Engineering Transactions, **74**(2): 125–144, 2026, https://doi.org/10.24423/engtrans.2026.3610
Institute of Fundamental Technological Research Polish Academy of Sciences (IPPT PAN)
Université de Lorraine • Poznan University of Technology

Research Paper

On the Out-of-Plane Deviation of Bending Deformation States in Moderately Thick Bars with Asymmetric Cross-Sections

Radosław Czubacki*, Zofia Kozyra, Tomasz Lewiński

Faculty of Civil Engineering, Warsaw University of Technology
Warsaw, Poland

*Corresponding Author: radoslaw.czubacki@pw.edu.pl

A characteristic feature of the six-parameter theories of bars is the coupled form of the constitutive equations; in particular the equations linking transverse forces with transverse shear deformations cannot be, in general, decoupled while keeping a separate form of the remaining constitutive equations. The mentioned feature of the constitutive equations implies that, within the six-parameter theories of straight elastic prismatic bars, there do not exist, in general, plane states of bending/shearing deformations. Thus, any vertical load causes lateral deflections, the only exception being the pure bending problem. The present paper delivers analytical solutions: closed-form formulae for shape functions, i.e., deformation states associated with kinematic loads at the ends, and solutions to selected static problems corresponding to transverse span load. Although elementary, the presented solutions seem to be derived for the first time. In particular, the hitherto published shape functions concerned the theories of moderately thick bars in which all the constitutive equations are decoupled.

Keywords: six-parameter theory of bars, moderately thick bars, shape functions.

1. Introduction

Among the theories of linearly elastic straight prismatic bars the most important are: the theory by Vlasov [1] for bars with thin-walled and open profiles and the six-parameter theories of moderately thick bars attributed to Timoshenko, see Bazoune *et al.* [2], or to Saint-Venant, see Petrolo and Casciaro [3]. A peculiar feature of the Vlasov theory is the decoupling of the constitutive equations: the axial force is linked to the axial strain, the bending moments are proportional to the corresponding bending strains, the torsional moment is linked to the measure of torsion and the bimoment is proportional to the measure of warping due to torsion. This decoupling is possible by applying special measures:

– assuming the principal axes y and z of the cross-section $\mathcal{A}$,
– an appropriate choice of the starting point of the sectorial coordinate,
– an appropriate choice of the position of the pole.

The formalism of the Vlasov theory can be extended to the case of straight prismatic bars with arbitrary cross-sections, see LEWIŃSKI and CZARNECKI [4, Sec. 9], still preserving the mentioned decoupling of the constitutive equations by making appropriate measures concerning the position of the axes to which the internal forces are referred.

The theory of Vlasov neglects the transverse shear deformations thus removing the transverse forces from the set of internal forces of the theory. However, including transverse shear effects is possible since the Saint-Venant theory provides analytical expressions for the functions modeling warping due to shear, see LOVE [5] and IEŞAN [6]. Such a bar theory has been proposed by LIBRESCU and SONG [7]. By neglecting in this theory the contribution of warping due to torsion to the elastic energy one arrives at a six-parameter theory. Its form in the 3D setting is not unique. In a standard approach, all the internal forces are referred to the neutral axis, i.e., the axis linking the centroids of the cross-sections, see, e.g., PETROLO and CASCIARO [3]. In this model, the axial force is proportional to the axial strain, the bending moments are linked to the corresponding measures of bending while the triple $(T_y, T_z, \mathcal{M})$ or the transverse forces and the torsional moment are linked with the triple: $(\gamma_y, \gamma_z, \rho)$, where γ_y and γ_z stand for the measures of transverse shear and ρ is the measure of torsional deformation. The 3×3 matrix linking these quantities is fully populated, see PETROLO and CASCIARO [3, Eq. (40)].

The papers discussing more complicated models of bars, such as DIKAROS *et al.* [8], El Fatmi [9, 10], show that, as in the Vlasov theory, it is expedient to shift the transverse forces to the axis $x_{(s)}$ linking the shear centers S. By an appropriate choice of functions modeling warping due to torsion and shear one can derive a bar model in which the torsional moment is only linked to the strain ρ, and the pair (T_y, T_z) is linked to the pair (γ_y, γ_z), the latter 2×2 matrix being, in general, fully filled up, i.e., the off-diagonal components are in general non-zero. If we decouple the latter 2×2 system by a certain rotation of the axes y and z, we introduce coupling in the constitutive equations for the bending moments. Indeed, there is no reason to change the parametrization y and z referred to the principal axes of the cross-section. We conclude: coupling of the constitutive equations linking (T_y, T_z) with the pair (γ_y, γ_z) is an immanent feature of the six-parameter theory of bars in all its versions.

The six-parameter model is the simplest theory of deformation of bars in space in which the kinematic unknowns are: u, v, w, θ, φ, β or, subsequently, the axial displacement, displacements of the shear center in the y- and z-directions, the angle of torsion, and the angles of rotations around the axes $(-z)$ and y. The internal forces of this theory are: N, T_y, T_z, $\mathcal{M}$, M_y, M_z or, respectively, the axial force, the transverse forces, torsional moment, and the bending moments in the y- and z-directions. The internal forces are linked to the measures of deformation: ϵ, γ_y, γ_z, ρ, κ_y, κ_z, or, respectively, the measures of axial strain, transverse shear strains, the measure of torsion, and measures of bending. The main feature of this theory is equality between the number of internal forces (always equal to the number of strains) and the number of kinematic fields. Thanks to this equality there exists a family of statically determinate problems, which paves the way for the force method, a helpful tool of structural mechanics.

If the cross-section of the bar is monosymmetric (or bisymmetric, in particular) then the problem of bending/transverse shearing in 3D decouples into two planar problems in the $x-y$ and $x-z$ planes. If the shape of the domain $\mathcal{A}$ of the cross-section is arbitrary then there do not exist plane states of bending/transverse shearing deformation. In particular, a load in the z-direction causes bending in two directions: z and y. It is precisely this problem that is studied in the present paper. The aim is to deliver explicit formulae for the deformation states of a bar subjected to kinematic loads and to selected transverse span loads.

The present paper draws upon the six-parameter theory of straight prismatic bars made of an isotropic and homogeneous material developed in [4, Sec. 10], called there the Timoshenko-like theory. A prerequisite of the theory is the construction of solutions to the three auxiliary elliptic problems posed on the domain $\mathcal{A}$ [4, Secs. 2–4]. Upon solving these problems one can fix the position of the shear center S with coordinates y_S, z_S referred to the principal axes and then determine all the required characteristics and stiffnesses of the bar. This algorithm will not be repeated in the present paper.

