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Research Paper

Fundamental Solutions in the Generalized Theory of Thermoelastic Diffusion with Triple Porosity

Tarun KANSAL

Department of Mathematics, Markanda National College
Shahabad(M.)-136135, India; e-mail: tarun1_kansal@yahoo.co.in

The main aim of this paper is to derive the basic governing equations for an anisotropic thermoelastic medium with mass diffusion and triple porosity. Additionally, the fundamental solutions of a system of equations for steady, pseudo-, quasi-static oscillations and equilibrium are constructed.

Keywords: thermoelastic diffusion; triple porosity; pores; steady oscillations.

1. Introduction

Diffusion is the transfer process of the mass of a substance from the high concentration regions to low concentration regions. Nowacki [1–4] developed the theory of thermoelasticity with mass diffusion based on classical Fourier's and Fick's laws. Sherief *et al.* [5] established a generalized theory of thermoelasticity with mass diffusion by modifying Fourier's and Fick's laws.

The first model for single porosity deformable solid was given by Biot [6]. Aifantis and colleagues [7–9] developed the theory for deformable materials with double porosity. In a double porosity elastic material, there are macro pores in the body but in addition there is a micro porosity arising because of fissures or cracks in the solid skeleton. Khalili and Selvadurai [10, 11] and Gelet *et al.* [12] established the basic governing homogeneous equations in the linear theory of thermoelasticity for solids with double porosity. Considerable research has been conducted in this field. While all the theories developed by the above mentioned authors were based on Darcy's law. Ieşan and Quintanilla [13] derived a non-linear theory of thermoelastic solids with double porosity structure without using this law. In a similar manner, Kansal [14] developed a linear generalized theory of thermoelastic diffusion with double porosity. Svanadze [15] developed the classical potential method in the linear theory of thermoelasticity for materials with a double porosity structure based on the mechanics of materials with voids. Marin *et al.* [16] approached transient elastic processes and steady-behavior state in a cylinder consisting of a linear elastic body with a dipolar structure only subjected to some boundary restrictions at a plane end. Amin *et al.* [17] obtained new uniqueness results for anisotropic thermoelastic bodies with a double porosity structure based on the Betti reciprocity relation that involve some thermoelastic processes.

In a triple porosity elastic material, the body possesses three levels of pore structures. The first is the largest visible porosity known as macro porosity, the second represents an intermediate case which is known as meso porosity, and the final scenario is referred to as a micro porosity. Svanadze [18] and Straughan [19] presented the governing equations for the theories of elasticity and thermoelasticity with triple porosity, respectively. Svanadze [20–25] studied various boundary value problems concerning elastic solids and thermoelastic solids with triple porosity.

The need for theories addressing multiple porosity elasticity and the associated mathematical, physical and numerical analyze is undoubtedly driven by the myriad of applications that currently exist and continue to emerge. The first application area is in mathematical biology and the associated field of health. Replacement of damaged long bones in humans is a major problem for a surgeon because the porosity of the bone can vary from 14% in the outer layer of bone to 42% in the inner layer. Indeed, to adequately model a long bone one may require a multi-porosity theory applicable to a graded porosity material. Another very important area of application for multiple porosity elasticity is geophysics. For example, a careful description of landslides may require employing the double porosity theory. Straughan [26] discussed various applications of multiple porosity in his book.

Fundamental solutions play an important role in solving various boundary value problems. The reason is that an integral representation of the solution of a boundary value problem using a fundamental solution is often more easily solved by numerical methods rather than a differential equation with specified boundary and initial conditions. When investigating boundary value problems of the theories of elasticity and thermoelasticity by the potential method, it is necessary to construct fundamental solutions of corresponding systems of partial differential equations and to establish their basic properties. Numerous authors [27–33] constructed fundamental solutions by means of elementary functions in different theories of elasticity and thermoelasticity involving double and triple porosity.

In this paper, the constitutive relations and field equations for anisotropic generalized thermoelastic diffusion with triple porosity are derived in Sec. 2. After reducing the anisotropic system of equations into an isotropic system of equations, the fundamental solution for steady oscillations is constructed in terms of elementary functions in Secs. 3 and 4. In Sec. 5, the fundamental solutions for pseudo-, quasi-static oscillations and equilibrium are constructed. Finally, some basic properties of fundamental matrix are established in Sec. 6.

2. Basic equations

Based on the work of Ieşan and Quintanilla [13], the law of conservation of energy for an arbitrary material volume V bounded by a surface B at time t can be written as

$$\int_V \rho[\dot{u}_i\ddot{u}_i + \kappa_1\dot{\nu}_1\ddot{\nu}_1 + \kappa_2\dot{\nu}_2\ddot{\nu}_2 + \kappa_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}]\,\mathrm{d}V = \int_V \rho[F_i\dot{u}_i + \Lambda_i\,\dot{\nu}_i]\,\mathrm{d}V + \int_B [f_i\dot{u}_i + \Omega_{ij}\varpi_j\dot{\nu}_i - q_i\varpi_i]\,\mathrm{d}B, \tag{2.1}$$

where U is the internal energy per unit mass, ρ is the density, q_i are the components of heat flux vector $\mathbf{q}$, F_i are the components of the external force per unit mass, u_i are the components of displacement vector $\mathbf{u}$, f_i are the components of surface traction vector $\mathbf{f}$ occurring on the surface B, ν_i are the volume fraction fields corresponding to macro-, meso-, micro-pores respectively, κ_i are the coefficients of equilibrated inertia, Λ_i are the extrinsic equilibrated body forces per unit mass associated with macro-, meso-, micro-pores, respectively, Ω_{ij} are the components of equilibrated stress vectors corresponding to ν_i measured per unit area of surface B, respectively, and ϖ_i are the components of outward unit normal vector $\boldsymbol{\varpi}$ to the surface B.

The components f_i are related to the stress vectors by the relation:

$$f_i = \sigma_{ji}\varpi_j, \tag{2.2}$$

where $\sigma_{ji}(=\sigma_{ij})$ are the components of the stress tensor.

