Engineering Transactions, **74**(2): 293–326, 2026, https://doi.org/10.24423/engtrans.2026.3505
Institute of Fundamental Technological Research Polish Academy of Sciences (IPPT PAN)
Université de Lorraine • Poznan University of Technology

Review Paper

Torsion of Functionally Graded Material Structures: An Overview

Mateusz Kumor[1)*], Artur Ganczarski[2)]

[1)] *Doctoral School*
[2)] *Chair of Applied Mechanics and Biomechanics*

Cracow University of Technology
Kraków, Poland

*Corresponding Author: mateusz.kumor@doktorant.pk.edu.pl

Examining torsion in functionally graded materials (FGMs) is crucial because their properties vary spatially. FGMs with continuously graded architectures provide a robust basis for investigating mechanical behavior. Current understanding of torsional response draws on analytical, numerical, and experimental approaches. This review synthesizes how material gradation influences stress distribution, stiffness, and failure modes, and compares advances in FGM torsion across diverse models and geometries. The theoretical background is framed by classical torsion theories, including the Saint-Venant theory, the Prandtl membrane analogy, and the Vlasov formulations. We further discuss modeling with isoparametric finite elements and summarize established homogenization schemes for FGMs. A tabulated overview of torsion-related results is also provided. The novelty of this review lies in its exclusive focus on torsion in FGMs, the systematic tabulation of prior contributions, and a coherent exposition of homogenization models and torsion theories tailored to FGM structures. To our knowledge, this is among the first reviews to focus specifically on torsion of FGM structures, distinguishing it from prior overviews that address torsion only briefly. Methodologically, we conduct a structured scoping review that screens peer-reviewed sources, classifies studies by geometry, torsion theory, homogenization scheme, and numerical strategy, and synthesizes observed trends. Finally, we present concise conclusions and future research directions. This review covers analytical, numerical, and experimental studies of torsion in FGMs, identified via a structured Google Scholar search and prioritized by citation impact and relevance.

Keywords: torsion, functionally graded materials, FGM structures, torsional stiffness.

1. Introduction

The concept of functionally graded materials (FGMs) was developed in Japan in the mid-1980s [1]. The established idea of materials providing a high through-thickness thermal barrier was crucial for space shuttle construction; Japanese engineers and scientists proposed functionally graded variations in thermal coefficients [2]. Since then, the FGM concept has advanced steadily. The literature on bending, tension, and compression is relatively extensive, whereas torsion remains comparatively underexplored. The spatial variation of graded structure in such materials underpins their usefulness in many automotive, aerospace, and biomedical applications. Figure 1 presents an example of property gradation in the x- and y-directions.

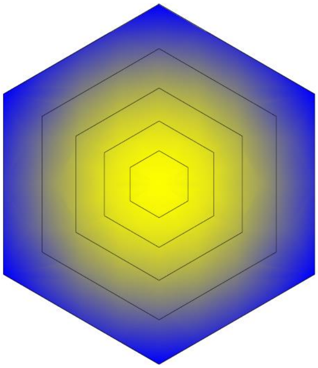

Fig. 1. Example of an FGM structure.

In the study of FGMs, one of the most important aspects is the behavior of such materials under torsional loads. A thorough exploration of the FGM torsion problem may yield substantial improvements and enable greater use of FGM layers in the design of parts that transmit significant twisting moments. These may include structural elements such as support beams with various cross-sectional areas, machine shafts, or even aircraft wings. Furthermore, the development of precise numerical torsion models could reduce design costs and time, thereby encouraging broader adoption of FGM structures in place of conventional materials. Therefore, a substantial task is to consider existing torsion theories and reflect on their potential improvements.

Valuable issues related to torsion in FGM structures have been presented in many articles. An analysis of the available literature shows that most torsion cases are approximated as linear elastic composites, often treated as isotropic models. Many of these works are based on anisotropic elasticity models developed by Lekhnitskii [3]. Horgan and Chan [4] investigated the influence of material inhomogeneity on the torsional behavior of linear elastic isotropic rods. They extended the work of Rooney *et al.* [5] and Lekhnitskii [3] by formulating the shear modulus as a function of the cross-sectional position. Batra [6] solved the torsion problem for an FGM cylinder in compressible and incompressible linear elastic materials with spatially varying moduli only in the axial direction. Arghavan and Hematiyan [7] formulated numerical models of FGM hollow tubes with arbitrary non-circular shapes. In another study, Horgan [8] extended the notation introduced by Chen and Wai [9], deriving unified formulas for the absence of warping effects in rods with elliptical cross sections. Barretta and Luciano [10] demonstrated a novel analogy between the Kirchhoff plate problem and the Saint-Venant torsion problem.

In recent years, numerous papers have examined the influence of torsion on FGM nanotubes and nanobars. Li and Hu [11] analyzed the behavior of 2D FGM microtubes under torsion using the modified couple-stress theory. Barretta *et al.* [12] investigated the torsion of FGM nanobeams based on the Eringen nonlocal elasticity theory. Moreover, many recent studies employ the Saint-Venant torsion theory, as evidenced by the work of Nikmehr and Lashkarbolok [13] and Darilmaz *et al.* [14]. The number of studies on beams with circular and square cross sections is substantial, whereas cases involving shafts with triangular, regular polygonal, and other non-circular cross sections are less frequent. The results of Akinlabi *et al.* [15] on torsion in triangular cross sections indicate clear room for expansion of this topic.

Furthermore, many investigations rely on finite element methods (FEM). Ganczarski *et al.* [16] offered a different perspective by solving torsion for Al-Ti FGM non-circular shafts using the finite difference method (FDM). Their work highlights significant potential for future research employing methods other than the FEM. In addition to aerospace and automotive applications, the torsion and shear-stress behavior of FGM structures is also relevant to the medical and energy sectors. Consequently, research in this area can support the design of innovative components across these industries.

Considering the current advances in the torsional behavior of functionally graded (FG) materials, we would like to highlight several of the most important findings from studies conducted in recent years. Hao *et al.* [99] analyzed bursting oscillations arising from bending–torsion coupling in cantilevered FGM conical sandwich panels driven by a static preload and slow in-plane harmonic forcing, and demonstrates with a nonautonomous, temperature-graded model that the onset is governed by symmetry-breaking pitchfork bifurcations. In turn, subsequent studies address the torsional behavior of nanotubes, nanorods, and microtubes. Using modified couple stress theory with radial, axial gradation, Aghazadeh *et al.* [100] derived, numerically solved, and validated torsion equations for bidirectional FG microtubes, quantifying how phase profile and geometry control twist and shear under distributed torque.

In turn, Civalek *et al.* [101] presented an exact nonlocal-elasticity solution for the torsional free vibration of restrained FGM nanotubes – modeling end restraints with torsional springs, deriving a characteristic matrix for natural frequencies, validating the results against prior studies, and quantifying the effects of the FG index and the length scale. Shakhlaviet *et al.* [102] studied von Kármán nonlinear torsional vibrations of FGM carbon nanotubes via nonlocal elasticity, deriving Hamiltonian equations, computing clamped–clamped free natural frequencies using the method of multiple scales, and quantifying the FG index, size, amplitude, and mode effects for design applications. Beni [103] examined the size-dependent, coupled electromechanical torsional behavior of porous, functionally graded flexoelectric microtubes and nanotubes. Barati and Norouzi [104] presented a nonlocal model for the static torsion of bidirectional FG microtubes under a longitudinal magnetic field – deriving the governing equation via the principle of minimum potential energy, validating the generalized differential quadrature method (GDQM) against a Galerkin solution, and showing that the torsional angle depends on the nonlocal parameter. Finally, Zarezadeh *et al.* [105] developed a nonlocal elasticity model for an FG nanorod on a torsional foundation under an axial magnetic field deriving the Navier equations and applying the Hamilton principle, solving them with GDQM, and showing that size effects, introduced through the nonlocal length scale, soften the response and reduce the natural frequencies.

Taking these considerations into account, issues related to the torsion problem, existing torsion theories, and torsion modeling methods are discussed and compared, together with a critical perspective on the topic.

2. FGM torsion problem

The torsional behavior of FGM structures is crucial for understanding their response and for designing proper and effective elements. The use of FGMs can provide improved performance and a more favorable stress gradient, which may result in better-designed components. The torsion of highly anisotropic or orthotropic materials differs markedly from that of isotropic structures. Nonhomogeneity, and thus variation of properties in all directions, makes the modeling process significantly more difficult. Another important problem is the definition of the graded composition. The most popular homogenization methods are based on linear approximations of the modulus of elasticity and the Poisson ratio. For the torsion of FGMs, it is possible to obtain the Kirchhoff modulus through homogenization, as shown by Reuss, Voigt, Hashin–Strikman, and Mori–Tanaka in their works. In contrast, isotropic torsion is much easier to analyze. The distribution of shear stresses is relatively straightforward for shafts with an isotropic and homogeneous microstructure. For a circular shaft, shear stresses are distributed linearly across the cross section and are an increasing function of the radius. This is not the case for twisting shafts with a graded structure, since differences in elastic moduli, and consequently various Kirchhoff modulus, lead to a significantly different distribution of shear stresses across the cross section (see Fig. 2).

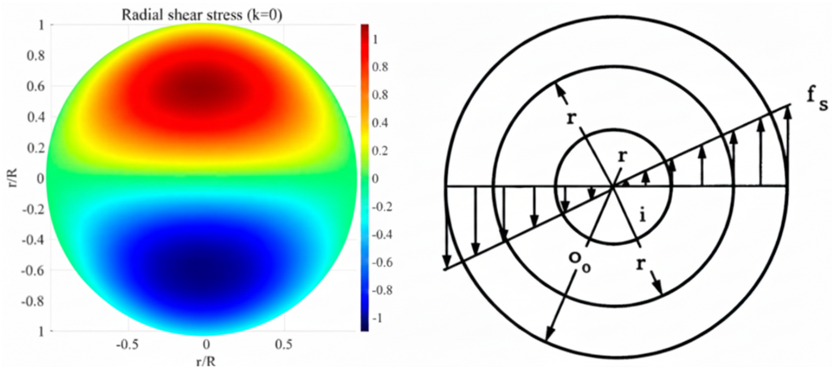

FIG. 2. Shear stress distribution in an isotropic round shaft and a FGM shaft with a metal core and ceramic inner surface, after [17].

The next issue of concern is the angle of twist. For an isotropic shaft with uniform torsional stiffness, the angle of twist is identical at every point on the surface of a circular bar. By contrast, for a functionally graded structure – even under linear homogenization – the torsional stiffness of each layer differs, which in turn affects the total angle of twist of a shaft subjected to a torque M (see Fig. 3).