One of the aims of the present paper is to provide explicit formulae for the so-called shape functions or the deformation forms of a bar subjected to arbitrary kinematic loads. These functions are given in the compact Eq. (4.14). Interestingly, these formulae are not available in the literature. There is only one paper, namely the paper by Schramm *et al.* [11] in which this problem is considered at a similar level of accuracy, but the explicit form of the solution (4.14) has not been published there. Other papers, such as [12–14], present solutions corresponding to the special case when the coupling of the constitutive equations is absent. This means that either these papers refer to monosymmetric cross-sections or the authors assume, usually tacitly, that the constitutive equations can be accepted in their decoupled form. This assumption paves the way for planar forms of the shape functions, which are incorrect in general or refer to the special case of the cross-section being mono- or bisymmetric.

Moreover, another aim of the present paper is to derive the specific shapes of deformation caused by span loads. In particular, the paper shows that the point loads generate rather unexpected lateral deflections in the form of zigzag lines.

2. Equations of the theory

Consider a straight prismatic bar with the cross-section $\mathcal{A}$, for which we construct the principal axes y and z, the centroid being its center. The axis x is orthogonal to the domain $\mathcal{A}$; this axis links the centroids of all cross-sections. They are x-independent, which means that the bar is prismatic. The domain of the bar is filled with a homogeneous and isotropic elastic material with Young's modulus E and shear modulus G. The shear center S has the coordinates (y_S, z_S); their construction is explained in [4, Sec. 2]. According to the underlying theory this center coincides with the center of torsion; in more complicated formulations these centers do not coincide, see comments in [4]. By linking points S we form the straight line $x_{(s)}$. Having solved the elliptic problems set up [4, Secs. 2–4] one can compute: the area A of $\mathcal{A}$, the principal moments of inertia J_y, J_z, and the shear correction factors k_y, $k_{yz} = k_{zy}$, k_z forming the matrix $\mathbf{k}$; its inverse is denoted by $\boldsymbol{\alpha}$:

$$\mathbf{k} = \begin{bmatrix} k_y & k_{yz} \\ k_{zy} & k_z \end{bmatrix}, \qquad \boldsymbol{\alpha} = \mathbf{k}^{-1}, \qquad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_y & \alpha_{yz} \\ \alpha_{zy} & \alpha_z \end{bmatrix}. \tag{2.1}$$

Both the matrices $\mathbf{k}$ and $\boldsymbol{\alpha}$ are positive definite. One can compute also the torsional constant J and the torsional stiffness GJ.

The state of deformation of the bar is determined by the kinematic fields $u(x)$, $v(x)$, $w(x)$, $\theta(x)$, $\varphi(x)$, $\beta(x)$ given along the bar, see Fig. 1a. The field u is an average of the displacements $u_x(x, y, z)$ over $\mathcal{A}$, while θ is the angle of rotation of $\mathcal{A}$ around the axis x, the functions v and w represent the displace-

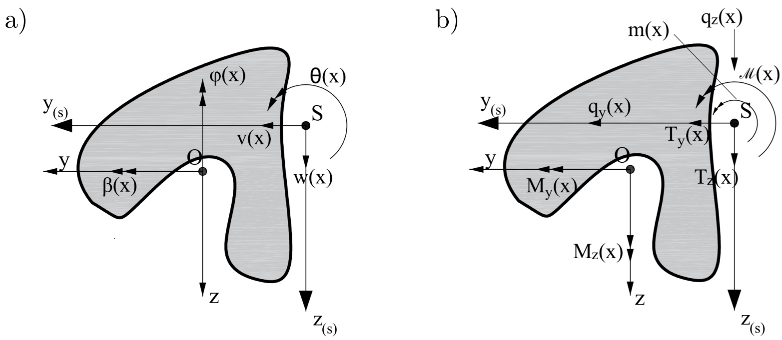

Fig. 1. Sign conventions concerning the kinematic unknowns (a) and the static unknowns (b) at the cross-section $x = \text{const}$.

ments of the point S along the y- and z-axes, respectively, φ and β are defined by the averages:

$$\beta(x) = \frac{1}{J_y} \int_{\mathcal{A}} z u_x \, \mathrm{d}\mathcal{A}, \qquad \varphi(x) = \frac{1}{J_z} \int_{\mathcal{A}} y u_x \, \mathrm{d}\mathcal{A}. \tag{2.2}$$

Their signs are chosen such that they cause positive displacements $u_x(x, y, z)$ if $y > 0$ and $z > 0$.

The loads acting along the bar are reduced to: the axial load of intensity $p(x)$, the transverse loads $q_y(x)$, $q_z(x)$ acting along the axes $y_{(s)}$, $z_{(s)}$, and the distributed moments $m(x)$ acting along the axis $x_{(s)}$.

The following internal forces (or stress and couple resultants) appear in the bar: the axial force $N(x)$, the transverse forces $T_y(x)$, $T_z(x)$ acting along the $y_{(s)}$, and $z_{(s)}$ directions, the bending moments $M_y(x)$, $M_z(x)$ acting along the axes y and z, and $\mathcal{M}(x)$ or the torsional moment acting along the axis $x_{(s)}$, see Fig. 1b.

The following strains are defined within the theory:

$$\begin{aligned} \epsilon &= \frac{\mathrm{d}u}{\mathrm{d}x}, & \gamma_y &= \frac{\mathrm{d}v}{\mathrm{d}x} + \varphi, & \gamma_z &= \frac{\mathrm{d}w}{\mathrm{d}x} + \beta, \\ \kappa_y &= \frac{\mathrm{d}\beta}{\mathrm{d}x}, & \kappa_z &= -\frac{\mathrm{d}\varphi}{\mathrm{d}x}, & \rho &= \frac{\mathrm{d}\theta}{\mathrm{d}x}, \end{aligned} \tag{2.3}$$

where ϵ represents the relative elongation of the bar, γ_y, γ_z are the measures of transverse shear strains in the planes: $x-y$ and $x-z$, respectively; κ_z, κ_y are the measures of bending in the planes $x-y$ and $x-z$, and ρ is the measure of torsion.

By virtue of special measures, i.e., specific interpretations of the internal forces, strains, and kinematic fields, the problem of statics of a bar decomposes into three independent problems:

($\mathcal{P}_1$) The tension/compression problem:
find $N(x)$, $\epsilon(x)$, $u(x)$ such that

$$\frac{\mathrm{d}N}{\mathrm{d}x} + p = 0, \qquad N = EA\epsilon, \qquad \epsilon = \frac{\mathrm{d}u}{\mathrm{d}x}, \tag{2.4}$$

while at the ends $x = 0$, $x = l$, either N or u is given.