By using Eq. (2.2) in Eq. (2.1) and applying the divergence theorem, we acquire

$$\int_V \rho[\dot{u}_i\ddot{u}_i + \kappa_1\dot{\nu}_1\ddot{\nu}_1 + \kappa_2\dot{\nu}_2\ddot{\nu}_2 + \kappa_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}]\,\mathrm{d}V = \int_V \rho[F_i\dot{u}_i + \Lambda_i\,\dot{\nu}_i]\,\mathrm{d}V + \int_V [\sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij,j}\dot{\nu}_i + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i}]\,\mathrm{d}V. \tag{2.3}$$

Since Eq. (2.3) is valid for every part of the body, therefore the local form of the conservation of energy is obtained as follows:

$$\rho[\dot{u}_i\ddot{u}_i + \kappa_1\dot{\nu}_1\ddot{\nu}_1 + \kappa_2\dot{\nu}_2\ddot{\nu}_2 + \kappa_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}] = \rho[F_i\dot{u}_i + \Lambda_i\,\dot{\nu}_i] + \sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij,j}\dot{\nu}_i + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i}. \tag{2.4}$$

Let us consider a second motion that differs from the given motion only by a constant superposed rigid body translational velocity. We assume that κ_i, U, Λ_i, ρ, ν_i, Ω_{ij}, q_i, F_i, σ_{ji} do not vary due to this superposed rigid body velocity. Equation (2.4) is also true when $\dot{u}_i$ is replaced by $\dot{u}_i + \wp_i$, where $\wp_i$ are arbitrary constants, and all other terms remain unchanged. Therefore, from Eq. (2.4), we arrive at

$$\rho[(\dot{u}_i + \wp_i)\ddot{u}_i + \kappa_1\dot{\nu}_1\ddot{\nu}_1 + \kappa_2\dot{\nu}_2\ddot{\nu}_2 + \kappa_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}] = \rho[F_i(\dot{u}_i + \wp_i) + \Lambda_i\,\dot{\nu}_i] + \sigma_{ji,j}(\dot{u}_i + \wp_i) + \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij,j}\dot{\nu}_i + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i}. \tag{2.5}$$

By subtracting Eq. (2.4) from Eq. (2.5), we obtain

$$\wp_i[\sigma_{ji,j} + \rho F_i - \rho\ddot{u}_i] = 0. \tag{2.6}$$

Because the quantities in the square brackets are independent of $\wp_i$, from Eq. (2.6) we obtain

$$\sigma_{ji,j} + \rho F_i = \rho\ddot{u}_i. \tag{2.7}$$

Equation (2.4) with the assistance of Eq. (2.7) yields a simplified form of the conservation of energy

$$\rho\dot{U} = \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i} - \Upsilon_i\dot{\nu}_i, \tag{2.8}$$

where $\Upsilon_i, i = 1, 2, 3$, satisfy the following relations

$$\begin{aligned}
\Omega_{1j,j} + \Upsilon_1 + \rho\Lambda_1 &= \rho\kappa_1\ddot{\nu}_1,\\
\Omega_{2j,j} + \Upsilon_2 + \rho\Lambda_2 &= \rho\kappa_2\ddot{\nu}_2,\\
\Omega_{3j,j} + \Upsilon_3 + \rho\Lambda_3 &= \rho\kappa_3\ddot{\nu}_3.
\end{aligned} \tag{2.9}$$

Following Nowacki [34], the balance of entropy can be expressed as

$$\int_V \rho\dot{S}\,\mathrm{d}V + \int_B \left(\frac{q_i}{T}\right)\varpi_i\,\mathrm{d}B - \int_B \left(\frac{P\eta_i}{T}\right)\varpi_i\,\mathrm{d}B = \int_V \left[-\frac{q_i}{T^2}T_{,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i}\right]\mathrm{d}V, \tag{2.10}$$

where S and P are entropy and chemical potential per unit mass respectively, η_i are the components of mass diffusion flux vector $\boldsymbol{\eta}$, and T is the absolute temperature.

Equation (2.10) can be written in the local form

$$\rho\dot{S} + \left(\frac{q_i}{T}\right)_{,i} - \left(\frac{P\eta_i}{T}\right)_{,i} = -\frac{q_i}{T^2}T_{,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i}. \tag{2.11}$$

The right-hand side of Eq. (2.11) is the entropy source

$$\Re = -\frac{q_i}{T^2}T_{,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i} \geq 0.$$

Based on the above relations, Eq. (2.11) can be represented in the form of an inequality called the Clausius-Duhem inequality

$$\rho\dot{S} + \frac{q_{i,i}}{T} - \frac{q_i}{T^2}T_{,i} - \frac{P}{T}\eta_{i,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i} \geq 0. \tag{2.12}$$

The equation of conservation of mass is

$$\eta_{j,j} = -\dot{C}, \tag{2.13}$$

where C is the concentration of the diffusion material in the elastic body.

Inequality (2.12) using Eqs. (2.8) and (2.13) becomes

$$\rho T\dot{S} - \rho\dot{U} + \sigma_{ij}\dot{e}_{ij} + \varOmega_{ij}\dot{\nu}_{i,j} - \varUpsilon_i\dot{\nu}_i - \frac{q_i}{T}T_{,i} + P\dot{C} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0, \tag{2.14}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are the components of the strain tensor.

