. It is worth noticing, however, that the function is undefined for $\overset{(\alpha)}{\omega} = \overset{(\alpha+1)}{\omega}$.

Each equation of type (11.1) comprises as many functions $\Delta\hat{\phi}(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega})$ as many lines $\mathcal{L}^{\alpha,\alpha+1}$ exist in the field. Then, properties of each particular function $\Delta\hat{\phi}$, its singularities and possible undefined points will repeat in the same number and will be superimposed on one another. It may give some idea what is the level of technical difficulties in solving each discontinuous, limit field of stresses.

14.2. Structural degenerations

The previously described properties of function $\Delta\hat{\phi}$ may explain, to some extent, the reasons why the systems of stress discontinuity lines in complex fields do not exhibit any regularity, and why various singularities appear in such fields.

One of the most visible effects that accompany the singularities are structural degenerations, which consist in disappearing of one or more stress discontinuity lines from the field. Degenerations have been examined so far only for the fields created around isolated nodes, and this case will be shortly presented here ([2]).

In order to do so, let us consider – as in the previous examples – the data in external regions $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \chi\}$ and two limit fields having different numbers of homogeneous regions, $N_1 < N_2$. Assume that both fields satisfy all conditions of the problem. For the field with N_1 regions and determined data in the external regions, one can always select such $N_2 > N_1$ that the field's solutions exist for both structures. However, one can also meet cases when N_1 and N_2 may not be a pair of arbitrary natural numbers.

We say that structural singularity (degeneration) takes place when the increase of the number of homogeneous regions from N_1 to N_2 causes that the solution ceases to exist for the framework of the structure of N_2 regions.

The boundary conditions $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \chi\}$ for which degeneration takes place are called the singular conditions. In the space $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \chi\}$, singular boundary conditions form a set whose dimension is lower than the dimension of the space itself. Such conditions are then difficult to detect, and create difficulties when applying numerical analyses, which – because of the complexity of the problem – are the only methods of analysis that could be used.

One of the simple cases of structural degeneration can be found in the fields consisting of only $N = 3$ homogeneous regions. It is known that the construction of field with only one line $\mathcal{L}^{1,3}$ is not permissible for an arbitrary ternary set of numbers $\{\overset{(1)}{\omega}, \overset{(3)}{\omega}, \chi\}$ from the domain. It turns out, however, that when $\{\overset{(1)}{\omega}, \overset{(3)}{\omega}, \chi\}$ are chosen in such a way that they can be separated with one line (it means $(\overset{(1)}{\omega}, \overset{(3)}{\omega}) \in \Lambda$, $|\chi| = |\Delta\hat{\phi}(\overset{(1)}{\omega}, \overset{(3)}{\omega})|$), then the construction of a field with two lines $\mathcal{L}^{1,2}$, $\mathcal{L}^{2,3}$ and one intermediate region 2 is impossible. The effect of increasing the number of regions from $N_1 = 2$ to $N_2 = 3$ is that the solution of the field would not exist for the structure of $N_2 = 3$ regions.

14.3. *Example*

Let us first consider a field consisting of two homogeneous regions 1, 2, separated by line $\mathcal{L}^{1,2}$. Let there be given: $\overset{(1)}{\omega} = 60$, $\overset{(2)}{\omega} = 90$. By substituting these data into (7.2) we obtain $\Delta\hat{\phi} = 27.36781°$. For the considered field of two regions, the value $\Delta\phi(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2})$ that we seek for (according to (7.1), with $q^{1,2} = 1$, there is $\Delta\phi(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2}) = \Delta\hat{\phi}$) is the angle χ between principal directions of stress in the external regions, because the regions 1 and 2 are the external ones in the considered field.

In the second case, we consider a field of three regions 1, 2, 3, and assume identical data in the external regions: $\overset{(1)}{\omega} = 60$, $\overset{(3)}{\omega} = 90$, $\chi = 27.36781°$. Substituting these data into equation of type (11.1), and assuming now $N = 3$, (equation (c) in the examples) we get the equation:

$$\Delta\phi\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, Q^{1,2}\right) + \Delta\phi\left(\overset{(2)}{\omega}, \overset{(3)}{\omega}, Q^{2,3}\right) = \chi$$

from which we determine two roots: $\overset{(2)}{\omega} = 60$, $\overset{(2)}{\omega} = 90$.

Then, identical states of stress exist either in the regions 1 and 2, or in the regions 2 and 3. The lines $\mathcal{L}$ drawn between these regions actually would not be the stress discontinuity lines, because they would separate identical states of stress.

The conclusion we can draw from this example is that the determined degeneration points belong to a surface set $|\chi| = |\Delta\hat{\phi}(\overset{(1)}{\omega}, \overset{(3)}{\omega})|$ in the space $\{\overset{(1)}{\omega}, \overset{(3)}{\omega}, |\chi|\}$, and this set is exactly equivalent to the function $|\Delta\hat{\phi}|$ shown in Fig. 18a. When the point-image $B(\overset{(1)}{\omega}, \overset{(3)}{\omega}, |\chi|)$ of the boundary conditions is situated on this surface, then one of the lines $\mathcal{L}$ disappears. To describe it in a graphic way we say that a structural collapse takes place there.

14.4. *Reduction of the set of variables and partial autonomy of component problems*

In the description presented in this paper, the field parameters are determined on the smallest, irreducible set of variables $\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}\}$. Moreover, in this set all the roots are explicitly specified by means of parameters $q^{\alpha,\alpha+1} = 1..4$. On top of that, when we use functions $\Delta\gamma(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1})$ ((7.3)), and when the equality conditions of existence (defined by functions $\Delta\phi(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1})$) are satisfied on each line, then the equilibrium equations on lines $\mathcal{L}$ are identically satisfied. Then, what remains to be done is to formulate only one equation of type (11.1) for each field around a node.

Besides of the properties presented so far, this equation has other interesting features, quite important for the methods of solving complex fields. First of all, it is defined exclusively on functions $\Delta\phi$. Consequently, it can be solved independently of the geometrical characteristics of line $\mathcal{L}$, and one can initially determine the unknown states of stress which, nonetheless, can be separated by admissible discontinuity lines. The increments $\Delta\gamma$ and the directions of lines $\mathcal{L}$ can be calculated separately later, using the equations (7.3), and taking into account, of course, the structural (10.1) and geometrical conditions.

The latter property is very valuable, because not only the reduction of the set of variables, but – first of all – the possibility of dividing the problems into partly autonomous ones, has an essential meaning for the effectiveness of solving the nonlinear problems encountered in the limiting fields. This property is then exploited everywhere, whenever it is possible.

In Part IV of this study, we will present yet another kind of partial autonomy, the one often exploited in analyses of fields of great complexity.

All the properties described here for the fields around nodes can be transferred, in a natural way, to the cases of the complex fields that contain more nodes, homogeneous regions and lines. Besides obvious multiplication of the effects already shown, the effects resulting from various couplings and mutual interactions between adjacent nodes will be also superimposed.

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