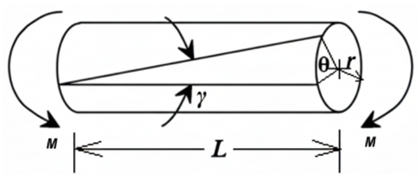

FIG. 3. Angle of twist in an isotropic shaft.

Therefore, when a round shaft is considered, the distribution of shear stresses and the angle of twist present a substantial challenge. An even greater problem arises for noncircular cross sections such as rectangular, elliptical, polygonal, or asymmetrical shapes, where, even for isotropic and homogeneous materials, obtaining accurate calculations and distributions of shear stresses and angles of twist requires multiple approximations and experimental methods. The first assumptions regarding torsion are based on the Prandtl membrane analogy. The membrane analogy is used to visualize the Prandtl stress function for any contour of a twisted shaft's cross section. The values of the Prandtl function at specific points within the cross section that follows a defined contour are related to the distance from this cross section to a membrane surface. This membrane is stretched across the cross-sectional contour and subjected to a uniform pressure acting perpendicular to the cross section. The Prandtl membrane analogy plays a significant role in describing the torsion of FGM structures.

The torsion equation is derived from the stress in a thin membrane subjected to the applied pressure p, which is always perpendicular to the membrane surface (see Fig. 4).

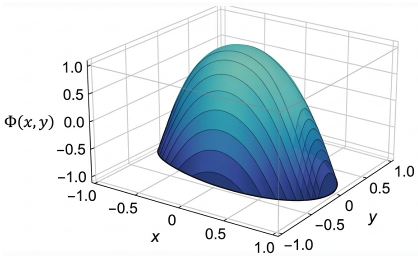

FIG. 4. Prandtl membrane analogy for anh elliptic isotropic cross section, visualization based on [18].

The analysis of an isotropic material is based on a Poisson-type partial differential equation that describes the torsional behavior of a shaft, thereby relating it to the membrane stress T. The following equation applies:

$$\nabla^2 \omega = -\frac{p}{T}, \tag{2.1}$$

where p is the distributed pressure across the membrane, analogous of torque, and T is the equivalent torsional stiffness.

In the case of FGMs, the mathematical formulation is more complicated, since the material properties, such as the Young modulus E, the Poisson ratio, and consequently the Kirchhoff modulus G must depend on the functions $p(y,z)$ and $T(y,z)$, which describe the change in material properties along the directions of gradation, according to the following equation:

$$\nabla^2 \omega = -\frac{p(y,z)}{T(y,z)}, \tag{2.2}$$

where $p(y,z)$ and $T(y,z)$ are functions describing the local material properties.

The application of Eq. (2.2) allows the changes in mechanical properties to vary in both the x- and y-directions. This approach is crucial for tailoring material behavior under mechanical stresses to specific directional requirements, thereby enhancing the design and functionality of advanced material systems such as FGMs, as shown in Fig. 5.

Numerous analogies for various FGMs can be found, among others, in the work of BARRETTA and LUCIANO [10], who established a new analogy between the orthotropic FGM Saint-Venant beam and the Kirchhoff plate. These studies demonstrated the aforementioned relationship and expanded upon earlier assumptions, see [19–27]. During the simulation of torsion processes in graded

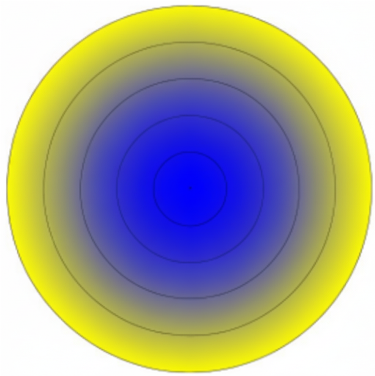

Fig. 5. Material gradation in a circular cross section.

materials, a number of approximation-related issues arise. To address these, numerous theories are employed, typically assuming material heterogeneity in one direction of the coordinate system. These approaches are discussed in the following sections.

3. Analytical torsion modeling methods

Modeling of FGMs primarily involves applying variable material properties along a single direction. To describe behavior under a twisting moment, existing torsion models are predominantly used with extensions that account for the property gradient in one direction and the dependence of changes in specific properties such as the Young modulus, Kirchhoff modulus, Poisson ratio, and density on variations along the chosen direction. The most widely used developments in torsion modeling include the classical Saint-Venant torsion model, the Prandtl membrane analogy, and the Vlasov model. These models are described further in the context of their application to FGMs.

3.1. Saint-Venant theorem

The beginnings of mathematical modeling of structures made from gradient materials can be traced back to issues raised by Saint-Venant. The Saint-Venant principle, although originally developed for homogeneous and isotropic materials, can also be applied to the analysis of FGMs. These materials are characterized by a gradual change in composition or structure, which leads to variations in their mechanical and thermal properties along a specified direction. The Saint-Venant formulation follows several assumptions. The shaft cross section rotates approximately as a rigid entity around a twist axis. This implies that during torsion the cross section retains its shape with minimal distortion, and every point on it moves in the circular trajectory around the twist axis.

The shaft features a prismatic cross section, meaning it is constant and uniform along its entire length. This uniformity simplifies the analysis since the same geometric and material properties apply to every cross section. Additionally, there is warping of the cross section that remains constant across all sections along the shaft length. Warping refers to the out-of-plane displacement of points in the cross section, accounting for the fact that in real materials the cross section is not perfectly rigid. These restrictions impose several limitations on the torsion model itself, especially when considering nonhomogeneous materials such as FGMs. Overcoming some of these limitations allows for a more comprehensive simulation of shaft torsion.

In FGMs, the Saint-Venant principle is particularly useful because it allows simplification of stress analysis in regions far from the point of load application. Despite the variable characteristics of the material, this principle assumes that local effects of loads, such as stress concentration or detailed stress distribution around the points of force application, diminish quickly as one moves away from the source of the load. Most examples are based on torsion analysis introduced by Saint-Venant. The characterization of torsional behavior is challenging for FGM structures. The original theory of torsion is based on isotropic shafts with a constant angle of twist. When it is applied to FGM structures, it is crucial to account properly for the spatial variation of material properties across the volume. FGMs are designed so that their mechanical, thermal, or electrical properties change gradually in response to specific application requirements. The Poisson-type equation is presented as follows [28, 29]:

$$\nabla^2\Phi + \left(\frac{\operatorname{grad} G}{G}\right)\cdot\nabla\Phi = -2G\theta. \tag{3.1}$$

The Poisson-type torsion equation is a partial differential equation in which, to describe changes in the Kirchhoff modulus G, the material properties must depend on variations in one direction. $\nabla^2\Phi$ denotes the Laplacian of the Prandtl function in the shaft cross section, $(\nabla G/G)\cdot\nabla\Phi$ represents the spatial variability of the Kirchhoff modulus G across the volume, and $-2G\theta$ captures the effect of the Kirchhoff modulus and the boundary-condition terms. The Saint-Venant torsion formulation relies on the dependence of the torsion function on material properties that vary in one or more directions. Because Saint-Venant theory is limited for FGM materials, it is necessary to improve and extend it to describe their torsional behavior more comprehensively and accurately. An important aspect is the proper treatment of material properties that vary with coordinate direction (usually a single direction), together with a fuller account of material inhomogeneity, which requires consideration of the equilibrium equations and boundary conditions. Another promising direction is the development of validation experiments to confirm and calibrate the theory.

3.2. Prandtl membrane analogy

The Prandtl torsion model addresses the torsion of prismatic shafts and primarily describes the distribution of shear stresses in prismatic shafts subjected to a twisting moment. Initially, the membrane analogy was applied only to isotropic and homogeneous materials, but over time it has been extended to nonhomogeneous materials such as FGMs. The Prandtl membrane analogy relates the torsion of a prismatic rod to a thin elastic membrane that is hypothetically stretched and conformed to the given cross section subjected to torsion. In the case of FGMs, applying this theory requires accounting for material dependence along the directions of gradation, x and y. For isotropic and homogeneous shafts, the Poisson-type equation is given by [30, 31]:

$$\nabla^2 \Phi = -2G\theta, \tag{3.2}$$

where Φ is the Prandtl function, G is the shear modulus (constant for homogeneous materials), and θ is the angle of twist per unit length.

In the case of FGM materials, to obtain the correct Prandtl function it is necessary to treat the Kirchhoff modulus as dependent on the spatially varying material properties, $G(x, y)$. This requires modifying the classical Prandtl equation by allowing G to depend on the x- and y-directions. After this modification, the Poisson-type equation for FGMs is as follows:

$$\nabla \cdot [G(x, y)\nabla \Phi] = -2G(x, y)\theta. \tag{3.3}$$

After modifying the differential equation and making it dependent on the derivatives of the material properties in the x- and y-directions, the equation is as follows [31]:

$$\frac{\partial}{\partial x}\left[G(x, y)\frac{\partial \Phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[G(x, y)\frac{\partial \Phi}{\partial y}\right] = -2G(x, y)\theta. \tag{3.4}$$

The membrane analogy offers several advantages. It enables a relatively straightforward determination of shear stress distributions in cross sections of rods subjected to torsion and provides a clear visualization. It is applicable to various cross-sectional shapes, which makes it highly useful. For FGMs, however, applying the membrane analogy introduces computational complexity. The use of complex material models can make analytical solutions difficult to obtain and, in some cases, unattainable.

3.3. Vlasov torsion model

Another commonly used torsion model is the Vlasov torsion model. It extends the Saint-Venant model by incorporating the effects of warping restraint.

In this model, warping is not negligible, which introduces additional dependencies required to obtain accurate results. The theory is mainly applied to thin-walled beam elements, where the warping effect is particularly evident and significant.

The Vlasov torsion theory primarily accounts for warping effects in cross sections and for the interaction between cross-sectional torsion and bending deformation. Another key concept is the shear center, defined as the point where shear stresses do not induce additional torsional effects [32, 33]. The torsion equation for a thin-walled beam with cross-sectional warping, as presented by Vlasov, is given by [33–35]:

$$\frac{\partial}{\partial x}\left(G(x)J_t\frac{\partial\theta}{\partial x}\right)+E(x)I_w\frac{\partial^3\theta}{\partial x^3}=0, \tag{3.5}$$

where $G(x)$ is the shear modulus dependent on the direction of property changes, J_t is the polar moment of inertia of the cross section, $E(x)$ is the Young modulus dependent on the direction of material property changes, I_w is the warping constant (warping moment of inertia) for the characteristic cross section, and θ is the angle of twist of the given cross section. Many studies address the torsion problem for FG thin-walled beams. A few representative works are outlined further.