The Helmholtz free energy function $\varGamma$ is defined as

$$\varGamma = U - TS. \tag{2.15}$$

By applying Eq. (2.15) into inequality (2.14), we obtain

$$-\rho[\dot{\varGamma} + \dot{T}S] + \sigma_{ij}\dot{e}_{ij} + \varOmega_{ij}\dot{\nu}_{i,j} - \varUpsilon_i\dot{\nu}_i - \frac{q_i}{T}T_{,i} + P\dot{C} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0. \tag{2.16}$$

The function $\varGamma$ can be expressed in terms of independent variables e_{ij}, ν_i, $\nu_{i,j}$, T, $T_{,i}$, C, and $C_{,i}$. Therefore, we have

$$\dot{\varGamma} = \frac{\partial\varGamma}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\varGamma}{\partial\nu_i}\dot{\nu}_i + \frac{\partial\varGamma}{\partial\nu_{i,j}}\dot{\nu}_{i,j} + \frac{\partial\varGamma}{\partial T}\dot{T} + \frac{\partial\varGamma}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial\varGamma}{\partial C}\dot{C} + \frac{\partial\varGamma}{\partial C_{,i}}\dot{C}_{,i}. \tag{2.17}$$

Inequality (2.16) using Eq. (2.17) becomes

$$\left[\sigma_{ij} - \rho\frac{\partial\varGamma}{\partial e_{ij}}\right]\dot{e}_{ij} + \left[\varOmega_{ij} - \rho\frac{\partial\varGamma}{\partial\nu_{i,j}}\right]\dot{\nu}_{i,j} - \left[\varUpsilon_i + \rho\frac{\partial\varGamma}{\partial\nu_i}\right]\dot{\nu}_i - \rho\left[S + \frac{\partial\varGamma}{\partial T}\right]\dot{T}$$
$$+ \left[P - \rho\frac{\partial\varGamma}{\partial C}\right]\dot{C} - \rho\frac{\partial\varGamma}{\partial T_{,i}}\dot{T}_{,i} - \rho\frac{\partial\varGamma}{\partial C_{,i}}\dot{C}_{,i} - \frac{q_i}{T}T_{,i} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0.$$

The inequality should hold all rates $\dot{e}_{ij}$, $\dot{\nu}_i$, $\dot{\nu}_{i,j}$, $\dot{T}$, $\dot{T}_{,i}$, $\dot{C}$, and $\dot{C}_{,i}$. Hence, the coefficients of the above variables must vanish, that is,

$$
\begin{aligned}
&\sigma_{ij} = \rho\frac{\partial\Gamma}{\partial e_{ij}}, \qquad \Omega_{ij} = \rho\frac{\partial\Gamma}{\partial \nu_{i,j}}, \qquad \Upsilon_i = -\rho\frac{\partial\Gamma}{\partial \nu_i},\\
&S = -\frac{\partial\Gamma}{\partial T}, \qquad P = \rho\frac{\partial\Gamma}{\partial C}, \qquad \frac{\partial\Gamma}{\partial T_{,i}} = \frac{\partial\Gamma}{\partial C_{,i}} = 0,\\
&-\frac{q_i}{T}T_{,i} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0.
\end{aligned}
\tag{2.18}
$$

Let us introduce the notations

$$\boldsymbol{\phi} = \boldsymbol{\nu} - \boldsymbol{\nu}_0, \qquad \theta = T - T_0,$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, T_0 is the reference temperature of the body chosen such that $|\frac{\theta}{T_0}| \ll 1$, and $\boldsymbol{\nu}_0$ represents the volume fraction fields in the reference configuration.

In the linear theory, the independent variables are e_{ij}, ϕ_i, $\phi_{i,j}$, θ, and C. It is assumed that the undeformed body is free from stresses and has zero intrinsic equilibrated body forces and entropy. If the body has a center of symmetry, then we have

$$
\begin{aligned}
2\rho\Gamma = {}& c_{ijkl}e_{ij}e_{kl} - 2a_{ij}e_{ij}\theta - 2b_{ij}e_{ij}C + 2c_{ij}e_{ij}\phi_1 + 2d_{ij}e_{ij}\phi_2\\
&+ 2f_{ij}e_{ij}\phi_3 + \alpha_1\phi_1^2 + \alpha_2\phi_2^2 + \alpha_3\phi_3^2 + 2\alpha_4\phi_1\phi_2 + 2\alpha_5\phi_2\phi_3 + 2\alpha_6\phi_3\phi_1\\
&+ A_{ij}\phi_{1,i}\phi_{1,j} + B_{ij}\phi_{2,i}\phi_{2,j} + C_{ij}\phi_{3,i}\phi_{3,j} + 2D_{ij}\phi_{1,i}\phi_{2,j}\\
&+ 2E_{ij}\phi_{2,i}\phi_{3,j} + 2F_{ij}\phi_{3,i}\phi_{1,j} - 2\ell_i\phi_i\theta - 2\varepsilon_i\phi_i C - \frac{\rho C_e\theta^2}{T_0} - 2a\theta C + bC^2.
\end{aligned}
$$

Using the above equation in the system of Eqs. (2.18), the following constitutive equations are obtained:

$$\sigma_{ij} = c_{ijkl}e_{kl} + c_{ij}\phi_1 + d_{ij}\phi_2 + f_{ij}\phi_3 - a_{ij}\theta - b_{ij}C, \tag{2.19}$$

$$
\begin{aligned}
\Omega_{1j} &= A_{ij}\phi_{1,i} + D_{ij}\phi_{2,i} + F_{ij}\phi_{3,i},\\
\Omega_{2j} &= D_{ij}\phi_{1,i} + B_{ij}\phi_{2,i} + E_{ij}\phi_{3,i},\\
\Omega_{3j} &= F_{ij}\phi_{1,i} + E_{ij}\phi_{2,i} + C_{ij}\phi_{3,i},
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\Upsilon_1 &= -c_{ij}e_{ij} - \alpha_1\phi_1 - \alpha_4\phi_2 - \alpha_6\phi_3 + \ell_1\theta + \varepsilon_1 C,\\
\Upsilon_2 &= -d_{ij}e_{ij} - \alpha_4\phi_1 - \alpha_2\phi_2 - \alpha_5\phi_3 + \ell_2\theta + \varepsilon_2 C,\\
\Upsilon_3 &= -f_{ij}e_{ij} - \alpha_6\phi_1 - \alpha_5\phi_2 - \alpha_3\phi_3 + \ell_3\theta + \varepsilon_3 C,
\end{aligned}
\tag{2.21}
$$

$$\rho S = a_{ij}e_{ij} + \ell_i\phi_i + \frac{\rho C_e\theta}{T_0} + aC, \tag{2.22}$$