($\mathcal{P}_2$) The torsion problem:
find $\mathcal{M}(x)$, $\rho(x)$, $\theta(x)$ such that

$$\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}x} + m = 0, \qquad \mathcal{M} = GJ\rho, \qquad \rho = \frac{\mathrm{d}\theta}{\mathrm{d}x}, \tag{2.5}$$

while at the ends $x = 0$, $x = l$, either $\mathcal{M}$ or θ is given

($\mathcal{P}_3$) The bending/transverse shearing problem:
find $T_y(x)$, $T_z(x)$, $M_y(x)$, $M_z(x)$, $\kappa_z(x)$, $\kappa_y(x)$, $\gamma_y(x)$, $\gamma_z(x)$, $v(x)$, $w(x)$, $\varphi(x)$, $\beta(x)$ such that

$$\frac{\mathrm{d}T_z}{\mathrm{d}x} + q_z = 0, \qquad T_z = \frac{\mathrm{d}M_y}{\mathrm{d}x}, \qquad M_y = EJ_y\kappa_y, \tag{2.6}$$
$$\kappa_y = \frac{\mathrm{d}\beta}{\mathrm{d}x}, \qquad \gamma_y = \varphi + \frac{\mathrm{d}v}{\mathrm{d}x},$$

$$\frac{\mathrm{d}T_y}{\mathrm{d}x} + q_y = 0, \qquad T_y = -\frac{\mathrm{d}M_z}{\mathrm{d}x}, \qquad M_z = EJ_z\kappa_z, \tag{2.7}$$
$$\kappa_z = -\frac{\mathrm{d}\varphi}{\mathrm{d}x}, \qquad \gamma_z = \beta + \frac{\mathrm{d}w}{\mathrm{d}x},$$

$$T_y = GA(k_y\gamma_y + k_{yz}\gamma_z), \qquad T_z = GA(k_{zy}\gamma_y + k_z\gamma_z). \tag{2.8}$$

At the ends $x = 0$ or $x = l$, the following are given: either w or T_z, either β or M_y, either v or T_y, and either φ or M_z.

Let us stress once again that in the standard setting the bending/shearing problem and the torsion problem are coupled, see PETROLO and CASCIARO [3].

3. SIMPLIFICATIONS IN THE CASE OF MONO-SYMMETRIC PROFILES

If the axis $y = 0$ is a symmetry axis of the domain $\mathcal{A}$ then $k_{yz} = k_{zy} = 0$, $\alpha_{yz} = \alpha_{zy} = 0$, and the bending/shearing problem splits up into two problems:

– find $T_y(x)$, $M_z(x)$, $\kappa_z(x)$, $\gamma_y(x)$, $v(x)$, $\varphi(x)$, such that

$$\begin{aligned} &\frac{\mathrm{d}T_y}{\mathrm{d}x} + q_y = 0, && T_y = -\frac{\mathrm{d}M_z}{\mathrm{d}x}, \\ &T_y = k_yGA\gamma_y, && M_z = EJ_z\kappa_z, \\ &\gamma_y = \varphi + \frac{\mathrm{d}v}{\mathrm{d}x}, && \kappa_z = -\frac{\mathrm{d}\varphi}{\mathrm{d}x}. \end{aligned} \tag{3.1}$$

At the ends $x = 0$, $x = l$, either v or T_y; either φ or M_z are prescribed;

– find $T_z(x)$, $M_y(x)$, $\kappa_y(x)$, $\gamma_z(x)$, $w(x)$, $\beta(x)$, such that

$$\begin{aligned} &\frac{\mathrm{d}T_z}{\mathrm{d}x} + q_z = 0, && T_z = \frac{\mathrm{d}M_y}{\mathrm{d}x}, \\ &T_z = k_zGA\gamma_z, && M_y = EJ_y\kappa_y, \\ &\gamma_z = \beta + \frac{\mathrm{d}w}{\mathrm{d}x}, && \kappa_y = \frac{\mathrm{d}\beta}{\mathrm{d}x}. \end{aligned} \tag{3.2}$$

At the ends $x = 0$, $x = l$, either w or T_z; either β or M_y are prescribed.

The majority of analyses available in the literature concern the aforementioned problems.

If one assumes that, additionally, $z = 0$ is a symmetry axis, then the domain $\mathcal{A}$ is bisymmetric and S coincides with the centroid; $(y_S = 0, z_S = 0)$; the axes x and $x_{(s)}$ coincide. Then, the loads q_z, q_y applied along the x-axis do not cause torsion. However, Eq. (3.1) and Eq. (3.2) do not change. Solving the static problems of bars with a mono-symmetric cross-section is no more complicated than in the case of bisymmetric cross-sections.

4. Deformations caused by kinematic loads

Assume that the cross-section $\mathcal{A}$ is of arbitrary shape, and the bar is clamped at both ends: $x = 0$, $x = l$. The span load is absent. The kinematic boundary conditions have the form:

$$\begin{aligned} &u(0) = {}^*u, \qquad && u(l) = u^*, \qquad && \theta(0) = {}^*\theta, \qquad && \theta(l) = \theta^*, \\ &w(0) = {}^*w, && w(l) = w^*, && \beta(0) = {}^*\beta, && \beta(l) = \beta^*, \\ &v(0) = {}^*v, && v(l) = v^*, && \varphi(0) = {}^*\varphi, && \varphi(l) = \varphi^*. \end{aligned} \tag{4.1}$$

The solutions to problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are elementary:

$$\begin{aligned} u(x) &= {}^*u \cdot (1 - \xi) + u^*\xi, \\ \theta(x) &= {}^*\theta \cdot (1 - \xi) + \theta^*\xi, \qquad \xi = \frac{x}{l}. \end{aligned} \tag{4.2}$$

In order to solve problem $(\mathcal{P}_3)$, let us note that the deflections satisfy the following uncoupled system of equations:

$$\begin{aligned} EJ_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} &= q_y - \frac{EJ_z}{GA}\left(\alpha_y \frac{\mathrm{d}^2 q_y}{\mathrm{d}x^2} + \alpha_{yz}\frac{\mathrm{d}^2 q_z}{\mathrm{d}x^2}\right), \\ EJ_y \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} &= q_z - \frac{EJ_y}{GA}\left(\alpha_{yz} \frac{\mathrm{d}^2 q_y}{\mathrm{d}x^2} + \alpha_z\frac{\mathrm{d}^2 q_z}{\mathrm{d}x^2}\right), \end{aligned} \tag{4.3}$$

provided that the span loads $q_z(x)$, $q_y(x)$ are smooth. In our case $q_y = 0$, $q_z = 0$, hence we see that both deflections are expressed by polynomials of degree 3, while the angles of rotation β, φ are expressed by polynomials of degree 2. Let us introduce the non-dimensional parameters:

$$\varkappa_y = \frac{12EJ_y}{l^2GA}, \qquad \varkappa_z = \frac{12EJ_z}{l^2GA}. \tag{4.4}$$

