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Vibrations of a Circular Plate Supported on a Rigid Concentric Ring with Translational Restraint Boundary

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This paper deals with frequency analysis of a circular plate supported on a rigid concentric ring with translational restrained boundary. Natural frequencies of such a circular plate are computed for different sets of elastic translational restraints, and for various values of the radius of the internal ring support. Results for different modes of plate vibrations are computed and presented in a tabular form suitable for use in design. The effect of plate boundary conditions such as translational restraints and the radius of concentric ring support on natural frequencies of the circular plate are studied. Exact frequency values presented in this paper are expected to serve as benchmark solutions for assessing the accuracy of other numerical methods being used in the literature.

Key words: circular plate, frequency, translational restraint, rigid ring, mode switching.

1. Introduction

Vibration analysis of plates of all shapes and sizes is a challenging subject of research and many classical problems were studied using both exact [1] and approximate methods such as finite element method [2], boundary element method [3] and differential quadrature method. A thorough review of those studies is not undertaken in this paper. However, it is worth to emphasize that with the continuous efforts of researchers such as Katsikadelis [4–8] and others such as Gospodinov and Liutskanov [9] and Guminiak [10] fruitful advances have been made in effectively applying the boundary element method to plate vibration problems.

Continuous circular plates have applications in various fields of engineering. Studies on the frequency analysis of circular plates with various edge conditions and internal strengthening have been extensively reviewed in the literature [11–21]. Bodine [22] studied axisymmetric free vibrations of the circular plate with classical boundary conditions, and Laura *et al.* [23] presented more accurate results for the case of axi-symmetric mode of vibration. However, the fundamental frequency under investigation does not need to be axisymmetric all the time. Bodine [24] studied the case of a circular plate supported on concentric ring-type support, and observed a change of the fundamental mode from symmetric to asymmetric in certain cases where the radius of the support approaches smaller values. Wang [19, 20] carried out frequency analysis of a free edge circular plate supported on a ring and showed that the fundamental frequency corresponds to an asymmetric mode as the concentric ring radius becomes lower.

It is now a widely accepted fact that the condition of plates on a periphery often tends to be a part of the classical boundary conditions and may correspond more closely to some form of elastic restraints, i.e., rotational and translational restraints. Free vibration analysis of circular plates with such boundary conditions was already discussed in [1–16, 21]. However, to the authors' best knowledge there is no other research available that would address the case of a circular plate supported by a rigid ring-type structure having translational restraints along the boundary of the plate. The main aim of this paper, therefore, is to study the effect of the radius of the rigid ring support of a thin circular plate being translationally restrained along the outer edge using the exact method of solving the problem. The natural frequencies of the circular plate for varying values of radius of rigid ring support and non-dimensional translational restraint parameters are computed and presented in the form suitable for use in the design of such circular plates, which has wider applications in the engineering industry.

2. Definition and formulation of the problem

The plate under consideration has the radius R, Poisson's ratio v, density ρ, thickness h, and elastic constant E. Figure 1 presents the plate, which has an outer boundary translational restrained edge (at radius R) and rigid ring support (at radius bR).

$b \leq r \leq 1$ (outer zone) is denoted by subscript I and $0 \leq r \leq b$ (inner zone) is denoted by subscript II respectively. All the lengths are normalized by R, i.e., the radius of the outer zone is 1 and that of the inner zone is b. In circular plate

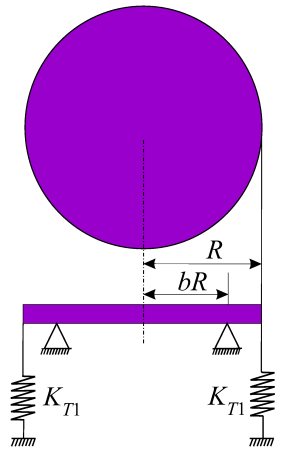

FIG. 1. Internal concentric rigid ring-supported circular plate with a translational restrained edge.