Addessi *et al.* [35] presented a comparison of the impact of warping effects on various thin-walled cross sections according to the Vlasov and Benscoter theories. They developed numerical models and compared the resulting predictions. Additional contributions include three works by Murín *et al.* [36–38], which present a series of extensions on the application of thin-walled theory to FGM beams, the role of warping during torsion, and the influence of graded property variation on the distribution of mode shapes, bimoment, and shear stresses in thin-walled beam cross sections.

4. Modeling methods

Due to the complexity of calculations and the sophisticated material models required to represent variable material properties, FGMs are of great interest in the contemporary scientific community. The potential for continuous improvements and the development of stiffer, stronger materials that can be applied across diverse industrial environments motivates researchers to conduct new experiments and studies on modeling the mechanics of various FGM structures.

The most commonly used methods for modeling structures with graded properties are the FEM and the FDM. With advanced numerical models, researchers can represent the behavior of FGM beams under loads, isolate the effects of cross-sectional warping, and analyze shear stress distributions during torsion, shear deformations, and mode shapes with reasonable accuracy. Every numerical model involves some degree of approximation and assumptions; therefore, it is never fully consistent with reality. FGMs, and the simulation of property variation in at least one direction, require homogenization and property approximation using established cross-sectional homogenization formulas. The accuracy of such models and simulations may be questioned because many publications do not include validation or calibration of the models. Further, issues related to these problems and the current state of knowledge in the scientific community are presented.

The internal structure of each FGM can be designed in different ways. Owing to variations in properties and their spatial distribution within a given structure, one can distinguish materials with different gradient architectures. Achieving a specific gradient profile depends on the manufacturing method used for the FGM. The following types of gradients are distinguished: (c), (f) – composition gradient; (d), (g) – orientation gradient; and (e), (h) – fraction gradient, as shown in Fig. 6.

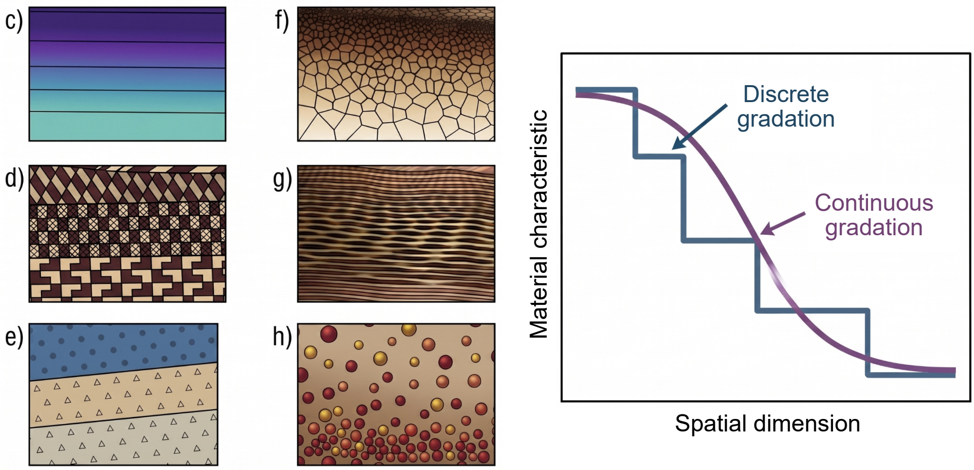

Fig. 6. FGM with various types of gradient, visualization based on [53, 54].

FGMs can be categorized into discontinuous and continuous types, as illustrated schematically in Fig. 6a and Fig. 6b. In discontinuous FGMs, the composition and microstructures change in a stepwise manner, typically across an interface. In continuous FGMs, the composition and microstructures vary gradually and continuously with position. Figure 6 schematically depicts various types of FGMs. Additionally, graded structures may occur throughout the entire material or within specific localized regions [54].

4.1. FEM modeling

Modeling of FGMs can be performed using two different approaches within the finite element formulation. The first is the classical method, which models finite elements that are homogeneous and isoparametric, following the principles used in commercial software such as Ansys. This approach is based on the formulation of isoparametric finite elements, which are defined by a prescribed shape functions and then used to approximate both the element geometry and the unknown field. In this method, the displacement vector and the coordinate vector are expressed as functions [39, 40, 42]:

$$\mathbf{u}_i^e = \sum_{i=1}^{n} \mathbf{N}_i \mathbf{u}^e, \qquad \mathbf{x} = \sum_{i=1}^{n} \mathbf{N}_i \mathbf{x}_i, \tag{4.1}$$

where $\mathbf{N}_i$ is the shape function, $\mathbf{u}_i$ is the nodal displacement, and n is the number of nodal points of the element. Here are examples of shape functions for a triangular element:

$$N_1 = 1 - \xi - \eta, \qquad N_2 = \xi, \qquad N_3 = \eta, \tag{4.2}$$

where ξ and η are the natural coordinates within the triangular element.

The constitutive relation between the stress tensor $\boldsymbol{\sigma}^{\mathrm{e}}$ and the strain is given by [40, 42]:

$$\boldsymbol{\sigma}^{\mathrm{e}} = \mathbf{D}^{\mathrm{e}} \boldsymbol{\epsilon}^{\mathrm{e}}, \tag{4.3}$$

where $\mathbf{D}^{\mathrm{e}}$ is the constitutive matrix and $\boldsymbol{\epsilon}^{\mathrm{e}}$ is the strain obtained from displacement. Thus, $\boldsymbol{\epsilon}^{\mathrm{e}}$ can be formulated as [40]:

$$\boldsymbol{\epsilon}^{\mathrm{e}} = \mathbf{B}^{\mathrm{e}} \mathbf{u}^{\mathrm{e}}, \tag{4.4}$$

where $\mathbf{B}^{\mathrm{e}}$ is the strain-displacement matrix of the shape function, and $\mathbf{u}^{\mathrm{e}}$ is the nodal displacement vector.

The main static equation based on the principle of virtual work is given by [40, 42]:

$$\mathbf{F}^{\mathrm{e}} = \mathbf{k}^{\mathrm{e}} \mathbf{u}^{\mathrm{e}}, \tag{4.5}$$

where $\mathbf{k}^{\mathrm{e}}$ is the force vector described with integration formula [40, 42]:

$$\mathbf{k}^{\mathrm{e}} = \int_{\Omega^{\mathrm{e}}} (\mathbf{B}^{\mathrm{e}})^{\mathrm{T}} \mathbf{D}^{\mathrm{e}} \mathbf{B}^{\mathrm{e}} \mathrm{d}\Omega^{\mathrm{e}}, \tag{4.6}$$

where the superscript $^{\mathrm{T}}$ describes the transpose and Ω^{e} is the domain of element (e). This type of classical formulation provides homogenous material properties, thus stiffness matrix remains constant. When the domain is discretized into finite elements, continuity between elements with different material properties is preserved, ensuring consistency of the properties at the Gauss integration points [39].

The next method of numerical representation is based on [41] and called isoparametric graded finite elements. This method involves interpolating material properties at each integration point from the material properties at each node using isoparametric shape functions, which have identical properties in the given coordinate system (x, y):

$$x = \sum_{i=1}^{n} N_i x_i, \qquad y = \sum_{i=1}^{n} N_i y_i, \tag{4.7}$$

and for the displacements:

$$u = \sum_{i=1}^{n} N_i u_i, \qquad v = \sum_{i=1}^{n} N_i v_i. \tag{4.8}$$

Using these properties, the Young modulus $E = E(x)$ and the Poisson ratio $\nu = \nu(x)$ functions can be interpolated using the isoparametric concept, as shown in Fig. 7:

$$E = \sum_{i=1}^{n} N_i E_i, \qquad \nu = \sum_{i=1}^{n} N_i \nu_i. \tag{4.9}$$

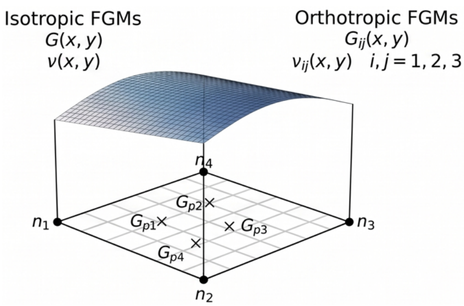

Fig. 7. Isoparametric formulation for isotropic and orthotropic FGMs, visualization based on [41].

This consideration presents the isoparametric formulation for an isotropic material. For an orthotropic material, it involves formulating four elastic parameters: the Young modulus $E_{11} = E_{11}(x)$, $E_{12} = E_{12}(x)$, the shear modulus

$G_{12} = G_{12}(x)$, and the Poisson ratio $\nu_{12} = \nu_{12}(x)$. The final formulation is given as [41, 43], as shown in Fig. 7:

$$E_{11} = \sum_{i=1}^{n} N_i (E_{11})_i, \qquad E_{22} = \sum_{i=1}^{n} N_i (E_{22})_i,$$
$$G_{12} = \sum_{i=1}^{n} N_i (G_{12})_i, \qquad \nu_{12} = \sum_{i=1}^{n} N_i (\nu_{12})_i. \tag{4.10}$$

Another considered model is one that varies depending on the volume fraction V and the material phase p. For this type of model, the isoparametric formulation is carried out in the classical manner according to a given function with exponent V_i^p, where the values V^p correspond to the values at the nodal points [41, 43]:

$$V^p = \sum_{i=1}^{n} N_i V_i^p. \tag{4.11}$$

4.2. Homogenization rules

Due to the variable internal structure in terms of mechanical properties such as the Young modulus, the Poisson ratio, and the Kirchhoff modulus in torsion, each object must be approximated according to spatially varying properties. This is necessary to obtain accurate analyses and results. Material approximations or FGM models that represent property variation are described in various ways. To model material behavior under torsion correctly, it is essential to idealize mathematically the internal substructures of the materials and, consequently, the variation in volume fractions of its constituents within each unit length or volume of the FGM. The material model may be represented as a particulate model with defined mixing phases, or as a multilayer model in which each layer follows a graded approximation of properties. The first model presented is the Voigt material model [47], which describes changes in material properties in terms of changes in volume fraction. The expression for the Young modulus is given by the following equation [47]:

$$\overline{E}^{\mathrm{V}} = E_1 V_1 + E_2 V_2, \tag{4.12}$$

where V_1 and V_2 are the volume fractions of the two materials. Thus, the sum of the fractions must satisfy:

$$V_f = 1 - V_1. \tag{4.13}$$

According to Voigt, the fraction V_f can be written as

$$V_f = \left(0.5 + \frac{z}{h}\right)^k, \tag{4.14}$$

where k is the positive power-law exponent and z/h is the dimensionless coordinate ratio. The resulting relation for the Young modulus is

$$\overline{E}^{\mathrm{V}}(z) = E_2 + (E_1 - E_2)\left[1 - \left(0.5 + \frac{z}{h}\right)^k\right]. \tag{4.15}$$

Accordingly, the dependence of the Kirchhoff modulus on the phases is [47]:

$$\overline{G}^{\mathrm{V}}(V_f) = G_1 V_f + G_2(1 - V_f), \tag{4.16}$$

where G_1 and G_2 are the Kirchhoff moduli of the two constituent materials.