Let (α_1, α_2) and $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ be solutions to the systems:

$$\begin{bmatrix} k_y + \varkappa_z & k_{yz} \\ k_{yz} & k_z + \varkappa_y \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varkappa_y \end{bmatrix}, \tag{4.5}$$

$$\begin{bmatrix} k_z + \varkappa_y & k_{yz} \\ k_{yz} & k_y + \varkappa_z \end{bmatrix} \begin{bmatrix} \widetilde{\alpha}_1 \\ \widetilde{\alpha}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varkappa_z \end{bmatrix}. \tag{4.6}$$

Let us introduce the polynomials:

$$\begin{gathered} a(\xi) = 1 - 3\xi^2 + 2\xi^3, \quad b(\xi) = \xi - 3\xi^2 + 2\xi^3, \quad c(\xi) = \xi - 2\xi^2 + \xi^3, \\ d(\xi) = 6\xi - 6\xi^2, \quad e(\xi) = 1 - 4\xi + 3\xi^2, \end{gathered} \tag{4.7}$$

with plots for $0 \le \xi \le 1$ presented in Fig. 2.

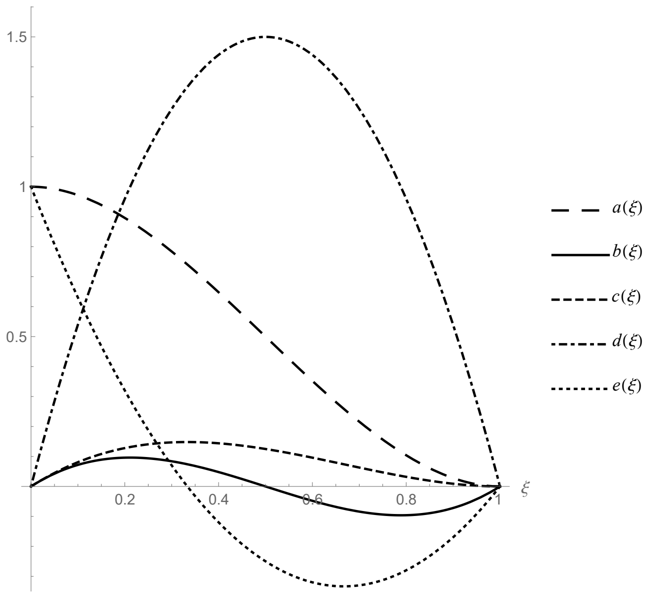

FIG. 2. Plots of the polynomials a, b, c, d, e.

The factorized forms of these polynomials read:

$$\begin{gathered} a(\xi) = (1-\xi)^2(1+2\xi), \quad b(\xi) = \xi(1-\xi)(1-2\xi), \quad c(\xi) = \xi(1-\xi)^2, \\ d(\xi) = 6\xi(1-\xi), \quad e(\xi) = (1-\xi)(1-3\xi). \end{gathered} \tag{4.8}$$

Let us introduce the angles of slopes in the $x - z$ and $x - y$ planes:

$$\psi = \frac{1}{l}\left(w^* - {}^*w\right), \qquad \chi = \frac{1}{l}\left(v^* - {}^*v\right), \tag{4.9}$$

as well as the mean values of the angles of rotation at the ends of the bar:

$$\beta_o = \frac{1}{2}\left({}^*\beta + \beta^*\right), \qquad \varphi_o = \frac{1}{2}\left({}^*\varphi + \varphi^*\right). \tag{4.10}$$

According to the Bernoulli–Euler theory (i.e., the theory of thin bars), the deflection functions and the functions representing variation of the angles of rotations of the bar's cross-sections are expressed as follows:

$$\begin{aligned}
\widehat{w}(x) &= l\left[\frac{{}^*w}{l}a(\xi) + \frac{w^*}{l}a(1-\xi) - {}^*\beta c(\xi) + \beta^* c(1-\xi)\right],\\
\widehat{v}(x) &= l\left[\frac{{}^*v}{l}a(\xi) + \frac{v^*}{l}a(1-\xi) - {}^*\varphi c(\xi) + \varphi^* c(1-\xi)\right],\\
\widehat{\beta}(x) &= {}^*\beta e(\xi) + \beta^* e(1-\xi) - \psi d(\xi),\\
\widehat{\varphi}(x) &= {}^*\varphi e(\xi) + \varphi^* e(1-\xi) - \chi d(\xi),
\end{aligned} \tag{4.11}$$

where

$$a(1-\xi) = 3\xi^2 - 2\xi^3, \quad c(1-\xi) = \xi^2 - \xi^3, \quad e(1-\xi) = -2\xi + 3\xi^2. \tag{4.12}$$

Thus, the angles of rotation are linked to the deflection functions by:

$$\widehat{\beta} = -\frac{\mathrm{d}\widehat{w}}{\mathrm{d}x}, \qquad \widehat{\varphi} = -\frac{\mathrm{d}\widehat{v}}{\mathrm{d}x}, \tag{4.13}$$

since the theory of thin bars imposes constraints on the angles of rotation of cross-sections, this assures the zero values of the transverse shear deformations in both the planes: $x-z$, $x-y$.

The deflection functions and the functions representing the variation of the angles of rotation within the six-parameter theory of bars differ from the mentioned solutions by terms involving the quantities $\psi + \beta_o$, $\chi + \varphi_o$, namely:

$$\begin{aligned}
w(x) &= \widehat{w}(\xi) + l\left[\alpha_2(\psi + \beta_o) + \widetilde{\alpha}_1(\chi + \varphi_o)\right] b(\xi),\\
\beta(x) &= \widehat{\beta}(\xi) + \left[\alpha_2(\psi + \beta_o) + \widetilde{\alpha}_1(\chi + \varphi_o)\right] d(\xi),\\
v(x) &= \widehat{v}(\xi) + l\left[\alpha_1(\psi + \beta_o) + \widetilde{\alpha}_2(\chi + \varphi_o)\right] b(\xi),\\
\varphi(x) &= \widehat{\varphi}(\xi) + \left[\alpha_1(\psi + \beta_o) + \widetilde{\alpha}_2(\chi + \varphi_o)\right] d(\xi).
\end{aligned} \tag{4.14}$$

Note that the quantities $\psi + \beta_o$, $\chi + \varphi_o$ are discrete deformation measures in the natural approach by Argyris (see comments in Petrolo and Casciaro [3]).