supported (CPT) [1], a fourth-order differential equation describes free flexural vibrations of the circular uniform plate as follows:

$$D \cdot \nabla^4 w + \rho \cdot h \cdot \frac{\partial^2 w}{\partial t^2} = 0, \tag{2.1}$$

where D represents the flexural rigidity. The general form of lateral displacement of vibration of the plate can be expressed as $w = u(r)\cos(n\theta)e^{i\omega t}$, where (r, θ) are polar coordinates, w is transverse displacement, n is the number of modal diameters, ω is the frequency, t is time, and $k = R(\rho\omega^2/D)^{1/4}$ is the square root of non-dimensional frequency [3]. Here, function $u(r)$ is a linear combination of Bessel functions $J_n(kr)$, $Y_n(kr)$, $I_n(kr)$, and $K_n(kr)$, where $J_n(kr)$ is the Bessel function of the first kind, $Y_n(kr)$ is the Bessel function of the second kind, $I_n(kr)$ is the modified Bessel function of the first kind and $K_n(kr)$ is the modified Bessel function of the second kind. General solutions for two zones can be expressed as

$$u_{I,rr}(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr), \tag{2.2}$$

$$u_{II,rr}(r) = C_5 J_n(kr) + C_6 I_n(kr). \tag{2.3}$$

Boundary conditions at the edge of the plate can be formulated as follows:

$$M_r(r,\theta) = 0, \tag{2.4}$$

$$V_r(r,\theta) = -K_{T1} w_I(r,\theta). \tag{2.5}$$

Here, bending and shear force can be expressed as

$$M_r(r,\theta) = -\frac{D}{R}\left[\frac{\partial^2 w_I(r,\theta)}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_I(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_I(r,\theta)}{\partial \theta^2}\right)\right], \tag{2.6}$$

$$V_r(r,\theta) = -\frac{D}{R^3}\left[\frac{\partial}{\partial r}\nabla^2 w_I(r,\theta) + (1-\nu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 w_I(r,\theta)}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_I(r,\theta)}{\partial \theta}\right)\right]. \tag{2.7}$$

From Eqs. (2.4), (2.5), (2.6) and, (2.7) we obtain the following:

$$\left[\frac{\partial^2 w_I(r,\theta)}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_I(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_I(r,\theta)}{\partial \theta^2}\right)\right] = 0, \tag{2.8}$$

$$\left[\frac{\partial}{\partial r}\nabla^2 w_I(r,\theta) + (1-\nu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 w_I(r,\theta)}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_I(r,\theta)}{\partial \theta}\right)\right] = T_{11} w_I(r,\theta). \tag{2.9}$$

Equations (2.8) and (2.9) can be written as

$$u_I''(r) + \nu\left[u_I'(r) - n^2 u_I(r)\right] = 0, \tag{2.10}$$

$$u_I'''(r) + u_I''(r) - \left[1 + n^2(2-\nu)\right]u_I'(r) + n^2(3-\nu)u_I(r) = -T_{11}u_I(r). \tag{2.11}$$

The boundary at the edge (at $r = 1$) is as follows:

$$u_I''(1) + \nu\left[u_I'(1) - n^2 u_I(1)\right] = 0, \tag{2.12}$$

$$u_I'''(1) + u_I''(1) - \left[1 + n^2(2-\nu)\right]u_I'(1) + n^2(3-\nu)u_I(1) = -T_{11}u_I(1), \tag{2.13}$$

where $T_{11} = \frac{K_{T1}R^3}{D}$ is normalized constant K_{T1} of the translational spring at the outer edge.