$$P = -b_{ij}e_{ij} - \varepsilon_i\phi_i - a\theta + bC. \tag{2.23}$$

Equations (2.7) and (2.9) with the use of Eqs. (2.19)–(2.21) become

$$c_{ijkl}e_{kl,j} + c_{ij}\phi_{1,j} + d_{ij}\phi_{2,j} + f_{ij}\phi_{3,j} - a_{ij}\theta_{,j} - b_{ij}C_{,j} + \rho F_i = \rho\ddot{u}_i, \tag{2.24}$$

$$\begin{aligned}
&-c_{ij}e_{ij} + A_{ij}\phi_{1,ij} + D_{ij}\phi_{2,ij} + F_{ij}\phi_{3,ij} - \alpha_1\phi_1 - \alpha_4\phi_2 - \alpha_6\phi_3 \\
&\qquad\qquad + \ell_1\theta + \varepsilon_1 C + \rho\Lambda_1 = \rho\kappa_1\ddot{\phi}_1, \\
&-d_{ij}e_{ij} + D_{ij}\phi_{1,ij} + B_{ij}\phi_{2,ij} + E_{ij}\phi_{3,ij} - \alpha_4\phi_1 - \alpha_2\phi_2 - \alpha_5\phi_3 \\
&\qquad\qquad + \ell_2\theta + \varepsilon_2 C + \rho\Lambda_2 = \rho\kappa_2\ddot{\phi}_2, \\
&-f_{ij}e_{ij} + F_{ij}\phi_{1,ij} + E_{ij}\phi_{2,ij} + C_{ij}\phi_{3,ij} - \alpha_6\phi_1 - \alpha_5\phi_2 - \alpha_3\phi_3 \\
&\qquad\qquad + \ell_3\theta + \varepsilon_3 C + \rho\Lambda_3 = \rho\kappa_3\ddot{\phi}_3.
\end{aligned} \tag{2.25}$$

The linearized form of Eq. (2.11) is

$$\rho T_0\dot{S} = -q_{i,i}. \tag{2.26}$$

Using Eq. (2.22) in Eq. (2.26), we obtain

$$T_0\big[a_{ij}\dot{e}_{ij} + \ell_i\dot{\phi}_i + a\dot{C}\big] + \rho C_e\dot{\theta} = -q_{i,i}. \tag{2.27}$$

The generalized Fourier's law of heat conduction equation is

$$q_i + \tau_0\dot{q}_i = -K_{ij}\theta_{,j}, \tag{2.28}$$

where K_{ij} are the coefficients of thermal conductivity tensor, τ_0 is the thermal relaxation time that will ensure that the heat conduction equation will predict finite speeds of heat propagation.

Equation (2.28) with the help of Eq. (2.27) becomes

$$\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\big[T_0(a_{ij}e_{ij} + \ell_i\phi_i + aC) + \rho C_e\theta\big] = K_{ij}\theta_{,ij}. \tag{2.29}$$

Similar to Eq. (2.28), the generalized Fick's law of mass diffusion is

$$\eta_i + \tau^0\dot{\eta}_i = -J_{ij}P_{,j}, \tag{2.30}$$

where J_{ij} are coefficients of diffusion tensor, and τ^0 is the diffusion relaxation time ensuring that the equation satisfied by the concentration will also predict finite speeds of propagation of matter from one medium to the other.

Using Eqs. (2.13) and (2.23) in Eq. (2.30), we obtain

$$-J_{ij}[b_{kl}e_{kl,ij}+\varepsilon_k\phi_{k,ij}+a\theta_{,ij}-bC_{,ij}]=\dot{C}+\tau^0\ddot{C}. \tag{2.31}$$

If we take

$$c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\delta_{ik}\delta_{jl}+\mu\delta_{il}\delta_{jk},\qquad a_{ij}=\vartheta_1\delta_{ij},\qquad b_{ij}=\vartheta_2\delta_{ij},$$

$$c_{ij}=\Re_1\delta_{ij},\qquad d_{ij}=\Re_2\delta_{ij},\qquad f_{ij}=\Re_3\delta_{ij},\qquad A_{ij}=A_1\delta_{ij},$$

$$B_{ij}=A_2\delta_{ij},\qquad C_{ij}=A_3\delta_{ij},\qquad D_{ij}=A_4\delta_{ij},\qquad E_{ij}=A_5\delta_{ij},$$

$$F_{ij}=A_6\delta_{ij},\qquad K_{ij}=K\delta_{ij},\qquad J_{ij}=D\delta_{ij},$$

where δ_{ij} is Kronecker's delta and λ, μ, ϑ_1, ϑ_2, $\Re_1$, $\Re_2$, $\Re_3$, A_1, ..., A_6, K, D are material constants, in Eqs. (2.24), (2.25), (2.29), and (2.31), the governing equations for homogeneous isotropic generalized thermoelastic diffusion with triple porosity in the absence of body forces are obtained as

$$\begin{aligned}
&\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\,\mathbf{u}+\Re_i\nabla\phi_i-\vartheta_1\nabla\theta-\vartheta_2\nabla C=\rho\ddot{\mathbf{u}},\\
&-\Re_1\operatorname{div}\mathbf{u}+(A_1\,\Delta-\alpha_1)\phi_1+(A_4\,\Delta-\alpha_4)\phi_2+(A_6\,\Delta-\alpha_6)\phi_3\\
&\qquad\qquad+\ell_1\theta+\varepsilon_1C=\rho\kappa_1\ddot{\phi}_1,\\
&-\Re_2\operatorname{div}\mathbf{u}+(A_4\,\Delta-\alpha_4)\phi_1+(A_2\,\Delta-\alpha_2)\phi_2+(A_5\,\Delta-\alpha_5)\phi_3\\
&\qquad\qquad+\ell_2\theta+\varepsilon_2C=\rho\kappa_2\ddot{\phi}_2,\\
&-\Re_3\operatorname{div}\mathbf{u}+(A_6\,\Delta-\alpha_6)\phi_1+(A_5\,\Delta-\alpha_5)\phi_2+(A_3\,\Delta-\alpha_3)\phi_3\\
&\qquad\qquad+\ell_3\theta+\varepsilon_3C=\rho\kappa_3\ddot{\phi}_3,\\
&\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left[T_0(\vartheta_1\operatorname{div}\mathbf{u}+\ell_i\phi_i+aC)+\rho C_e\theta\right]=K\Delta\theta,\\
&D\Delta[\vartheta_2\operatorname{div}\mathbf{u}+\varepsilon_i\phi_i+a\theta-bC]+\left(\frac{\partial}{\partial t}+\tau^0\frac{\partial^2}{\partial t^2}\right)C=0,
\end{aligned} \tag{2.32}$$

where Δ, ∇ are respectively, Laplacian and Del operators.