The next model is the Reuss model [44–46], which assumes a uniform stress distribution throughout the material. The Young modulus $E = E(V_f)$ is given by

$$\overline{E}^{\mathrm{R}}(V_f) = \frac{E_1 E_2}{E_1 V_f + E_2(1 - V_f)}, \tag{4.17}$$

where E_1 and E_2 are the Young moduli of the two materials and V_f is the same volume-fraction function as in the Voigt model. The corresponding Kirchhoff modulus is [46]:

$$\overline{G}^{\mathrm{R}}(V_f) = \frac{G_1 G_2}{G_1 V_f + G_2(1 - V_f)}. \tag{4.18}$$

The next model considered is the Mori–Tanaka material model. In this approach, the homogenized composite consists of two phases: inclusions that are uniformly distributed and assumed spherical, and a matrix that is randomly oriented. The influence of the Poisson ratio on the overall behavior is typically considered negligible and taken as constant. The Kirchhoff modulus according to the Mori–Tanaka scheme is [48, 49]:

$$\overline{G}^{\mathrm{MT}}(V_f) = G_1 + \frac{V_f(G_2 - G_1)G_1}{(1 - V_f)(G_2 - G_1)\beta_1 + G_1}, \tag{4.19}$$

where

$$\beta_1 = \frac{6(K_1 + 2G_1)}{5(3K_1 + 4G_1)}, \tag{4.20}$$

and K is the bulk modulus.

Hashin and Shtrikman [51] proposed narrower bounds using the principle of minimum potential energy and polarization concepts, thereby defining rigorous upper and lower bounds for homogenized properties. The upper bound of the Kirchhoff modulus is [50, 51]:

$$\overline{G}^{\mathrm{HS}+}(V_f) = G_2 + \frac{(G_1 - G_2)V_f}{1 + \zeta(1 - V_f)(G_1/G_2 - 1)}, \tag{4.21}$$

where

$$\zeta = \frac{1+\nu}{3(1-\nu)}. \tag{4.22}$$

The lower bound is [50, 51]:

$$\overline{G}^{\mathrm{HS}-}(V_f) = G_1 + \frac{(G_2 - G_1)(1 - V_f)}{1 + \zeta V_f(G_2/G_1 - 1)}. \tag{4.23}$$

Each of these theories specifies the Kirchhoff modulus as a function of the volume fraction. As shown in Fig. 8, the Voigt and Reuss models yield substantially wider bounds than the Hashin–Shtrikman bounds, which provide tighter constraints. These bounds depend solely on the volume fractions of the phases and are therefore scale-independent.

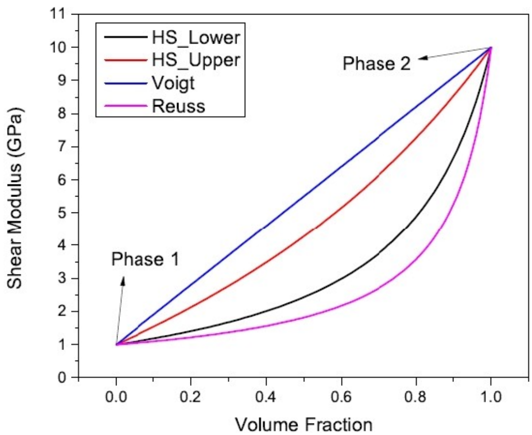

Fig. 8. Shear modulus between two phases as a function of volume fraction, after [52].

Another material model is the exponential material model (EMM). In this formulation, properties vary according to an exponential function, producing smooth transitions through the FGM thickness. This approximation can be applied to properties such as the Young modulus, the Kirchhoff modulus, or thermal conductivity. If the property variation through the thickness is exponential, the Kirchhoff modulus can be approximated as [53]:

$$\overline{G}^{\mathrm{EMM}}(z) = G_2 \exp(\beta z), \tag{4.24}$$

where $\beta = \frac{1}{h} \ln\left(\frac{G_1}{G_2}\right)$, and G_1 and G_2 are the Kirchhoff moduli of the two materials, h is the thickness, and z is the coordinate varying from 0 to h.

The power-law material model is also widely used for FGMs. The volume-fraction variation of an FGM layer can be represented as

$$V_f = f(z) = \left(\frac{z + h/2}{h}\right)^n, \tag{4.25}$$

where $f(z)$ denotes the volume fraction of one constituent; accordingly, the other constituent has fraction $1 - V_f$. Once the local volume fraction is known, pointwise properties follow from the rule of mixtures. In particular, the Kirchhoff modulus $G(z)$ within the layer is [53]:

$$\overline{G}^{\mathrm{PLMM}}(z) = G_1 V_f + G_2(1 - V_f), \tag{4.26}$$

where G_1 and G_2 are the Kirchhoff moduli of the two materials.

Another important model is the linear variation model. Here, the Poisson ratio $\nu(x)$ is taken as constant, while the Young modulus and the Kirchhoff modulus vary linearly with the gradation. The elastic modulus $E(x)$ and the Kirchhoff modulus $G(x)$ are expressed as [41]:

$$\overline{E}^{\mathrm{LVM}}(x) = E + \gamma x, \qquad \overline{G}^{\mathrm{LVM}}(x) = G + \gamma x, \tag{4.27}$$

where x is the graded coordinate, and γ is the parameter of nonhomogeneity defined by [41]:

$$\gamma = \frac{G(W) - G(0)}{W}, \qquad \gamma = \frac{E(W) - E(0)}{W}, \tag{4.28}$$

where W denotes the width of the graded region.

The final model is the sigmoid material model. It uses two functions to represent the change in properties with gradation, effectively partitioning the beam into two regions and describing the behavior in each. For one coordinate direction, changes are defined on the domains $-h/2$ to 0 and 0 to $h/2$, where h is the graded height. The expressions for the Young modulus are

$$\overline{E}^{\mathrm{SMM}}(z) = E_2 + (E_1 - E_2)\left[1 - \frac{1}{2}\left(\frac{z}{h} + \frac{1}{2}\right)^k\right] \tag{4.29}$$

for $-h/2 \leq z \leq 0$, and

$$\overline{E}^{\mathrm{SMM}}(z) = E_1 + \frac{E_1 - E_2}{2}\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{4.30}$$

for $0 \leq z \leq h/2$.

4.3. Accuracy of results

Modeling FGM properties typically uses either homogeneous or heterogeneous finite elements. Kim and Paulino [41] were the first to model FGMs using heterogeneous elements, moving beyond classical homogeneous elements for, which only midpoint responses match the FGM solution.

Partitioning the domain into equal segments and assigning homogeneous elements does not accurately represent heterogeneous behavior, as shown by Hernik [39]. In contrast, using heterogeneous elements that account for spatial variation in the Young modulus and Kirchhoff modulus at each point along the graded layer leads to markedly improved simulation outcomes in FGM regions. Figure 9 and Fig. 10 show x-direction displacements and the differences between homogeneous and heterogeneous finite elements.

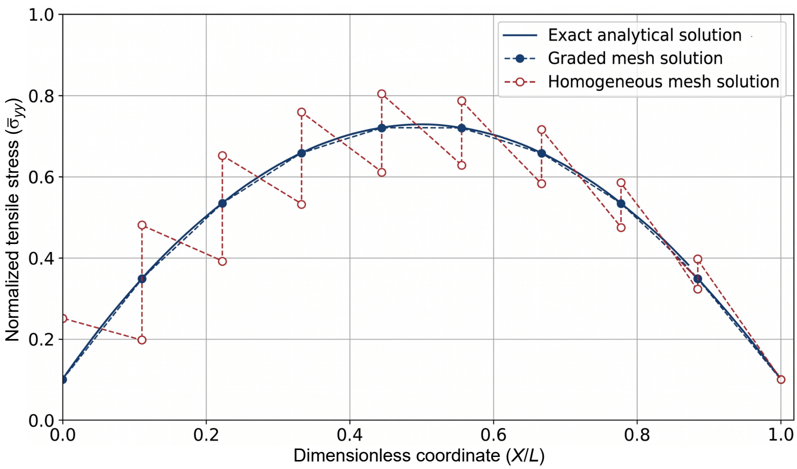

Fig. 9. Stress-distribution differences among exact, graded, and homogeneous elements, visualization based on [41].

According to Kim and Paulino [41], employing heterogeneous elements yields more accurate stress and deformation predictions in tensile tests. Mesh refinement reduces discrepancies between homogeneous and heterogeneous formulations for all considered cases [41].

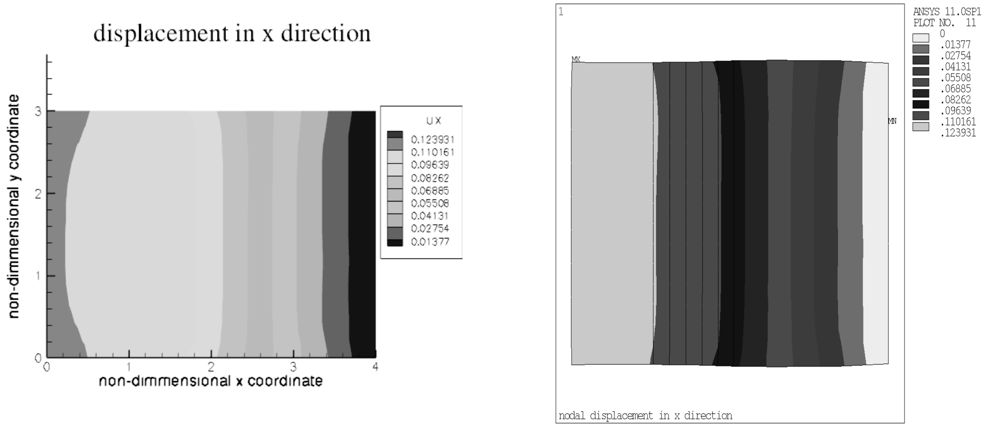

Fig. 10. Differences in x-direction displacement for homogeneous versus heterogeneous finite elements, after [39].