The continuity requirement at the ring support can be expressed as

$$u_I(b) = 0, \tag{2.14}$$

$$u_{II}(b) = 0, \tag{2.15}$$

$$u_I'(b) = u_{II}'(b), \tag{2.16}$$

$$u_I''(b) = u_{II}''(b). \tag{2.17}$$

Next, non-trivial solutions to Eqs. (2.12)–(2.17) are sought. From Eqs. (2.1), (2.2), (2.12)–(2.17) we obtain the following:

$$\begin{aligned}(2.18)\quad & \left[\frac{k^2}{4}P_2+\frac{k\nu}{2}P_1-\left(\frac{k^2}{2}+\nu n^2\right)J_n\left(k\right)\right]C_1\\ &+\left[\frac{k^2}{4}Q_2+\frac{k\nu}{2}Q_1-\left(\frac{k^2}{2}+\nu n^2\right)Y_n(k)\right]C_2\\ &+\left[\frac{k^2}{4}R_2+\frac{k\nu}{2}R_1+\left(\frac{k^2}{2}-\nu n^2\right)I_n\left(k\right)\right]C_3\\ &-\left[\frac{k^2}{4}S_2-\frac{k\nu}{2}S_1+\left(\frac{k^2}{2}-\nu n^2\right)K_n(k)\right]C_4=0,\end{aligned}$$

$$\begin{aligned}(2.19)\quad & \left[\frac{k^3}{8}P_3+\frac{k^2}{4}P_2-\frac{k}{2}\left(\frac{3}{4}k^2+n^2\left(2-\nu\right)+1\right)P_1\right.\\ &\left.+\left(n^2\left(3-\nu\right)-\frac{k^2}{2}-T_{11}\right)J_n(k)\right]C_1\\ &+\left[\frac{k^3}{8}Q_3+\frac{k^2}{4}Q_2-\frac{k}{2}\left(\frac{3}{4}k^2+n^2\left(2-\nu\right)+1\right)Q_1\right.\\ &\left.+\left(n^2\left(3-\nu\right)-\frac{k^2}{2}-T_{11}\right)Y_n(k)\right]C_2\\ &+\left[\frac{k^3}{8}R_3+\frac{k^2}{4}R_2+\frac{k}{2}\left(\frac{3}{4}k^2-n^2\left(2-\nu\right)+1\right)R_1\right.\\ &\left.+\left(n^2\left(3-\nu\right)+\frac{k^2}{2}-T_{11}\right)I_n(k)\right]C_3\\ &+\left[-\frac{k^3}{8}S_3+\frac{k^2}{4}S_2+\frac{k}{2}\left(-\frac{3}{4}k^2+n^2\left(2-\nu\right)+1\right)S_1\right.\\ &\left.+\left(n^2\left(3-\nu\right)+\frac{k^2}{2}-T_{11}\right)K_n(k)\right]C_4=0,\end{aligned}$$

$$(2.20)\qquad J_n(kb)C_1+Y_n(kb)C_2+I_n(kb)C_3+K_n(kb)C_4=0,$$

$$(2.21)\qquad J_n(kb)C_5+I_n(kb)C_6=0,$$

$$(2.22)\qquad \left[\frac{k}{2}P_1'\right]C_1+\left[\frac{k}{2}Q_1'\right]C_2+\left[\frac{k}{2}R_1'\right]C_3-\left[\frac{k}{2}S_1'\right]C_4-\left[\frac{k}{2}P_1'\right]C_5-\left[\frac{k}{2}R_1'\right]C_6=0,$$

$$\left[\frac{k^2}{4}P_2' - \frac{k^2}{2}J_n(kb)\right]C_1 + \left[\frac{k^2}{4}Q_2' - \frac{k^2}{2}Y_n(kb)\right]C_2 + \left[\frac{k^2}{4}R_2' + \frac{k^2}{2}I_n(kb)\right]C_3 + \left[\frac{k^2}{4}S_2' + \frac{k^2}{2}K_n(kb)\right]C_4 - \left[\frac{k^2}{4}P_2' - \frac{k^2}{2}J_n(kb)\right]C_5 - \left[\frac{k^2}{4}R_2' + \frac{k^2}{2}I_n(kb)\right]C_6 = 0, \tag{2.23}$$