Since $T_y = \text{const}$ and $T_z = \text{const}$, the deformation measures $\gamma_y = \gamma_y^o$, $\gamma_z = \gamma_z^o$ are constant and equal:

$$\begin{aligned} \gamma_y^o &= \alpha_1(\psi + \beta_0) + \widetilde{\alpha}_2(\chi + \varphi_0), \\ \gamma_z^o &= \alpha_2(\psi + \beta_0) + \widetilde{\alpha}_1(\chi + \varphi_0). \end{aligned} \tag{4.15}$$

The shear deformations are expressed by the functions:

$$\begin{aligned} w_T(x) &= l\gamma_z^o b\left(\frac{x}{l}\right), \\ \beta_T(x) &= \gamma_z^o d\left(\frac{x}{l}\right), \\ v_T(x) &= l\gamma_y^o b\left(\frac{x}{l}\right), \\ \varphi_T(x) &= \gamma_y^o d\left(\frac{x}{l}\right). \end{aligned} \tag{4.16}$$

Thus, the final deformation (4.14) is also expressed as the superposition as follows:

$$\begin{aligned} w(x) &= \widehat{w}(x) + w_T(x), \\ \beta(x) &= \widehat{\beta}(x) + \beta_T(x), \\ v(x) &= \widehat{v}(x) + v_T(x), \\ \varphi(x) &= \widehat{\varphi}(x) + \varphi_T(x). \end{aligned} \tag{4.17}$$

The shear load is skew–symmetric with respect to the plane $x = l/2$, hence the functions $w_T(x)$, $v_T(x)$ are skew–symmetric, while the functions $\beta_T(x)$, $\varphi_T(x)$ are symmetric with respect to $x = l/2$. The extremal values of the latter functions are attained in the middle of the bar and read:

$$\beta_T\left(\frac{l}{2}\right) = \frac{3}{2}\gamma_z^o, \qquad \varphi_T\left(\frac{l}{2}\right) = \frac{3}{2}\gamma_y^o. \tag{4.18}$$

Let us conclude that the final deformation is a sum of the bending deformation predicted by the thin bar theory and the shear deformation added by the six-parameter theory.

The bar theoretical results (4.17) determine the shapes of deformation of the bar viewed as a 3D body; in particular one can predict the transverse deformation of the cross-sections $x = \text{const}$; it is given by the function $u_x(x, y, z)$ which represents the displacement along the x-axis of the point (x, y, z). According to El-Fatmi's kinematical hypothesis, this function has the form, see [4]:

$$u_x = u(x) + y\varphi(x) + z\beta(x) + \omega(y,z)\rho(x) + [\eta(y,z) - y]\gamma_y(x) + [\zeta(y,z) - z]\gamma_z(x), \tag{4.19}$$

where ρ is the measure of torsion, $\omega(y,z)$ is the warping function due to torsion, and $\eta(y,z)$, $\zeta(y,z)$ characterize warping due to transverse shear in the $x-y$ and $x-z$ planes. In the problem discussed here, axial deformation and torsion are not present (which is a justified assumption due to the decoupling phenomena of the discussed theory). Since

$$\begin{aligned} &\int_{\mathcal{A}} y(\eta - y)\,\mathrm{d}\mathcal{A} = 0, \qquad \int_{\mathcal{A}} z(\zeta - z)\,\mathrm{d}\mathcal{A} = 0, \\ &\int_{\mathcal{A}} z(\eta - y)\,\mathrm{d}\mathcal{A} = 0, \qquad \int_{\mathcal{A}} y(\zeta - z)\,\mathrm{d}\mathcal{A} = 0, \end{aligned} \tag{4.20}$$

Eq. (2.2) holds; consequently the last two terms in Eq. (4.19) do not affect the angles of rotation β, φ understood as the averaged quantities, see Eq. (2.2). In Fig. 3, for simplicity, we shall not show the deformations generated by these

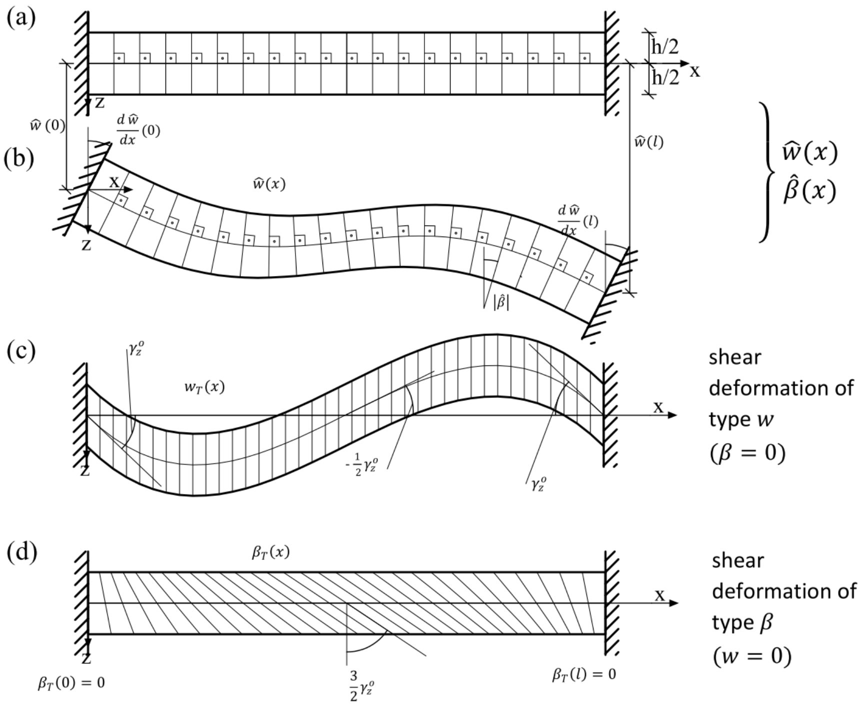

FIG. 3. Orthogonal net in the plane $x-z$ (a), bending deformation according to the thin bar theory (b), the deformation due to shear with $\beta = 0$ (c), the deformation due to shear with $w = 0$ (d).

quantities; the deformation of the orthogonal net of lines along x- and z-axes will be approximated by $u_x = y\varphi(x) + z\beta(x)$, $u_z = w(x)$, or within the assumption of planar cross-sections.

For the illustration of the deformation state of the bar assume that:

$$w(0) > 0, \qquad w(l) > w(0), \qquad \frac{\mathrm{d}w}{\mathrm{d}x}(0) > 0, \qquad \frac{\mathrm{d}w}{\mathrm{d}x}(l) > 0, \qquad \gamma_z^o > 0.$$

Before deformation, the net $x = \text{const}$, $z = \text{const}$ is orthogonal, see Fig. 3a. The fields $\widehat{w}$, $\widehat{\beta}$ determine bending deformation preserving orthogonality, see Fig. 3b. The transverse end forces act in a skew–symmetric manner and generate the deflection $w_T(x)$ characterizing shear deformation: the angles of rotation of tangents to the neutral line are γ_z^o at the ends and equal to $(-1/2)\gamma_z^o$ in the middle of the bar, see Fig. 3c. Here to the angles of rotation of the transverse cross-sections vanish. The angles of rotation generated by the shear forces are expressed by the function $\beta_T(x)$; this function vanishes at both the ends and is extremal in the middle of the bar, see Fig. 3d; the corresponding deflection is 0.