5. Comparison of achievements

5.1. Achievements

The torsional mechanics of structures made from FGMs presents a range of challenges across various formulations, including homogenization of the material model, mixing rules for individual graded phases, and approaches to homogenizing the gradation direction itself. Many studies model gradation along the longitudinal direction for beams or rods, while others focus on homogenization of the cross section. Over time, approaches to analyzing FGMs have evolved. Numerous limitations have been identified when using homogeneous elements in FEM, including inaccuracies in representing gradient variations. The comparison presented in Table 1 summarizes advances in the torsion of FGMs developed over the past several decades, with the aim of compiling these results in a concise form for comparison.

5.2. Comparison

The torsional response of structures and bars made from FGMs has been extensively studied, with the literature emphasizing their importance for understanding material behavior under complex loading conditions. Foundational work by Rooney and Ferrari [5] made a significant contribution through the analysis of FG shafts with rectangular cross sections. They examined the torsion and bending of bars with variable shear modulus, determined stiffness bounds, and presented solutions for specific cases such as laminates and cylindrical bars. The results were used to evaluate graded material properties.

Table 1. Comparison of key findings in torsion of FGMs over decades.

No.	Scientists	Landmark achievement
1	Rooney, Ferrari [55]	Investigation of the torsion of an FGM shaft with a rectangular cross section.
2	Rooney, Ferrari [56]	Analysis of torsional behavior in various classes of FG shafts.
3	Rooney, Ferrari [5]	Combination of torsion and flexure in inhomogeneous elements, exploring the relationship between material properties and structural response.
4	Horgan, Chan [4]	Analysis of isotropic, linearly elastic bars with a functional gradient, showing stress distribution across these bars.
5	Horgan, Chan [57]	Presentation of the stress response in rotating isotropic FGM disks, considering the effect of the gradient.
6	Ting *et al.* [58]	Demonstration that neutrality occurs when the geometric mean of the cylinder's shear moduli equals the shear modulus of the shaft, and out-lining criteria for preserving rigidity with embedded cylinders.
7	Singh *et al.* [59]	Analysis of torsional vibrations of graded cylinders, considering shear moduli and densities as functions of radius and axis.
8	Batra [6]	Examination of the torsion of cylindrical bars with material moduli varying along the axis.
9	Sofiyev, Schnack [60]	Study of the stability of FG cylindrical shells under dynamic torsional loading as a linear function of time.
10	Li *et al.* [61]	Examination of a cylindrical crack in a graded layer between coaxial cylinders subjected to dynamic torsional loading.
11	Hematiyan, Estakhrian [62]	Development of an approximate analytical method for analyzing the torsion of FG open-section members with uniform thickness.
12	Arghavan, Hematiyan [63]	Study of the torsion of hollow tubes with a functional gradient, identifying the impact of the gradient on stiffness and torsional resistance.
13	Gholami Bazehhour, Rezaeepazhand [64]	Analysis of the torsion of multilayered tubes with non-circular cross sections, focusing on the influence of material properties.

TABLE 1. [Cont.]

No.	Scientists	Landmark achievement
14	VASILIEV [65]	Analysis of the torsion of a circular punch on a half-space with a graded coating, reducing the problem to integral equations and deriving an ex-plicit solution.
15	WANG *et al.* [66]	Solution for torsional vibrations in FG finite hollow cylinders, focusing on the analysis of transient behavior and vibration characteristics.
16	SHEN *et al.* [67]	Development of a size-dependent gradient shaft model to study the effects of microstructure and material scale on torsional wave propagation, free vibration, and static torsion, considering radial variation of material properties.
17	BAYAT, EKHTERAEI TOUSSI [68]	Study of the elasto-plastic torsion of FGM shafts with a ceramic-metal structure, where the plastic zone may develop on the surfaces or within the thickness of the shaft, depending on material inhomogeneity and thickness.
18	HUANG *et al.* [69]	Analysis of the elasto-plastic buckling of cylindrical FGM shells under axial and torsional loads using the Tamura–Tomota–Ozawa (TTO) model and the Ritz method.
19	TSIATAS, BABOUSKOS [70]	Presentation of a new solution to the elasto-plastic torsion problem of FGM bars using an iterative numerical method based on BEM and AEM.
20	MURÍN *et al.* [71]	Presentation of an elastostatic analysis of spatial beam structures made of FGMs, considering smoothly varying material properties along the longitudinal direction and symmetric variations in the transverse and lateral directions.
21	BARRETTA *et al.* [12]	Formulation of the elastostatic problem of FG circular nanobeams under torsion, incorporating nonlocal elastic behavior based on the Eringens theory.
22	LIAGHAT *et al.* [72]	Investigation of material tailoring in FG hollow rods with arbitrary cross sections under torsion.
23	BARRETTA *et al.* [73]	Analysis of the torsion of linearly elastic, isotropic beams with inhomogeneities in the cross section and along the beam axis.
24	BAYAT *et al.* [74]	Presentation of torsion problem for hollow cylinders made of FGM, considering arbitrary variations of the Young modulus and the Poisson ratio in the radial direction.

Table 1. [Cont.]

No.	Scientists	Landmark achievement
25	Rizov [75]	Analysis of a cylindrical surface crack in circular shafts under torsional loading, considering nonlinear material behavior.
26	Rahaeifard [76]	Examination of the size-dependent behavior of FG microbars based on the modified couple stress theory, defining two length scale parameters to describe their mechanical properties.
27	Aminbaghai *et al.* [77]	Analysis of the impact of torsional warping and secondary deformations on the deformation and stress states in thin-walled FGM beams with longitudinally varying properties.
28	Murín *et al.* [78]	Analysis of warping torsion in FGM beams with spatially varying properties, analysis of torsional behavior.
29	Murín *et al.* [38]	Investigation of torsional warping eigenmodes in FGM beams with longitudinally varying properties, determining modal characteristics.
30	Guendouz *et al.* [79]	Torsional-bending in FGM beams using a 3D Saint-Venant refined beam theory.
31	Murín *et al.* [80]	Investigation of the effect of longitudinal variation in material properties on deformation and stresses in thin-walled FGM beams under non-uniform torsion.
32	Guendouz *et al.* [81]	Analysis of the static bending-torsion behavior of FG cantilever beams using an advanced 1D/3D beam theory.
33	Murín *et al.* [36]	Extension of the 3D FGM Timoshenko finite element to include the warping torsion effect for non-uniform torsion.
34	Barati *et al.* [82]	Solution for static torsion in a microtube composed of bidirectional FGMs (BDFGMs).
35	Naghibi *et al.* [83]	Determination of defects in hollow cylinders coated with FGMs under torsion, identifying their impact on stress results.
36	Singh *et al.* [84]	Analysis of shear stresses in FGM bars under pure torsional loading, considering different cross-sectional shapes (circular, square, triangular) and varying thicknesses.
37	Zhang *et al.* [85]	Study of the buckling of FGM cylindrical shells under torsional impact loading using the symplectic method, considering torsional stress waves.

Table 1. [Cont.]

No.	Scientists	Landmark achievement
38	Noroozi *et al.* [86]	Investigation of multiple cylindrical interface cracks between a homogeneous circular cylinder and its FGM coating under torsional impact loading.
39	Li, Hu [87]	Analysis of torsional statics of two-dimensionally FG microtubes, focusing on stress distribution and material properties.
40	Karaca, Alyavuz [88]	Examination of the torsional behavior of beams with one- and two-directional gradation under large displacements and angular deformations, considering power law and sinusoidal functions.
41	Barretta *et al.* [89]	Development of a nonlocal strain gradient theory of elasticity, combining the Eringen nonlocal integral convolution and the Lam strain gradient model through a variational approach.
42	Soltani, Asgarian [90]	Analysis of a lateral buckling of simply supported web- and/or flange-tapered I-beams made of axially FGMs under uniformly distributed loads.
43	Murín *et al.* [91]	Extension of previous research by investigating the effect of spatially varying material properties on torsional eigenvibrations of FGM beams.
44	Hajhashemkhani, Hematiyan [92]	Analysis of inflation, extension, and torsion in hyperelastic rods and tubes, focusing on rubber-like materials and soft biological tissues.
45	Nie *et al.* [93]	Presentation of analytical solutions for the torsion of bidirectional FG, linearly elastic truncated conical cylinders, considering six different functional forms of shear modulus variations in both radial and axial directions.
46	Baksa [94]	Study of analytical solution for Saint-Venant torsion of a circular bar with a slit extending radially from the boundary to the axis.
47	Rizov [95]	Analysis of cylindrical delamination in a multilayered FG circular shaft under torsional loading, based on the Ramberg–Osgood equation.

Their subsequent paper [56] focused on the torsion of bars with inhomogeneous shear modulus and arbitrary geometry, where the modulus varies across the cross section as a function of coordinates with the additional assumption of constancy at the boundary. Solutions were also presented for the torsion of a circular cylindrical bar with angular symmetry.

Further studies by Horgan and Chan [4] extended the analysis to isotropic, linearly elastic bars with functional gradients, providing deeper insight into stress and strain fields. Their additional works [8, 57] addressed rotating bodies and yielded exact solutions for a power-law variation of the Young modulus, showing that stress distributions differ from homogeneous cases and that maximum stresses are not always located at the center. Ting *et al.* [58] investigated the design of neutral cylinders under torsion, considering cylinders with multiple coatings or a graded shear modulus in the cross section. A multilayer cylinder with piecewise-varying shear modulus was generalized to a graded cylinder with continuous radial variation of the shear modulus. The warping field of the neutral graded cylinder is governed by a second-order differential equation, with solutions obtained using the Frobenius method. Singh *et al.* [59] studied torsional vibrations of FG finite cylinders and demonstrated resonance behavior influenced by material gradation, improving the understanding of vibrational responses in graded structures.

Batra [6] provided exact solutions for torsional behavior in FG cylinders, refining theoretical models, while Sofiyev and Schnack [60] examined the stability of FG cylindrical shells under dynamic torsional loads and highlighted the effects of transient stresses. Li *et al.* [61] analyzed a cylindrical crack in a FG interlayer between two coaxial elastic cylinders under torsional impact. The shear modulus and density of the FGM layer varied continuously. The problem was solved numerically, and the dynamic stress intensity factor was computed, showing that increasing the FGM gradient can significantly reduce this factor.

Continued contributions by Horgan [4, 8, 57] on anisotropic, linearly elastic bars with functional gradients, together with the study by Gholami Bazehhour and Rezaeepazhand [64] on multilayered tubes with non-circular cross sections, provided additional depth for complex geometries. Vasiliev [65] presented an analytical solution for the torsion of a circular punch on a transversely isotropic elastic half space with a FG coating. Wang *et al.* [66] provided exact solutions for transient torsional responses of a finitely long FG hollow cylinder under free–free, free–fixed, and fixed–fixed boundary conditions. Shen *et al.* [67] developed a size-dependent shaft model within the framework of nonlocal strain gradient theory, accounting for radial power-law variation in a two-constituent FGM and investigating small-scale effects on static and dynamic torsion, including material length scale and nonlocal parameters.