where

$$\begin{aligned}
P_1 &= J_{n-1}(k) - J_{n+1}(k); & P_2 &= J_{n-2}(k) + J_{n+2}(k);\\
P_3 &= J_{n-3}(k) - J_{n+3}(k); & Q_1 &= Y_{n-1}(k) - Y_{n+1}(k);\\
Q_2 &= Y_{n-2}(k) + Y_{n+2}(k); & Q_3 &= Y_{n-3}(k) - Y_{n+3}(k);\\
R_1 &= I_{n-1}(k) + I_{n+1}(k); & R_2 &= I_{n-2}(k) + I_{n+2}(k);\\
R_3 &= I_{n-3}(k) + I_{n+3}(k); & S_1 &= K_{n-1}(k) + K_{n+1}(k);\\
S_2 &= K_{n-2}(k) + K_{n+2}(k); & S_3 &= K_{n-3}(k) + K_{n+3}(k);\\
P_1' &= J_{n-1}(kb) - J_{n+1}(kb); & P_2' &= J_{n-2}(kb) + J_{n+2}(kb);\\
Q_1' &= Y_{n-1}(kb) - Y_{n+1}(kb); & Q_2' &= Y_{n-2}(kb) + Y_{n+2}(kb);\\
R_1' &= I_{n-1}(kb) + I_{n+1}(kb); & R_2' &= I_{n-2}(kb) + I_{n+2}(kb);\\
S_1' &= K_{n-1}(kb) + K_{n+1}(kb); & S_2' &= K_{n-2}(kb) + K_{n+2}(kb).
\end{aligned}$$

3. Results and discussions

Poisson's ratio used in our study is 0.3. Given the set of n, v, T_{11} and b, the above mentioned equations are solved to obtain an exact characteristic frequency equation by suitably eliminating the coefficients C_1, C_2, C_3, C_4, C_5 and C_6. The frequency parameter k can be determined from characteristic equation.

The fundamental frequency for $n = 0$ (axisymmetric) and $n = 1$ (asymmetric) modes for different sets of translational restraints ($T_{11} = 5$, 20, 50, 100, 500, 1000 and 10^{16}) are computed. Plate vibrations for the first three modes are obtained and presented in Figs. 2–9. Figure 2 represents, the curve that is composed of two segments for a given $T_{11} = 2.5$ due to vibration mode switching. For lower values of b, fundamental frequency is corresponding to asymmetric $n = 1$ mode. This mode is represented by the dotted line shown in Fig. 2, where fundamental frequency decreases as b decreases. For higher values of b, fundamental frequency is corresponding to $n = 0$ mode. This mode is represented by the continuous line shown in Fig. 2, where fundamental frequency increases as b decreases up to a peak point that corresponds to the maximum frequency and thereafter decreases as b decreases, as shown in Fig. 2.

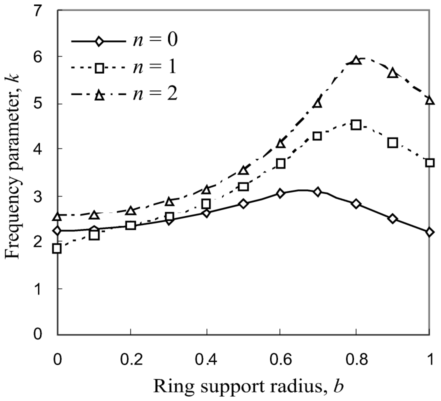

FIG. 2. Frequency of a circular plate and concentric rigid support radius b for $T_{11} = 2.5$.

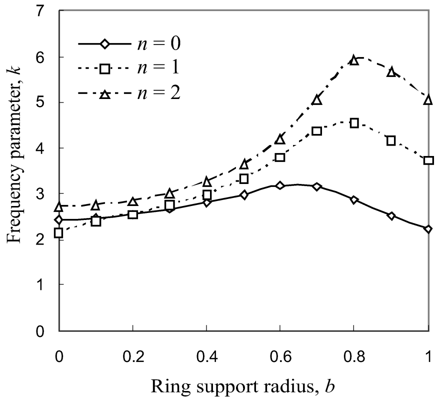

FIG. 3. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 5$.