Kinematic loads in the plane $x - z$ generate, in general, deformation in the plane $x - y$. This phenomenon will be discussed below by considering two kinds of kinematic loads.

4.1. Case 1

Consider the kinematic load:

$$(4.21) \qquad {}^*w = 1, \qquad w^* = {}^*v = v^* = 0, \qquad {}^*\beta = \beta^* = {}^*\varphi = \varphi^* = 0.$$

The deformation of the bar is given by:

$$(4.22) \qquad \begin{aligned} w(x) &= (1 - 3\xi^2 + 2\xi^3)\underline{-\alpha_2(\xi - 3\xi^2 + 2\xi^3)}, \\ \beta(x) &= \frac{1}{l}(1 - \alpha_2)(6\xi - 6\xi^2), \\ v(x) &= -\alpha_1(\xi - 3\xi^2 + 2\xi^3), \\ \varphi(x) &= -\frac{1}{l}\alpha_1(6\xi - 6\xi^2), \qquad \xi = x/l. \end{aligned}$$

To be specific let us fix the domains $\mathcal{A}$ as Z-shape, U-shape, and RI60 rail, denoted as A1, A2, A3, see Fig. 4. Their characteristics are set up in Table 1. Assume that the length of the bar is $l = 1\,\mathrm{m}$ for case (a), $l = 2\,\mathrm{m}$ for case (b); $E = 210\,\mathrm{GPa}$, $G = 81\,\mathrm{GPa}$. Their non-dimensional parameters $\varkappa_y$, $\varkappa_z$, α_1, α_2, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ are given in Table 2.

The shape of the line $x_{(s)}$ upon deformation, determined by Eq. (4.22), is spatial, see Fig. 5; the points (x, y_S, z_S) displace in two directions: z (let us call it transverse) and y (let us call it lateral).

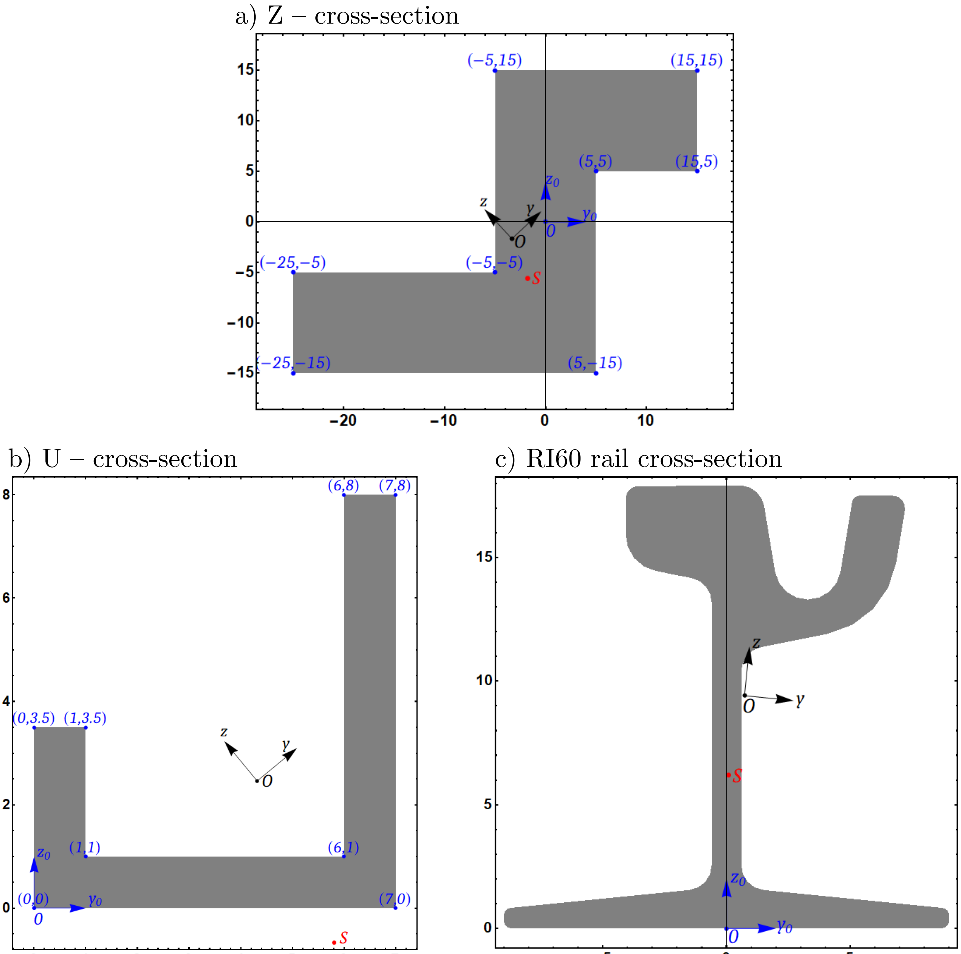

FIG. 4. Chosen cross-sections of the bar, called A1 (a), A2 (b), A3 (c).

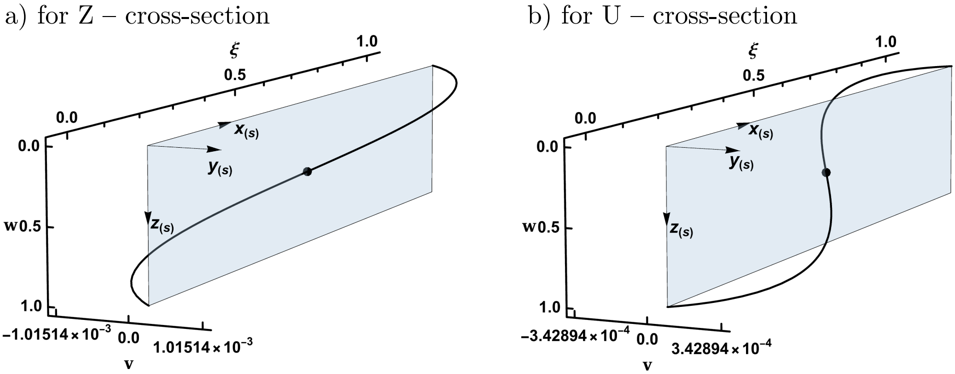

FIG. 5. Deflections caused by the kinematic load in case 1 for the length of the bar 1 m and the cross-sections A1 (a), A2 (b).

TABLE 1. Characteristics of the cross-sections shown in Fig. 4.