In the upcoming sections, the chemical potential has been used as a state variable rather than concentration. In an isotropic medium, Eq. (2.23) becomes

$$P=-\vartheta_2\operatorname{div}\mathbf{u}-\varepsilon_k\phi_k-a\theta+bC. \tag{2.33}$$

The system of Eqs. (2.32) with the aid of Eq. (2.33) can be rewritten as

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda' + \mu)\nabla\,\mathrm{div}\;\mathbf{u} + \sigma_i\nabla\phi_i - \zeta_1\nabla\theta - \zeta_2\nabla P = \rho\ddot{\mathbf{u}},\\
&-\sigma_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \beta_1)\phi_1 + (A_4\,\Delta - \beta_4)\phi_2 + (A_6\,\Delta - \beta_6)\phi_3\\
&\qquad + \xi_1\theta + v_1 P = \rho\kappa_1\ddot{\phi}_1,\\
&-\sigma_2\,\mathrm{div}\,\mathbf{u} + (A_4\,\Delta - \beta_4)\phi_1 + (A_2\,\Delta - \beta_2)\phi_2 + (A_5\,\Delta - \beta_5)\phi_3\\
&\qquad + \xi_2\theta + v_2 P = \rho\kappa_2\ddot{\phi}_2,\\
&-\sigma_3\,\mathrm{div}\,\mathbf{u} + (A_6\,\Delta - \beta_6)\phi_1 + (A_5\,\Delta - \beta_5)\phi_2 + (A_3\,\Delta - \beta_3)\phi_3\\
&\qquad + \xi_3\theta + v_3 P = \rho\kappa_3\ddot{\phi}_3,\\
&-\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)T_0\big[\zeta_1\,\mathrm{div}\,\mathbf{u} + \xi_i\phi_i + \eta\theta + \varsigma P\big] + K\Delta\theta = 0,\\
&-\left(\frac{\partial}{\partial t} + \tau^0\frac{\partial^2}{\partial t^2}\right)\big[\zeta_2\,\mathrm{div}\,\mathbf{u} + v_i\phi_i + \varsigma\theta + \varpi P\big] + D\Delta P = 0,
\end{aligned} \tag{2.34}$$

where

$$\begin{aligned}
&\varpi = b^{-1}, && \zeta_2 = \varpi\vartheta_2, && \zeta_1 = \vartheta_1 + a\,\zeta_2, && \sigma_i = \Re_i - \varepsilon_i\zeta_2,\\
&\lambda' = \lambda - \zeta_2\vartheta_2, && \varsigma = a\varpi, && v_i = \varepsilon_i\varpi, && \beta_i = \alpha_i - \varepsilon_i v_i,\\
&\beta_4 = \alpha_4 - \varepsilon_1 v_2, && \beta_5 = \alpha_5 - \varepsilon_2 v_3, && \beta_6 = \alpha_6 - \varepsilon_3 v_1, && \xi_i = \ell_i + \varsigma\varepsilon_i,
\end{aligned}$$

$$\eta = \frac{\rho C_e}{T_0} + a\varsigma, \qquad i = 1, 2, 3.$$

3. Steady oscillations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space $\mathbb{E}^3$, $|\mathbf{x}| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$, $\mathbf{D}_\mathbf{x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$.

The displacement vector, volume fraction fields, temperature change, and chemical potential functions are assumed as:

$$[\mathbf{u}(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x},t), \theta(\mathbf{x},t), P(\mathbf{x},t)] = \mathrm{Re}\left[(\mathbf{u}^*, \boldsymbol{\phi}^*, \theta^*, P^*)e^{-\iota\omega t}\right], \tag{3.1}$$

where ω is the oscillation frequency.

Using Eq. (3.1) in the system of Eqs. (2.34) and omitting asterisk (*) for simplicity, the system of equations of steady oscillations is obtained as

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + [(\lambda' + \mu)\nabla\operatorname{div} + \rho\omega^2]\mathbf{u} + \sigma_i\nabla\phi_i - \zeta_1\nabla\theta - \zeta_2\nabla P = \mathbf{0},\\
&-\sigma_1\operatorname{div}\mathbf{u} + (A_1\Delta - \gamma_1)\phi_1 + (A_4\Delta - \beta_4)\phi_2 + (A_6\Delta - \beta_6)\phi_3 + \xi_1\theta + v_1 P = 0,\\
&-\sigma_2\operatorname{div}\mathbf{u} + (A_4\Delta - \beta_4)\phi_1 + (A_2\Delta - \gamma_2)\phi_2 + (A_5\Delta - \beta_5)\phi_3 + \xi_2\theta + v_2 P = 0,\\
&-\sigma_3\operatorname{div}\mathbf{u} + (A_6\Delta - \beta_6)\phi_1 + (A_5\Delta - \beta_5)\phi_2 + (A_3\Delta - \gamma_3)\phi_3 + \xi_3\theta + v_3 P = 0,\\
&\tau_1 T_0[\zeta_1\operatorname{div}\mathbf{u} + \xi_i\phi_i] + [K\Delta + \tau_1\eta T_0]\theta + \tau_1\varsigma T_0 P = 0,\\
&\tau^1[\zeta_2\operatorname{div}\mathbf{u} + v_i\phi_i + \varsigma\theta] + [D\Delta + \tau^1\varpi]P = 0,
\end{aligned}
\tag{3.2}
$$

where

$$
\gamma_i = \beta_i - \rho\kappa_i\omega^2, \qquad \tau_1 = \iota\omega(1 - \iota\omega\tau_0), \qquad \tau^1 = \iota\omega(1 - \iota\omega\tau^0).
$$