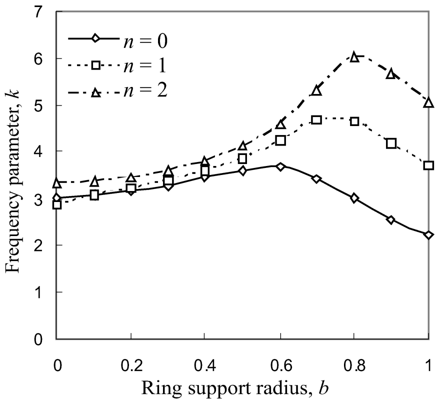

FIG. 4. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 20$.

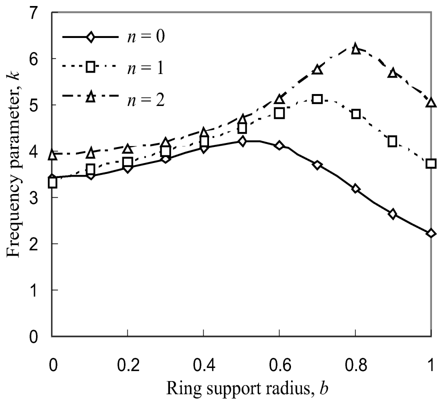

FIG. 5. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 50$.

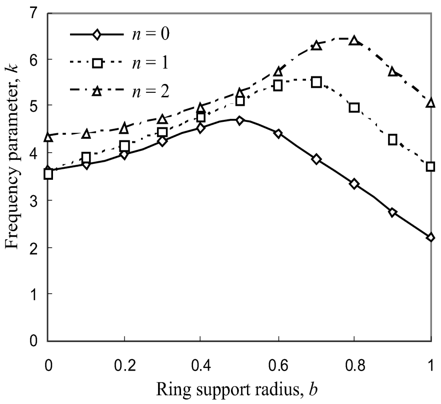

FIG. 6. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 100$.

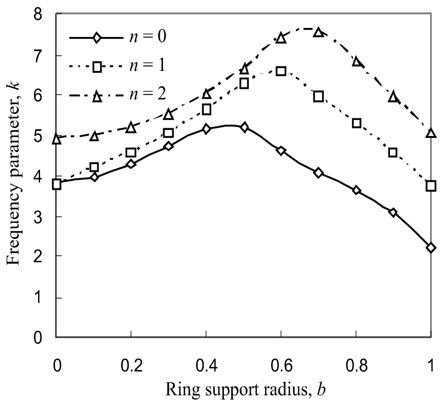

FIG. 7. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 500$.

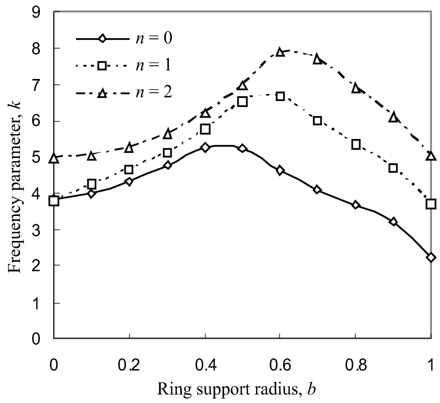

FIG. 8. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 10^3$.

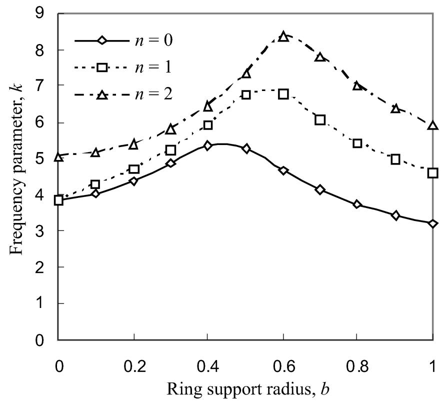

FIG. 9. Fundamental frequency of a circular plate and concentric rigid support radius b for $T_{11} = 10^{16}$.