	A [cm^2]	J_y [cm^4]	J_z [cm^4]	J [cm^4]	$y_{0(O)}$ [cm]	$z_{0(O)}$ [cm]	$y_{0(S)}$ [cm]	$z_{0(S)}$ [cm]	$\measuredangle(y_0, y)$ [°]	k_y [–]	$k_{yz} = k_{zy}$ [–]	k_z [–]
A1	600	19 082	92 585	19 338	−3.333	−1.667	−1.812	−5.598	43.0	0.555	−0.075	0.587
A2	16.5	50	140.43	5.438	4.318	2.462	5.815	−0.675	39.5	0.351	0.0669	0.471
A3	77.1	3367.3	902.05	129.67	0.727	9.415	0.0792	6.189	−5.83	0.491	0.0027	0.288

TABLE 2. Characteristics of the bars with the cross-sections shown in Fig. 4, for lengths of 1 m (Ana) and lengths 2 m (Anb), $n = 1, 2, 3$.

	$\varkappa_y$	$\varkappa_z$	α_1	α_2	$\widetilde{\alpha}_1$	$\widetilde{\alpha}_2$
A1a	0.09894	0.4801	0.01055	0.1455	0.05119	0.4674
A1b	0.02474	0.1200	0.004561	0.04102	0.02213	0.1802
A2a	0.009428	0.02648	−0.003563	0.02013	−0.01001	0.07186
A2b	0.002357	0.006620	−0.0009561	0.005116	−0.002685	0.01900
A3a	0.1359	0.03640	−0.001674	0.3209	−0.0004484	0.06907
A3b	0.03397	0.009100	−0.0005812	0.1056	−0.0001557	0.01821

TABLE 3. Maximal absolute values of horizontal displacement v and maximal absolute values of w_α for the analyzed bars.

	A1a	A1b	A2a	A2b	A3a	A3b
10^3 max $\lvert v \rvert$	1.015	0.4389	0.343	0.0920	0.1611	0.0559
10^3 max $\lvert w_\alpha \rvert$	14.00	3.948	1.937	0.4923	30.88	10.17

The term underlined in Eq. (4.22)$_1$ denoted here by $w_\alpha(x)$ represents the deflection due to transverse shear. The extremal values of this term are attained for $x = ((3-\sqrt{3})/6)l$ and $x = ((3+\sqrt{3})/6)l$, and are equal to $-(\sqrt{3}\alpha_2)/18$, $(\sqrt{3}\alpha_2)/18$, respectively. The lateral deflection $v(x)$ is skew–symmetric with respect to $x = l/2$. Maximal absolute values of v and w_a for the analyzed bars are given in Table 3.

4.2. Case 2

We shall consider the state of deformation caused by the kinematic load:

$$ {}^*w = 0, \quad w^* = 0, \quad {}^*v = v^* = 0, \quad {}^*\beta = 1, \quad \beta^* = 0, \quad {}^*\varphi = \varphi^* = 0. \tag{4.23}$$

The deformation of the bar is determined by:

$$\begin{aligned} w(x) &= -l(\xi - 2\xi^2 + \xi^3) + \frac{l}{2}\alpha_2(\xi - 3\xi^2 + 2\xi^3), \\ \beta(x) &= (1 - 4\xi + 3\xi^2) + \frac{1}{2}\alpha_2(6\xi - 6\xi^2), \\ v(x) &= \frac{l}{2}\alpha_1(\xi - 3\xi^2 + 2\xi^3), \\ \varphi(x) &= \frac{1}{2}\alpha_1(6\xi - 6\xi^2), \qquad \xi = x/l. \end{aligned} \tag{4.24}$$

The deformation is spatial, a lateral deflection $v(x)$ appears, which is skew–symmetric with respect to $x = l/2$, see Fig. 6.

a) for Z – cross-section b) for U – cross-section

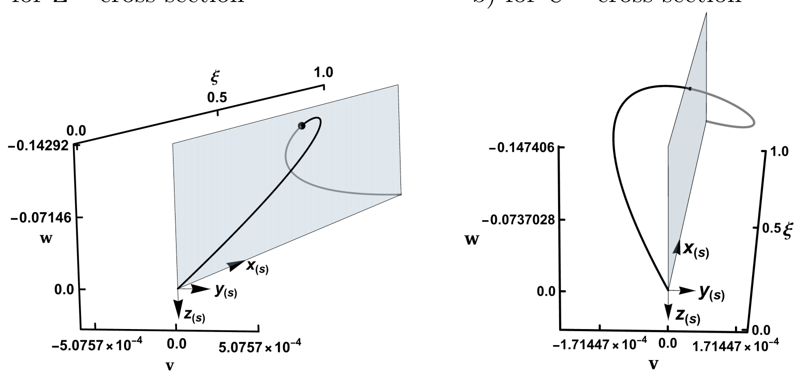

FIG. 6. Deflections caused by the kinematic load in case 2 for the length of the bar of 1 m and cross-sections A1 (a), A2 (b).

5. Deformations caused by span loads

5.1. Bar clamped at both ends, subjected to the transverse load $q_z = \mathrm{const}$

The load $q_z = q = \mathrm{const}$ acts along the axis $x_{(s)}$ linking the shear centers. The bar is clamped at both ends, i.e.:

$$\begin{aligned}
&w(0)=0, \qquad && w(l)=0, \qquad && v(0)=0, \qquad && v(l)=0,\\
&\beta(0)=0, && \beta(l)=0, && \varphi(0)=0, && \varphi(l)=0.
\end{aligned} \tag{5.1}$$

The solution to problem ($\mathcal{P}_3$) has the form:

$$\begin{aligned}
w &= \frac{ql^4}{24EJ_y}\xi^2(1-\xi)^2 + \alpha_z\frac{ql^2}{2GA}\xi(1-\xi),\\
\beta &= -\frac{ql^3}{12EJ_y}(\xi - 3\xi^2 + 2\xi^3),\\
v &= \alpha_{yz}\frac{ql^2}{2GA}\xi(1-\xi),\\
\varphi &= 0.
\end{aligned} \tag{5.2}$$

Let us note that $\beta_T = 0$, $\widehat{v} = 0$, $\widehat{\varphi} = 0$, $\varphi_T = 0$. The deformation is spatial, and there appears a lateral deflection $v(x)$ in the shape of a parabola. Its extremal value is $\alpha_{yz}ql^2/(8GA)$ and occurs at $x = l/2$. The distribution of internal forces is the same as in the thin bar theory and is given by:

$$\begin{aligned}
M_y &= -\frac{ql^2}{12}(1 - 6\xi + 6\xi^2),\\
T_z &= ql\left(\frac{1}{2} - \xi\right),\\
M_z &= 0, \qquad T_y = 0.
\end{aligned} \tag{5.3}$$