If we replace ω by $-\iota\tau$, where τ is a complex number and $\operatorname{Re}(\tau) > 0$ in Eqs. (3.2), we obtain the system of equations of pseudo-oscillations as

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + [(\lambda' + \mu)\nabla\operatorname{div} - \rho\tau^2]\mathbf{u} + \sigma_i\nabla\phi_i - \zeta_1\nabla\theta - \zeta_2\nabla P = \mathbf{0},\\
&-\sigma_1\operatorname{div}\mathbf{u} + (A_1\Delta - \widetilde{\gamma}_1)\phi_1 + (A_4\Delta - \beta_4)\phi_2 + (A_6\Delta - \beta_6)\phi_3 + \xi_1\theta + v_1 P = 0,\\
&-\sigma_2\operatorname{div}\mathbf{u} + (A_4\Delta - \beta_4)\phi_1 + (A_2\Delta - \widetilde{\gamma}_2)\phi_2 + (A_5\Delta - \beta_5)\phi_3 + \xi_2\theta + v_2 P = 0,\\
&-\sigma_3\operatorname{div}\mathbf{u} + (A_6\Delta - \beta_6)\phi_1 + (A_5\Delta - \beta_5)\phi_2 + (A_3\Delta - \widetilde{\gamma}_3)\phi_3 + \xi_3\theta + v_3 P = 0,\\
&\widetilde{\tau}_1 T_0[\zeta_1\operatorname{div}\mathbf{u} + \xi_i\phi_i] + [K\Delta + \widetilde{\tau}_1\eta T_0]\theta + \widetilde{\tau}_1\varsigma T_0 P = 0,\\
&\widetilde{\tau}^1[\zeta_2\operatorname{div}\mathbf{u} + v_i\phi_i + \varsigma\theta] + [D\Delta + \widetilde{\tau}^1\varpi]P = 0,
\end{aligned}
\tag{3.3}
$$

where

$$
\widetilde{\gamma}_i = \beta_i + \rho\kappa_i\tau^2, \qquad \widetilde{\tau}_1 = \tau(1 - \tau\tau_0), \qquad \widetilde{\tau}^1 = \tau(1 - \tau\tau^0).
$$

If we put $\rho = 0$, i.e., neglecting inertial effect in Eqs. (3.2), we obtain the system of equations of quasi-static oscillations as

$$
(3.4)\quad
\begin{aligned}
&\big[\mu\Delta + (\lambda' + \mu)\nabla\,\mathrm{div}\big]\,\mathbf{u} + \sigma_i\nabla\phi_i - \zeta_1\nabla\theta - \zeta_2\nabla P = \mathbf{0},\\
&-\sigma_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \beta_1)\phi_1 + (A_4\,\Delta - \beta_4)\phi_2 + (A_6\,\Delta - \beta_6)\phi_3\\
&\qquad + \xi_1\theta + v_1 P = 0,\\
&-\sigma_2\,\mathrm{div}\,\mathbf{u} + (A_4\,\Delta - \beta_4)\phi_1 + (A_2\,\Delta - \beta_2)\phi_2 + (A_5\,\Delta - \beta_5)\phi_3\\
&\qquad + \xi_2\theta + v_2 P = 0,\\
&-\sigma_3\,\mathrm{div}\,\mathbf{u} + (A_6\,\Delta - \beta_6)\phi_1 + (A_5\,\Delta - \beta_5)\phi_2 + (A_3\,\Delta - \beta_3)\phi_3\\
&\qquad + \xi_3\theta + v_3 P = 0,\\
&\tau_1 T_0[\zeta_1\,\mathrm{div}\,\mathbf{u} + \xi_i\phi_i] + [K\Delta + \tau_1 a\varsigma T_0]\theta + \tau_1\varsigma T_0 P = 0,\\
&\tau^1[\zeta_2\,\mathrm{div}\,\mathbf{u} + v_i\phi_i + \varsigma\theta] + [D\Delta + \tau^1\varpi]P = 0.
\end{aligned}
$$

If we place $\omega = 0$ in Eqs. (3.2), we obtain the system of equations in the equilibrium theory of thermoelastic diffusion with a triple porosity as

$$
(3.5)\quad
\begin{aligned}
&\big[\mu\Delta + (\lambda' + \mu)\nabla\,\mathrm{div}\big]\,\mathbf{u} + \sigma_i\nabla\phi_i - \zeta_1\nabla\theta - \zeta_2\nabla P = \mathbf{0},\\
&-\sigma_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \beta_1)\phi_1 + (A_4\,\Delta - \beta_4)\phi_2 + (A_6\,\Delta - \beta_6)\phi_3\\
&\qquad + \xi_1\theta + v_1 P = 0,\\
&-\sigma_2\,\mathrm{div}\,\mathbf{u} + (A_4\,\Delta - \beta_4)\phi_1 + (A_2\,\Delta - \beta_2)\phi_2 + (A_5\,\Delta - \beta_5)\phi_3\\
&\qquad + \xi_2\theta + v_2 P = 0,\\
&-\sigma_3\,\mathrm{div}\,\mathbf{u} + (A_6\,\Delta - \beta_6)\phi_1 + (A_5\,\Delta - \beta_5)\phi_2 + (A_3\,\Delta - \beta_3)\phi_3\\
&\qquad + \xi_3\theta + v_3 P = 0,\\
&K\Delta\theta = 0,\\
&D\Delta P = 0.
\end{aligned}
$$