Huang *et al.* [69] proposed a method for elastoplastic buckling of cylindrical shells made of FGMs under axial and torsional loads, with properties varying according to a power law. The Ritz method and stress-state analysis were used to determine the critical condition and the location of the elastoplastic interface. Tsiatas and Babouskos [70] provided solutions for the elastic-plastic torsion problem of FG bars with arbitrary cross sections and property variation across the section, introducing a simplified nonlinear procedure using the BEM and the AEM. Elastostatic analyses by Murín *et al.* [71], as well as the study by Barretta *et al.* [73] on torsion in nonlocal viscoelastic nanobeams, highlighted novel modeling techniques for FGMs. Liaghat *et al.* [72] examined material tailoring for FG rods under torsion, and Barretta *et al.* [73] presented closed-form solutions for the torsion of linearly elastic isotropic beams with axial and cross-sectional inhomogeneities. New solutions were derived by analyzing axial distributions of longitudinal and shear moduli, and the effects of warping and shear modulus variation on the torsional behavior of elliptic and equilateral triangular beams were discussed.

Ongoing efforts by Bayat *et al.* [74] on generalized solutions for hollow cylinders, Rizov's [75] elastic-plastic fracture analysis, and Rahaeifard's [76] studies on size-dependent torsion illustrate substantial advances in predicting FGM behavior under diverse conditions. Aminbaghai [77] analyzed the influence of torsional warping on the elastostatic behavior of thin-walled twisted FGM beams with longitudinal material variation and secondary deformations due to the twist angle. Murín *et al.* [78] examined the effect of spatially varying properties on the warping torsion of I-section FGM beams using the reference beam method and the FGM-warping tension (WT) finite element. Extended stress equations accounting for secondary torsional moment and warping were applied, reinforcing the importance of torsional analysis for the development of these materials.

Future research directions include expanding understanding and application of FGMs under torsional loads. Promising areas are the development of multifunctional FGMs that combine torsional resistance with enhanced thermal or electrical performance, and the advancement of modeling techniques to capture interactions among torsion, bending, and axial loads in complex geometries. Experimental validation and improved materials characterization are essential for representing gradients accurately in practice. Experimental torsion testing of FGM shafts faces significant reliability challenges. Residual stresses arising from manufacturing (e.g., additive manufacturing (AM), sintering, deposition) and non-ideal material interfaces across graded transitions can distort the measured response. Maintaining perfect coaxiality and geometric tolerances is difficult, even small deviations introduce parasitic bending, compromising data quality. In thin-walled specimens, warping effects are pronounced and often amplified by gradation further obscuring a pure shear state. Finally, the absence of standardized torsion test protocols for FGMs hinders cross-laboratory validation and benchmarking. Investigation of fatigue and long-term durability under cyclic torsional loading is also important, with direct implications for aerospace, automotive, and civil engineering. While significant progress has been achieved, substantial opportunities remain to improve the design and application of FGMs across engineering disciplines.

6. Conclusion and future research

Torsion in FGMs can be described reliably within classical theories, provided they are generalized to account for the spatial variability of elastic parameters. It is essential to introduce $G(x, y)$, $E(x, y)$, and $\nu(x, y)$ explicitly into the torsion equations and to model gradation consistently. Here are the concise conclusions and brief research pointers:

- generalized classical theories in the context of torsion: Saint-Venant, Prandtl, and Vlasov formulations remain applicable when the spatial variability of G, E, and ν is explicitly incorporated and appropriate boundary conditions are enforced;
- gradation modeling: Voigt, Reuss, Hashin–Shtrikman, and Mori–Tanaka schemes as well as power law or exponential profiles are useful; in practice, the tighter Hashin–Shtrikman bounds and Mori–Tanaka estimates typically yield more stable predictions of torsional stiffness;
- numerical methods: isoparametric graded finite elements (per Kim–Paulino formulation) capture heterogeneity more faithfully than classical piecewise-homogeneous meshes and provide a robust basis for torsion analyses of FGMs;
- future research: a promising direction is to implement torsion modeling in commercial solvers such as Ansys, moving beyond the prevalent Ansys parametric design language (APDL)-based studies. Representing material gradation with isoparametric graded finite elements offers a more rigorous, scalable path for complex geometries. Further steps include coupling torsion models with additive manufacturing process simulations to quantify porosity and residual-stress effects, and expanding experimental validation, especially for FGM shafts and thin-walled members where warping complicates the realization of pure shear state. Future work should also assess fatigue and creep under torsion, develop multifunctional FGMs [96] that add thermal, damping, or conductive capabilities, and advance modeling [97, 98] that can capture coupled torsion–bending–axial behavior with explicit material, geometric nonlinearities and anisotropy.

Fundings

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Conflict of interest

The authors declare that there are no known competing financial interests or personal relationships that could have influenced the work described in this paper.

Authors' contributions

Mateusz Kumor performed the analysis and wrote the original draft. Artur Ganczarski performed the analysis and contributed to data interpretation. All authors reviewed and approved the final manuscript.

References

1. Niino M., Hirai T., Watanabe R., Functionally gradient materials. In pursuit of super heat resisting materials for spacecraft, *Journal of the Japan Society for Composite Materials*, **13**(6): 257–264, 1987, https://doi.org/10.6089/jscm.13.257.
2. Pasha A., Rajaprakash B.M., Functionally graded materials (FGM) fabrication and its potential challenges & applications, *Materials Today: Proceedings*, **52**(3): 413–418, 2022, https://doi.org/10.1016/j.matpr.2021.09.077.
3. Lekhnitskii S.G., *Theory of Elasticity of an Anisotropic Body*, Mir Publishers, Moscow, 1981.
4. Horgan C.O., Chan A.M., Torsion of functionally graded isotropic linearly elastic bars, *Journal of Elasticity*, **52**: 181–199, 1998, https://doi.org/10.1023/A:1007544011803.
5. Rooney F.J., Ferrari M., Torsion and flexure of inhomogeneous elements, *Composites Engineering*, **5**(7): 901–911, 1995, https://doi.org/10.1016/0961-9526(95)00043-M.
6. Batra R.C., Torsion of a functionally graded cylinder, *AIAA Journal*, **44**(6): 1363–1365, 2006, https://doi.org/10.2514/1.19555.
7. Arghavan S., Hematiyan M.R., Torsion of functionally graded hollow tubes, *European Journal of Mechanics – A/Solids*, **28**(3): 551–559, 2009, https://doi.org/10.1016/j.euromechsol.2008.07.009.
8. Horgan C.O., On the torsion of functionally graded anisotropic linearly elastic bars, *IMA Journal of Applied Mathematics*, **72**(5): 556–562, 2007, https://doi.org/10.1093/imamat/hxm027.
9. Chen T., Wei C.-J., Saint-Venant torsion of anisotropic shafts: Theoretical frameworks, extremal bounds and affine transformations, *The Quarterly Journal of Mechanics and Applied Mathematics*, **58**(2): 269–287, 2005, https://doi.org/10.1093/qjmamj/hbi013.

10. Barretta R., Luciano L., Analogies between Kirchhoff plates and functionally graded Saint-Venant beams under torsion, *Continuum Mechanics and Thermodynamics*, **27**: 499–505, 2015, https://doi.org/10.1007/s00161-014-0385-2.

11. Li L., Hu Y., Torsional statics of two-dimensionally functionally graded microtubes, *Mechanics of Advanced Materials and Structures*, **26**(5): 430–442, 2019, https://doi.org/10.1080/15376494.2017.1400617.

12. Barretta R., Feo L., Luciano R., Torsion of functionally graded nonlocal viscoelastic circular nanobeams, *Composites Part B: Engineering*, **72**: 217–222, 2015, https://doi.org/10.1016/j.compositesb.2014.12.018.

13. Nikmehr N., Lashkarbolok M., A numerical investigation on the torsional rigidity of bars with functionally graded material (FGM) cross sections weakened by cracks, *Iranian Journal of Science and Technology, Transactions of Civil Engineering*, **43**: 117–123, 2019, https://doi.org/10.1007/s40996-018-0147-7.

14. Darilmaz K., Orakdöğen E., Girgin K., Saint-Venant torsion of arbitrarily shaped orthotropic composite or FGM sections by a hybrid finite element approach, *Acta Mechanica*, **229**: 1387–1398, 2018, https://doi.org/10.1007/s00707-017-2067-1.

15. Akinlabi E.T., Mikhin M.N., Murashkin E.V., Functionally graded prismatic triangular rod under torsion, *Journal of Physics: Conference Series*, **1474**(1): 012003, 2020, https://doi.org/10.1088/1742-6596/1474/1/012003.

16. Ganczarski A., Szubartowski D., Kumor M., Torsion of non-circular functionally graded material shafts, [in:] *State of the Art and Future Trends in Materials Modelling 2*, Altenbach H., Öchsner A. [Eds], Springer, pp. 115–131, 2024, https://doi.org/10.1007/978-3-031-72900-3_9.

17. Duan T.C., Li X.Y., Xiao Y., Zhang L., Chen C., Li Z.J., A modified lower-order theory for FG beam with circular cross-section, *Frontiers in Materials*, **9**: 1020820, 2022, https://doi.org/10.3389/fmats.2022.1020820.

18. Gil-Martín L.M., Palomares A., Hernández-Montes E., Approximate expression of the Prandtl membrane analogy in linear elastic pure torsion of open thin-walled cross sections and regular polygons, *International Journal of Solids and Structures*, **210–211**: 109–118, 2021, https://doi.org/10.1016/j.ijsolstr.2020.11.020.

19. Timoshenko S., A membrane analogy to flexure, *Proceedings of the London Mathematical Society*, **s2–20**(1): 398–407, 1922, https://doi.org/10.1112/plms/s2-20.1.398.

20. Irschik H., Analogies between bending of plates and torsion problem, *Journal of Engineering Mechanics*, **117**(11): 2503–2508, 1991, https://doi.org/10.1061/(ASCE)0733-9399(1991)117:11(2503).

21. Irschik H., Analogy between refined beam theories and the Bernoulli-Euler theory, *International Journal of Solids and Structures*, **28**(9): 1105–1112, 1991, https://doi.org/10.1016/0020-7683(91)90105-O.

22. Furukawa T., Irschik H., Body-force analogy for one-dimensional coupled dynamic problems of thermoelasticity, *Journal of Thermal Stresses*, **28**(4): 455–464, 2005, https://doi.org/10.1080/01495730590916713.