5.2. Bar simply supported in two directions at both ends, subjected to the transverse load $q_z = \mathrm{const}$

The load $q_z = q = \mathrm{const}$ acts along the axis $x_{(s)}$ linking the shear centers. The bar is simply supported in two directions at both the ends, i.e.:

$$\begin{aligned}
&w(0)=0, \qquad && w(l)=0, \qquad && M_y(0)=0, \qquad && M_y(l)=0,\\
&v(0)=0, && v(l)=0, && M_z(0)=0, && M_z(l)=0.
\end{aligned} \tag{5.4}$$

The solution to problem $(\mathcal{P}_3)$ has the form:

$$
\begin{aligned}
w &= \frac{ql^4}{24EJ_y}\xi(1-\xi)(1+\xi-\xi^2) + \alpha_z\frac{ql^2}{2GA}\xi(1-\xi),\\
\beta &= -\frac{ql^3}{24EJ_y}(1-6\xi^2+4\xi^3),\\
v &= \alpha_{zy}\frac{ql^2}{2GA}\xi(1-\xi),\\
\varphi &= 0.
\end{aligned}
\tag{5.5}
$$

We see that the lateral deflection is the same as in the previous example. The distribution of internal forces is the same as in the thin bar theory:

$$
\begin{aligned}
M_y &= \frac{1}{2}ql^2\xi(1-\xi),\\
T_z &= ql\left(\frac{1}{2}-\xi\right),\\
M_z &= 0, \qquad T_y = 0.
\end{aligned}
\tag{5.6}
$$

5.3. Cantilever under a point load P at its end

The boundary conditions read:

$$
\begin{aligned}
&w(0)=0, \qquad v(0)=0, \qquad \varphi(0)=0, \qquad \beta(0)=0,\\
&T_y(l)=0, \qquad T_z(l)=P, \qquad M_y(l)=0, \qquad M_z(l)=0.
\end{aligned}
\tag{5.7}
$$

The solution to problem $(\mathcal{P}_3)$ has the form:

$$
\begin{aligned}
w &= \frac{Pl^3}{6EJ_y}\xi^2(3-\xi) + \alpha_z\frac{Pl}{GA}\xi,\\
\beta &= -\frac{Pl^2}{2EJ_y}\xi(2-\xi),\\
v &= \alpha_{yz}\frac{Pl}{GA}\xi,\\
\varphi &= 0.
\end{aligned}
\tag{5.8}
$$

There appears a lateral deflection in the shape of a straight line; an extremal value of this deflection is $\alpha_{yz}Pl/(GA)$.

The distribution of internal forces is the same as in the thin bar theory, i.e.:

$$
M_y = -Pl(1-\xi), \qquad M_z = 0, \qquad T_z = P, \qquad T_y = 0.
\tag{5.9}
$$

5.4. Pure bending

Consider a bar simply supported in two directions at both ends, loaded by two end moments in opposite directions at both ends. The boundary conditions read:

$$\begin{aligned} &w(0)=0, && w(l)=0, && v(0)=0, && v(l)=0,\\ &M_y(0)=M, && M_y(l)=M, && M_z(0)=0, && M_z(l)=0. \end{aligned} \tag{5.10}$$

The solution to problem ($\mathcal{P}_3$) has the form:

$$\begin{aligned} &w=\frac{Ml^2}{2EJ_y}\xi(1-\xi),\\ &\beta=-\frac{Ml}{2EJ_y}(1-2\xi),\\ &v=0, \qquad \varphi=0,\\ &M_y=M, \qquad T_z=0, \qquad M_z=0, \qquad T_y=0, \end{aligned} \tag{5.11}$$

which coincides with that predicted by the thin bar theory.

6. Final remarks

The present paper discussed the predictions of the six-parameter theory of bars of arbitrary cross-sections proposed in [4, Sec. 10]. The presence of coupling terms in the constitutive equations linking the transverse forces to the transverse shear measures induces asymmetry in the solutions and results in deplanation of the deformation states. This effect is clearly seen in the shape functions corresponding to kinematic loads. In general, the response to the static load is not planar. There is only one exception in which the deplanation is absent: the problem of pure bending.

If the plane $x = l/2$ is the section of a symmetry plane of the static problem, a constant load applied in the z-direction along the $x_{(s)}$ axis generates a parabolic deflection in the y-direction, without causing the angles of rotation φ along the $-z$-axis, while the state of stress resultants remains planar, i.e., $M_z = 0$, $T_y = 0$. Moreover, the rotation angles β in y-direction coincide with those predicted by the thin bar theory, i.e., they are not affected by transverse shear.

If the static problem is asymmetric with respect to the plane $x = l/2$, a constant load along the z-direction induces all stress resultants and the deflections along the y-direction are given by polynomials of degree 3. For instance, in a bar subjected to the load q_z which is clamped at the left end and simply supported in both y- and z-directions at the right end there appear two non-zero reactions in the y- and z-directions at the right end, generating the stress resultants in both planes. Moreover, note that point loads applied to cantilevers in the z-direction cause lateral deflections of straight shape. Thus, the lateral deflection $v(x)$ of cantilever loaded in the z-direction by several point loads will assume the shape of a zigzag. These lateral deflections are caused by the transverse forces T_z which generate the shear deformations γ_y.

Let us emphasize that within the standard six-parameter theory of bars, e.g., Petrolo and Casciaro [3], a load along the z-direction will, in general, generate torsion. Moreover, kinematic loads will cause torsional deformation. This shows that the six-parameter theory discussed in the presented paper makes the static problems as simple as possible.

We conclude that, in general, none of the six-parameter theories admit planar bending/shearing states of deformation. This phenomenon is of vital importance when analyzing stability of bending states, since the standard description of lateral buckling phenomenon assumes an ideally planar initial bending configurations.

Let us stress here that this problem of occurring spatial deformation states is not observed in the theory by Vlasov (or the Vlasov-like theory developed in [4]), because transverse shear deformation is neglected and all constitutive equations may be decoupled by an appropriate choice of free parameters of this theory, such as the position of the pole. Thus, the standard theory of lateral buckling works correctly, as the initial planar bending states are admissible. This shows how difficult it is to extend the results concerning Vlasov thin bars to moderately thick bars. As usually, a direct extension of the known predictions is impossible.

Fundings

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Conflict of interests

The authors declare that there are no known competing financial interests or personal relationships that could have influenced the work described in this paper.

Authors' contributions

All authors contributed equally to this work, reviewed, and approved the final manuscript.

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Received July 10, 2025; revised September 17, 2025; accepted October 9, 2025; available online January 20, 2026; version of record April 16, 2026; published issue June 24, 2026.