We introduce the second-order matrix differential operators with constant coefficients

$$
\mathbf{F}^{(i)}(\mathbf{D_x}) = \Big(F^{(i)}_{gh}(\mathbf{D_x})\Big)_{8\times 8},
$$

where

$$
F^{(1)}_{pq}(\mathbf{D_x}) = [\mu\Delta + \rho\omega^2]\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p \partial x_q},
$$

$$
F^{(1)}_{p;q+3}(\mathbf{D_x}) = -F^{(1)}_{q+3;p}(\mathbf{D_x}) = \sigma_q\frac{\partial}{\partial x_p},
$$

$$
F^{(1)}_{p7}(\mathbf{D_x}) = -\zeta_1\frac{\partial}{\partial x_p}, \qquad F^{(1)}_{p8}(\mathbf{D_x}) = -\zeta_2\frac{\partial}{\partial x_p},
$$

$$F^{(1)}_{p+3;p+3}(\mathbf{D_x}) = A_p\Delta - \gamma_p,$$

$$F^{(1)}_{45}(\mathbf{D_x}) = F^{(1)}_{54}(\mathbf{D_x}) = A_4\Delta - \beta_4,$$

$$F^{(1)}_{46}(\mathbf{D_x}) = F^{(1)}_{64}(\mathbf{D_x}) = A_6\Delta - \beta_6,$$

$$F^{(1)}_{56}(\mathbf{D_x}) = F^{(1)}_{65}(\mathbf{D_x}) = A_5\Delta - \beta_5,$$

$$F^{(1)}_{p+3;7}(\mathbf{D_x}) = \xi_p, \qquad F^{(1)}_{p+3;8}(\mathbf{D_x}) = \upsilon_p,$$

$$F^{(1)}_{7p}(\mathbf{D_x}) = \tau_1\zeta_1 T_0\frac{\partial}{\partial x_p},$$

$$F^{(1)}_{7;p+3}(\mathbf{D_x}) = \tau_1\xi_p T_0,$$

$$F^{(1)}_{77}(\mathbf{D_x}) = K\Delta + \tau_1\eta T_0,$$

$$F^{(1)}_{78}(\mathbf{D_x}) = \tau_1\varsigma T_0,$$

$$F^{(1)}_{8p}(\mathbf{D_x}) = \tau^1\zeta_2\frac{\partial}{\partial x_p},$$

$$F^{(1)}_{8;p+3}(\mathbf{D_x}) = \tau^1\upsilon_p, \qquad F^{(1)}_{87}(\mathbf{D_x}) = \tau^1\varsigma,$$

$$F^{(1)}_{88}(\mathbf{D_x}) = D\Delta + \tau^1\varpi, \qquad p,q = 1,2,3.$$

Here $i = 1,2,3,4$ corresponds to static, pseudo-, quasi-static oscillations, and equilibrium theory of thermoelastic diffusion with a triple porosity, respectively. The matrices $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 2,3,4$, can be obtained from matrix $\mathbf{F}^{(1)}(\mathbf{D_x})$ by taking $\omega = -\iota\tau$, $\rho = 0$, and $\omega = 0$, respectively,
and

$$\widetilde{\mathbf{F}}(\mathbf{D_x}) = \left(\widetilde{F}_{gh}(\mathbf{D_x})\right)_{8\times 8},$$

where

$$\widetilde{F}_{pq}(\mathbf{D_x}) = \mu\Delta\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p\partial x_q},$$

$$\widetilde{F}_{p+3;p+3}(\mathbf{D_x}) = A_p\Delta,$$

$$\widetilde{F}_{45}(\mathbf{D_x}) = \widetilde{F}_{54}(\mathbf{D_x}) = A_4\Delta,$$

$$\widetilde{F}_{46}(\mathbf{D_x}) = \widetilde{F}_{64}(\mathbf{D_x}) = A_6\Delta,$$

$$\widetilde{F}_{56}(\mathbf{D_x}) = \widetilde{F}_{65}(\mathbf{D_x}) = A_5\Delta,$$

$$\widetilde{F}_{77}(\mathbf{D_x}) = K\Delta, \qquad \widetilde{F}_{88}(\mathbf{D_x}) = D\Delta,$$

$$\widetilde{F}_{p;q+3}(\mathbf{D_x}) = \widetilde{F}_{p+3;q}(\mathbf{D_x}) = 0,$$

$$\widetilde{F}_{kl}(\mathbf{D_x}) = \widetilde{F}_{lk}(\mathbf{D_x}) = \widetilde{F}_{78}(\mathbf{D_x}) = \widetilde{F}_{87}(\mathbf{D_x}) = 0,$$
$$p,q = 1,2,3, \quad l = 1,...,6, \quad k = 7,8.$$

The system of Eqs. (3.2)–(3.5) can be represented as

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \qquad i = 1,2,3,4,$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\phi}, \theta, P)$ is a eight-component vector function on $\mathbb{E}^3$. The matrix $\widetilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of operator $\mathbf{F}^{(i)}(\mathbf{D_x})$.

Definition 1. The operator $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 1,2,3,4$, is said to be elliptic if $|\widetilde{\mathbf{F}}(\mathbf{m})| \neq 0$, where $\mathbf{m} = (m_1, m_2, m_3)$.

Since $|\widetilde{\mathbf{F}}(\mathbf{m})| = \mu^2\widetilde{\lambda}KD\varrho|\mathbf{m}|^{16}$, $\widetilde{\lambda} = \lambda' + 2\mu$, $\varrho = \begin{vmatrix} A_1 & A_4 & A_6 \\ A_4 & A_2 & A_5 \\ A_6 & A_5 & A_3 \end{vmatrix}$, therefore operator $\mathbf{F}^{(i)}(\mathbf{D_x})$ is an elliptic differential operator iff

$$\mu\widetilde{\lambda}KD\varrho \neq 0. \tag{3.6}$$

Definition 2. The fundamental solutions of the system of Eqs. (2.36)–(2.39) (fundamental matrices of operators $\mathbf{F}^{(i)}$) are the matrices $\mathbf{G}^{(i)}(\mathbf{x}) = \big(G^{(i)}_{gh}(\mathbf{x})\big)_{8\times 8}$ satisfying conditions

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{G}^{(i)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}), \qquad i = 1,2,3,4, \tag{3.7}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I} = (\delta_{gh})_{8\times 8}$ is the unit matrix, and $\mathbf{x} \in \mathbb{E}^3$.