23. Irschik H., Gusenbauer M., Body force analogy for transient thermal stresses, *Journal of Thermal Stresses*, **30**(9–10): 965–975, 2007, https://doi.org/10.1080/01495730701499024.

24. Irschik H., Nader M., Actuator placement in static bending of smart beams utilizing Mohr's analogy, *Engineering Structures*, **31**(8): 1698–1706, 2009, https://doi.org/10.1016/j.engstruct.2009.02.026.

25. Irschik H., Krommer M., Zehetner C., A generalized body force analogy for the dynamic theory of thermoelasticity, *Journal of Thermal Stresses*, **35**(1–3): 235–247, 2012, https://doi.org/10.1080/01495739.2012.637826.

26. Romano G., Diaco M., Barretta R., Variational formulation of the first principle of continuum thermodynamics, *Continuum Mechanics and Thermodynamics*, **22**: 177–187, 2010, https://doi.org/10.1007/s00161-009-0119-z.

27. Barretta R., Marotti de Sciarra F., Analogies between nonlocal and local Bernoulli–Euler nanobeams, *Archive of Applied Mechanics*, **85**: 89–99, 2015, https://doi.org/10.1007/s00419-014-0901-7.

28. Ugural A.C., Fenster S.K., *Advanced Mechanics of Materials and Applied Elasticity*, Pearson Education, 2011.

29. Makai E., A proof of Saint-Venant's theorem on torsional rigidity, *Acta Mathematica Academiae Scientiarum Hungarica*, **17**: 419–422, 1966, https://doi.org/10.1007/BF01894885.

30. Nikmehr O., Lashkarbolok M., A numerical investigation on the torsional rigidity of bars with functionally graded material (FGM) cross sections weakened by cracks, *Iranian Journal of Science and Technology, Transactions of Civil Engineering*, **43**: 117–123, 2019, https://doi.org/10.1007/s40996-018-0147-7.

31. Prandtl L., Torsion of prismatic rods [in German: Zür torsion von prismatischen stäben], *Phys. Z*, **4**: 758–770, 1903.

32. Pegoraro F., Califano F., Manfredi C., Morrison P.J., Theory and applications of the Vlasov equation, *The European Physical Journal D*, **69**: 68, 2015, https://doi.org/10.1140/epjd/e2015-60082-y.

33. Vlasov V.Z., *Thin-Walled Elastic Beams*, Oldbourne Press, London, 1961.

34. Reddy J.N., *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, 2nd ed., CRC Press, 2003.

35. Addessi D., Di Re P., Cimarello G., Enriched beam finite element models with torsion and shear warping for the analysis of thin-walled structures, *Thin-Walled Structures*, **159**: 107259, 2021, https://doi.org/10.1016/j.tws.2020.107259.

36. Murín J., Hrabovský J., Aminbaghai M., Kutiš V., Paulech J., Kugler S., Extension of the FGM beam finite element by warping torsion, *Strojnícky časopis – Journal of Mechanical Engineering*, **69**(2): 57–76, 2019, https://doi.org/10.2478/scjme-2019-0017.

37. Murin J., Aminbaghai M., Hrabovsky J., Balduzzi G., Dorn M., Mang H.A., Torsional warping eigenmodes of FGM beams with longitudinally varying material properties, *Engineering Structures*, **175**: 912–925, 2018, https://doi.org/10.1016/j.engstruct.2018.08.048.

38. Murin J., Kugler S., Habrovsky J., Kutis V., Paulech J., Aminbaghai M., Influence of spatially varying material properties on the bimoment normal and shear stresses by warping torsion of FGM beams, *Composite Structures*, **256**: 113043, 2021, https://doi.org/10.1016/j.compstruct.2020.113043.

39. Hernik S., New concept of finite element method for FGM materials, *Czasopismo Techniczne. Mechanika*, **107**(8): 99–106, 2010.

40. Ansys, *Theory reference, Ansys 2021 R2*, Ansys INC, 2021, https://www.scribd.com/document/665578059/Ansys-Mechanical-APDL-Theory-Reference.

41. Kim J.-H., Paulino G.H., Isoparametric graded finite elements for nonhomogeneous isotropic and orthotropic materials, *Journal of Applied Mechanics*, **69**(4): 502–514, 2002, https://doi.org/10.1115/1.1467094.

42. Hughes T.J.R., *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Dover Publication, 2012.

43. Jin Z.-H., Paulino G.H., Dodds Jr. R.H., Finite element investigation of quasi-static crack growth in functionally graded materials using a novel cohesive zone fracture model, *Journal of Applied Mechanics*, **69**(3): 370–379, 2002, https://doi.org/10.1115/1.1467092.

44. Năstăsescu V., Marzavan S., An overview of functionally graded material models, *Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science*, **23**(3): 259–267, 2022.

45. Mathew C., Oluomachi E., Mechanics and computational homogenization of effective material properties of functionally graded (composite) material plate FGM, *International Journal of Scientific and Research Publications*, **13**(9): 128–150, 2023, https://doi.org/10.29322/IJSRP.13.09.2023.p14120.

46. Reuss A., Calculation of the yield point of mixed crystals on the basis of the plasticity condition for single crystals [in German: Berechnung der fließgrenze von mischkristallen auf grund der plastizitätsbedingung für einkristalle], *ZAMM – Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, **9**(1): 49–58, 1929, https://doi.org/10.1002/zamm.19290090104.

47. Voigt W., On the relationship between the two elasticity constants of isotropic bodies [in German: Über die Beziehung zwischen den beiden Elasticitätsconstanten isotroper Körper], *Annalen der Physik*, **274**(12): 573–587, 1889, https://doi.org/10.1002/andp.18892741206.

48. Aboudi J., Arnold S.M., Bednarcyk B.A., *Micromechanics of Composite Materials: A Generalized Multiscale Analysis Approach*, Butterworth-Heinemann, 2013, https://doi.org/10.1016/C2011-0-05224-9.

49. Mori T., Tanaka K., Average stress in matrix and average elastic energy of materials with misfitting inclusions, *Acta Metallurgica*, **21**(5): 571–574, 1973, https://doi.org/10.1016/0001-6160(73)90064-3.

50. Pobedra B., *Mechanics of Composite Materials* [in Russian], Moscow University, Moscow, 1984.

51. Hashin Z., Shtrikman S., A variational approach to the theory of the elastic behaviour of multiphase materials, *Journal of the Mechanics and Physics of Solids*, **11**(2): 127–140, 1963, https://doi.org/10.1016/0022-5096(63)90060-7.

52. Saharan A., *Homogenization and elastic-plastic transitions in random and FGM microstructures*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2015.

53. El-Galy I.M., Bassiouny I.S., Mahmoud H.A., Functionally graded materials classifications and development trends from industrial point of view, *SN Applied Sciences*, **1**: 1378, 2019, https://doi.org/10.1007/s42452-019-1413-4.

54. Zhang C. *et al.*, Additive manufacturing of functionally graded materials: A review, *Materials Science and Engineering: A*, **764**: 138209, 2019, https://doi.org/10.1016/j.msea.2019.138209.

55. Rooney F.J., Ferrari M., *Torsion of a Functionally Graded Shaft With Rectangular Cross Section*, American Society of Mechanical Engineers, New York, 1994, https://www.osti.gov/biblio/82883.

56. Rooney F.J., Ferrari M., Torsion of a class of functionally graded shafts, Conference paper, *American Society of Mechanical Engineers*, 1995, https://www.osti.gov/biblio/86253.

57. Horgan C.O., Chan A.M., The stress response of functionally graded isotropic linearly elastic rotating disks, *Journal of Elasticity*, **55**: 219–230, 1999, https://doi.org/10.1023/A:1007644331856.

58. Ting T.C.T., Chen T., Li K.S., Torsion of an isotropic shaft of arbitrary cross-section embedded with multicoated or graded circular cylinders of cylindrically orthotropic materials, *The Quarterly Journal of Mechanics and Applied Mathematics*, **57**(3): 347–362, 2004, https://doi.org/10.1093/qjmam/57.3.347.

59. Singh B.M., Rokne J., Dhaliwal R.S., Torsional vibrations of functionally graded finite cylinders, *Meccanica*, **41**: 459–470, 2006, https://doi.org/10.1007/s11012-006-0003-x.

60. Sofiyev A.H., Schnack E., The stability of functionally graded cylindrical shells under linearly increasing dynamic torsional loading, *Engineering Structures*, **26**(10): 1321–1331, 2004, https://doi.org/10.1016/j.engstruct.2004.03.016.

61. Li C., Weng G.J., Duan Z., Dynamic behavior of a cylindrical crack in a functionally graded interlayer under torsional loading, *International Journal of Solids and Structures*, **38**(42–43): 7473–7485, 2001, https://doi.org/10.1016/S0020-7683(01)00046-4.

62. Hematiyan M.R., Estakhrian E., Torsion of functionally graded open-section members, *International Journal of Applied Mechanics*, **4**(2): 1250020, 2012, https://doi.org/10.1142/S1758825112500202.

63. Arghavan S., Hematiyan M.R., Torsion of functionally graded hollow tubes, *European Journal of Mechanics – A/Solids*, **28**(3): 551–559, 2009, https://doi.org/10.1016/j.euromechsol.2008.07.009.

64. Gholami Bazehhour B., Rezaeepazhand J., Approximate torsional analysis of multi-layered tubes with non-circular cross-sections, *Applied Composite Materials*, **18**: 485–497, 2011, https://doi.org/10.1007/s10443-011-9213-z.

65. Vasiliev A., Sevostianov I., Aizikovich S., Jeng Y.-R., Torsion of a punch attached to transversely-isotropic half-space with functionally graded coating, *International Journal of Engineering Science*, **61**: 24–35, 2012, https://doi.org/10.1016/j.ijengsci.2012.06.006.

66. Wang H.M., Liu C.B., Ding H.J., Exact solution and transient behavior for torsional vibration of functionally graded finite hollow cylinders, *Acta Mechanica Sinica*, **25**: 555–563, 2009, https://doi.org/10.1007/s10409-009-0251-9.

67. Shen Y., Chen Y., Li L., Torsion of a functionally graded material, *International Journal of Engineering Science*, **109**: 14–28, 2016, https://doi.org/10.1016/j.ijengsci.2016.09.003.

68. Bayat Y., Ekhteraei Toussi H., Elastoplastic torsion of hollow FGM circular shaft, *Journal of Computational & Applied Research in Mechanical Engineering (JCARME)*, **4**(2): 165–180, 2015, https://doi.org/10.22061/jcarme.2015.274.