Figure 2 represents mode switching (cross-over radius) from asymmetric to axi-symmetric mode, in this case $b = 0.195325$. Fundamental frequency is governed by $n = 1$ mode when $b \leq 0.195325$, which is shown by the dotted line in Fig. 2. When b increases beyond 0.195325, the $n = 0$ mode gives correct fundamental frequency as shown by the continuous lines in Fig. 2. Optimal locations (concentric ring support and subsequent fundamental frequency) are $b = 0.70238$ and $k = 3.07535$ respectively, which are equal to nodal radius of axisymmetric mode and frequency.

Similarly, it can be observed in Figs. 3–9, for the given set of translational restraints ($T_{11} = 5$, 20, 50, 100, 500, 1000 and 10^{16}), that the curve is composed of two segments due to vibration mode switch. For lower values of b, fundamental frequency is corresponding to asymmetric mode. This mode is represented by the dotted line shown in Figs. 3–9, where fundamental frequency decreases as b decreases. For higher values of b, fundamental frequency is corresponding to axisymmetric mode. This mode is represented by the continuous line shown in Figs. 3–9, where fundamental frequency increases as b decreases up to a peak point which corresponds to the maximum frequency and thereafter decreases as b decreases, as shown in Figs. 3–9. The cross-over radius b_{cor} and the corresponding frequency parameters k_{cor} are computed and presented in Table 1.

Table 1. The cross-over radius b_{cor} and the corresponding frequency parameters k_{cor}.

T_{11}	2.5	5	20	50	100	500	1000	10^{16}
b_{cor}	0.19532	0.17964	0.08801	0.04724	0.02616	0.01072	0.008642	0.00712
k_{cor}	2.33696	2.51871	3.05530	3.45200	3.64852	3.81890	3.840800	3.86051

In addition, optimal locations (concentric ring support b_{opt} and subsequent fundamental frequency k_{opt}) are obtained and presented in Table 2.

Table 2. Optimal locations (ring support b_{opt} and subsequent frequency k_{opt}).

T_{11}	2.5	5	20	50	100	500	1000	10^{16}
b_{opt}	0.70238	0.59948	0.60000	0.50000	0.5000	0.50000	0.40000	0.40000
k_{opt}	3.07535	3.17442	3.67786	4.22808	4.7004	5.18757	5.27143	5.36056

The switching of mode changes (decreases) from 0.19532 to 0.00712 as T_{11} varies from 2.5 to 10^{16}. The optimal location varies (decreases) from 0.70238 for $T_{11} = 2.5$ to 0.4 for $T_{11} = 10^{16}$. The fundamental frequency increases from 3.07535 to 5.36056 at the respective optimal locations. Frequency values for $n = 0$ mode agree with that of Laura *et al.* [2]. Table 3 presents the exact fundamental frequency for a circular plate with a free boundary (by setting $T_{11} \to 0$ in the present problem), in agreement with that found by Wang [15].

Table 3. Comparison of fundamental frequencies for $v = 0.3$ with the ones obtained by Wang [15], for free edge.

Ring support radius, b	Wang [15]	Present
0.00	0.000	0.00000
0.02	1.501	1.50077
0.05	1.634	1.63422
0.10	1.789	1.78911
0.15	1.922	1.92226
0.20	2.051	2.05103

4. Conclusions

Fundamental frequency of a concentric ring-supported circular plate with a translational restrained boundary was studied in this work. In addition, frequencies were presented for different translational restraints T_{11} at the boundary, which simulate a free boundary when $T_{11} \to 0$. A fundamental frequency mode switching (from $n = 1$ to $n = 0$) was observed at a specific radius of the ring. This mode switch was computed. The optimal solutions (optimum internal *ring* support and the corresponding fundamental frequency) were computed. The obtained results are a closed form solution. Hence, the results can serve as a benchmark solution. These results can be useful in the design of support structures.

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