69. Huang H., Zhang Y., Han Q., Inelastic buckling of FGM cylindrical shells subjected to combined axial and torsional loads, *International Journal of Structural Stability and Dynamics*, **17**(9): 1771010, 2017, https://doi.org/10.1142/S0219455417710109.

70. Tsiatas G.C., Babouskos N.G., Elastic-plastic analysis of functionally graded bars under torsional loading, *Composite Structures*, **176**: 254–267, 2017, https://doi.org/10.1016/j.compstruct.2017.05.044.

71. Murín J., Aminbaghai M., Hrabovský J., Elastostatic analysis of the spatial FGM structures, *Strojnícky časopis – Journal of Mechanical Engineering*, **65**(1): 27–56, 2015, https://doi.org/10.1515/scjme-2016-0003.

72. Liaghat F., Hematiyan M.R., Khosravifard A., Material tailoring in functionally graded rods under torsion, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **228**(18): 3283–3295, 2014, https://doi.org/10.1177/0954406214529557.

73. Barretta R., Feo L., Luciano R., Some closed-form solutions of functionally graded beams undergoing nonuniform torsion, *Composite Structures*, **123**: 132–136, 2015, https://doi.org/10.1016/j.compstruct.2014.12.027.

74. Bayat Y., Alizadeh M., Bayat A., Generalized solution of functionally graded hollow cylinder under torsional load, *Journal of Computational & Applied Research in Mechanical Engineering (JCARME)*, **2**(2): 23–32, 2013, https://doi.org/10.22061/jcarme.2013.50.

75. Rizov V.I., Elastic-plastic fracture of functionally graded circular shafts in torsion, *Advances in Materials Research*, **5**(4): 299–318, 2016, https://doi.org/10.12989/amr.2016.5.4.299.

76. Rahaeifard M., Size-dependent torsion of functionally graded bars, *Composites Part B: Engineering*, **82**: 205–211, 2015, https://doi.org/10.1016/j.compositesb.2015.08.011.

77. Aminbaghai M., Murin J., Kutiš V., Hrabovsky J., Kostolani M., Mang H.A., Torsional warping elastostatic analysis of FGM beams with longitudinally varying material properties, *Engineering Structures*, **200**: 109694, 2019, https://doi.org/10.1016/j.engstruct.2019.109694.

78. Murín J., Kugler S., Habrovsky J., Kutiš V., Paulech J., Aminbaghai M., Warping torsion of FGM beams with spatially varying material properties, *Composite Structures*, **291**: 115592, 2022, https://doi.org/10.1016/j.compstruct.2022.115592.

79. Guendouz I., Khebizi M., Guenfoud H., Guenfoud M., El Fatmi R., Analysis of torsional-bending FGM beam by 3D Saint-Venant refined beam theory, *Structural Engineering and Mechanics*, **84**(3): 423–435, 2022, https://doi.org/10.12989/.2022.84.3.423.

80. Murín J., Hrabovsky J., Kugler S., Kutiš V., Aminbaghai M., Effect of longitudinal variation of material properties in warping torsion of FGM beams, [in:] *Analysis of Shells, Plates, and Beams: A State of the Art Report*, Altenbach H., Chinchaladze N., Kienzler R., Müller W.H. [Eds], pp. 243–296, 2020, https://doi.org/10.1007/978-3-030-47491-1_14.

81. Guendouz I., Khebizi M., Guenfoud H., Guenfoud M., Analysis of FGM cantilever beams under bending-torsional behavior using a refined 1D beam theory, *Periodica Polytechnica Civil Engineering*, **66**(4): 1262–1277, 2022, https://doi.org/10.3311/PPci.20595.

82. Barati A., Adeli M.M., Hadi A., Static torsion of bi-directional functionally graded microtube based on the couple stress theory under magnetic field, *International Journal of Applied Mechanics*, **12**(2): 2050021, 2020, https://doi.org/10.1142/S1758825120500210.

83. Naghibi S.R., Wang W., Ghavi M.R., Gollou R.M., Several defects in a hollow cylinder coated by a functionally graded material (FGM) subjected to torsional loading, *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*, **47**: 109–131, 2023, https://doi.org/10.1007/s40997-022-00492-2.

84. Singh P.K., Kumar M., Mishra S., Finite element analysis of functionally graded bar under torsional load, *Materials Today: Proceedings*, **56**(5): 2960–2966, 2022, https://doi.org/10.1016/j.matpr.2021.11.012.

85. Zhang J., Chen S., Chen L., Dynamic buckling of FGM cylindrical shells under torsional impact loads, [in:] *New Trends in Nonlinear Dynamics: Proceedings of the First International Nonlinear Dynamics Conference (NODYCON 2019), Volume III*, Lacarbonara W., Balachandran B., Ma J., Machado J.T.A., Stepan G. [Eds], Springer, 2020, https://doi.org/10.1007/978-3-030-34724-6_12.

86. Noroozi M., Ghassemi A., Atrian A., Vahabi M., Multiple cylindrical interface cracks in FGM coated cylinders under torsional transient loading, *Theoretical and Applied Fracture Mechanics*, **97**: 258–264, 2018, https://doi.org/10.1016/j.tafmec.2018.08.015.

87. Li L., Hu Y., Torsional statics of two-dimensionally functionally graded microtubes, *Mechanics of Advanced Materials and Structures*, **26**(5): 430–442, 2019, https://doi.org/10.1080/15376494.2017.1400617.

88. Karaca M., Alyavuz B., Non-linear behavior of functionally graded elastoplastic beam under torsion, *Journal of Scientific Reports-A*, **57**: 27–50, 2023, https://doi.org/10.59313/jsr-a.1415211.

89. Barretta R., Ali Faghidian S., Marotti de Sciarra F., Penna R., Pinnola F.P., On torsion of nonlocal Lam strain gradient FG elastic beams, *Composite Structures*, **233**: 111550, 2020, https://doi.org/10.1016/j.compstruct.2019.111550.

90. Soltani M., Asgarian B., Lateral-torsional stability analysis of a simply supported axially functionally graded beam with a tapered I-section, *Mechanics of Composite Materials*, **56**: 39–54, 2020, https://doi.org/10.1007/s11029-020-09859-5.

91. Murín J., Aminbaghai M., Hrabovsky J., Kutis V., Kugler S., Paulech J., Nonuniform Torsional Eigenmodes of FGM Beams, [in:] *Proceedings of the 14th International Conference on Vibration Problems*, 2021, https://doi.org/10.1007/978-981-15-8049-9_17.

92. Hajhashemkhani M., Hematiyan M.R., Inflation, extension and torsion analysis of compressible functionally graded hyperelastic tubes, *Acta Mechanica*, **231**: 3947–3960, 2020, https://doi.org/10.1007/s00707-020-02742-3.

93. Nie G.J., Pydah A., Batra R.C., Torsion of bi-directional functionally graded truncated conical cylinders, *Composite Structures*, **210**: 831–839, 2019, https://doi.org/10.1016/j.compstruct.2018.11.081.

94. Baksa A., Saint-Venant torsion of a functionally graded circular bar with a radial slit, *Journal of Theoretical and Applied Mechanics*, **50**: 83–101, 2020, https://doi.org/10.7546/JTAM.50.20.01.08.

95. RIZOV V.I., Nonlinear delamination analysis of multilayered functionally graded circular shafts in torsion, *Journal of Applied Mechanics and Technical Physics*, **59**: 1104–1110, 2018, https://doi.org/10.1134/S0021894418060160.

96. CHEN D., LIU L., CHU L., LIU Q., Analytical solution of thermo–mechanical properties of functionally graded materials by asymptotic homogenization method, *Materials*, **15**(9): 3073, 2022, https://doi.org/10.3390/ma15093073.

97. YADAV S., LIU S., KUMAR SINGH R., KUMAR SHARMA A., RAWAT P., A state-of-art review on functionally graded materials (FGMs) manufactured by 3D printing techniques: Advantages, existing challenges, and future scope, *Journal of Manufacturing Processes*, **131**: 2051–2072, 2024, https://doi.org/10.1016/j.jmapro.2024.10.026.

98. ALKUNTE S., FIDAN I., NAIKWADI V., GUDAVASOV S., ALSHAIKH ALI M., MAHMUDOV M., HASANOV S., CHEEPU M., Advancements and challenges in additively manufactured functionally graded materials: A comprehensive review, *Journal of Manufacturing and Materials Processing*, **8**(1): 23, 2024, https://doi.org/10.3390/jmmp8010023.

99. HAO Y.X., WANG M.X., ZHANG W., YANG S.W., LIU L.T., QIAN Y.H., Bending-torsion coupling bursting oscillation of a sandwich conical panel under parametric excitation, *Journal of Sound and Vibration*, **495**: 115904, 2021, https://doi.org/10.1016/j.jsv.2020.115904.

100. AGHAZADEH R., RAFIGHI M., KUMAR R., AL AWADH M., Material gradation effects on twisting statics of bi-directional functionally graded micro-tubes, *AIP Advances*, **14**(2): 025228, 2024, https://doi.org/10.1063/5.0194270.

101. CIVALEK Ö., UZUN B., YAYLI M.Ö., Torsional vibrations of functionally graded restrained nanotubes, *The European Physical Journal Plus*, **137**: 113, 2022, https://doi.org/10.1140/epjp/s13360-021-02309-8.

102. SHAKHLAVI S.J., HOSSEINI-HASHEMI S., NAZEMNEZHAD R., Torsional vibrations of functionally graded restrained nanotubes, *International Journal of Non-Linear Mechanics*, **124**: 103513, 2020, https://doi.org/10.1016/j.ijnonlinmec.2020.103513.

103. BENI Y.T., Size dependent coupled electromechanical torsional analysis of porous FG flexoelectric micro/nanotubes, *Mechanical Systems and Signal Processing*, **178**: 109281, 2022, https://doi.org/10.1016/j.ymssp.2022.109281.

104. BARATI A., NOROUZI S., Nonlocal elasticity theory for static torsion of the bi-directional functionally graded microtube under magnetic field, *Journal of Computational Applied Mechanics*, **51**(1): 30–36, 2020, https://doi.org/10.22059/jcamech.2019.294263.462.

105. ZAREZADEH E., HOSSEINI V., HADI A., Torsional vibration of functionally graded nanorod under magnetic field supported by a generalized torsional foundation based on nonlocal elasticity theory, *Mechanics Based Design of Structures and Machines*, **48**(4): 480–495, 2020, https://doi.org/10.1080/15397734.2019.1642766.

Received January 21, 2025; revised October 28, 2025; accepted November 27, 2025; available online January 20, 2026; version of record June 3, 2026; published issue June 24, 2026.