4. Construction of $\mathbf{G}^{(1)}(\mathbf{x})$ in terms of elementary functions

Let us consider the system of non-homogeneous equations

$$\mu\Delta\mathbf{u} + [(\lambda' + \mu)\nabla\,\mathrm{div} + \rho\omega^2]\mathbf{u} - \sigma_i\nabla\phi_i + \tau_1\zeta_1 T_0\nabla\theta + \tau^1\zeta_2\nabla P = \mathbf{H}, \tag{4.1}$$

$$\begin{aligned}\sigma_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \gamma_1)\phi_1 + (A_4\,\Delta - \beta_4)\phi_2 \\ + (A_6\,\Delta - \beta_6)\phi_3 + \tau_1 T_0\xi_1\theta + \tau^1 v_1 P = X_1,\end{aligned} \tag{4.2}$$

$$\sigma_2 \operatorname{div}\mathbf{u} + (A_4\,\Delta - \beta_4)\phi_1 + (A_2\,\Delta - \gamma_2)\phi_2 + (A_5\,\Delta - \beta_5)\phi_3 + \tau_1 T_0 \xi_2 \theta + \tau^1 v_2 P = X_2, \tag{4.3}$$

$$\sigma_3 \operatorname{div}\mathbf{u} + (A_6\,\Delta - \beta_6)\phi_1 + (A_5\,\Delta - \beta_5)\phi_2 + (A_3\,\Delta - \gamma_3)\phi_3 + \tau_1 T_0 \xi_3 \theta + \tau^1 v_3 P = X_3, \tag{4.4}$$

$$-\zeta_1 \operatorname{div}\mathbf{u} + \xi_i \phi_i + [K\Delta + \tau_1 \eta T_0]\theta + \tau^1 \varsigma P = Y, \tag{4.5}$$

$$-\zeta_2 \operatorname{div}\mathbf{u} + v_i \phi_i + \tau_1 \varsigma T_0 \theta + [D\Delta + \tau^1 \varpi]P = Z, \tag{4.6}$$

where $\mathbf{H}$ is a three-component vector function on $\mathbb{E}^3$, and X_i, Y, and Z are scalar functions on $\mathbb{E}^3$.

The system of Eqs. (4.1)–(4.6) may also be written in the form

$$\mathbf{F}^{(1)^{\mathrm{T}}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{4.7}$$

where $\mathbf{F}^{(1)^{\mathrm{T}}}$ is the transpose of matrix $\mathbf{F}^{(1)}$, $\mathbf{Q} = (\mathbf{H}, X_i, Y, Z)$, and $\mathbf{x} \in \mathbb{E}^3$.

After applying the operator div to Eq. (4.1), we obtain

$$[\widetilde{\lambda}\,\Delta + \rho\omega^2] \operatorname{div}\mathbf{u} - \sigma_i \Delta\phi_i + \tau_1 \zeta_1 T_0 \Delta\theta + \tau^1 \zeta_2 \Delta P = \operatorname{div}\mathbf{H}. \tag{4.8}$$

Equations (4.2)–(4.6) and (4.8) may be expressed in the form

$$\mathbf{N}^{(1)}(\Delta)\mathbf{S} = \widetilde{\mathbf{Q}}, \tag{4.9}$$

where $\mathbf{S} = (\operatorname{div}\mathbf{u}, \boldsymbol{\phi}, \theta, P)$, $\widetilde{\mathbf{Q}} = (\varphi_1, ..., \varphi_6) = (\operatorname{div}\mathbf{H}, X_i, Y, Z)$, and

$$\mathbf{N}^{(1)}(\Delta) = \left(N^{(1)}_{gh}(\Delta)\right)_{6\times 6}
= \begin{pmatrix}
\widetilde{\lambda}\Delta + \rho\omega^2 & -\sigma_1\Delta & -\sigma_2\Delta & -\sigma_3\Delta & \tau_1\zeta_1 T_0\Delta & \tau^1\zeta_2\Delta \\
\sigma_1 & A_1\Delta - \gamma_1 & A_4\Delta - \beta_4 & A_6\Delta - \beta_6 & \tau_1\xi_1 T_0 & \tau^1 v_1 \\
\sigma_2 & A_4\Delta - \beta_4 & A_2\Delta - \gamma_2 & A_5\Delta - \beta_5 & \tau_1\xi_2 T_0 & \tau^1 v_2 \\
\sigma_3 & A_6\Delta - \beta_6 & A_5\Delta - \beta_5 & A_3\Delta - \gamma_3 & \tau_1\xi_3 T_0 & \tau^1 v_3 \\
-\zeta_1 & \xi_1 & \xi_2 & \xi_3 & K\Delta + \tau_1\eta T_0 & \tau^1\varsigma \\
-\zeta_2 & v_1 & v_2 & v_3 & \tau_1\varsigma T_0 & D\Delta + \tau^1\varpi
\end{pmatrix}_{6\times 6}.$$

Equations (4.9) may also be written in determinant form as

$$\Gamma^{(1)}(\Delta)\mathbf{S} = \boldsymbol{\Psi}, \tag{4.10}$$

where

$$\mathbf{\Psi} = (\Psi_1, ..., \Psi_6), \qquad \Psi_p = \frac{1}{\widetilde{A}} \sum_{i=1}^{6} \widetilde{N}_{ip}^{(1)}(\Delta)\varphi_i,$$

$$\Gamma^{(1)}(\Delta) = \frac{1}{\widetilde{A}} |\mathbf{N}^{(1)}(\Delta)|, \qquad \widetilde{A} = \widetilde{\lambda} K D \varrho, \qquad p = 1, ..., 6,$$

and $\widetilde{N}_{ip}^{(1)}$ is the cofactor of the element $N_{ip}^{(1)}$ of the matrix $\mathbf{N}^{(1)}$.

On expanding $\Gamma^{(1)}(\Delta)$, we see that

$$\Gamma^{(1)}(\Delta) = \prod_{i=1}^{6} (\Delta + \lambda_i^2),$$

where λ_i^2, $i = 1, ..., 6$, are the roots of the equation $\Gamma^{(1)}(-m) = 0$ (with respect to m).

Applying operator $\Gamma^{(1)}(\Delta)$ to the Eq. (4.1) and using Eq. (4.10), we obtain

$$\Gamma^{(1)}(\Delta)(\Delta + \lambda_7^2)\mathbf{u} = \mathbf{\Psi}', \qquad \lambda_7^2 = \frac{\rho\omega^2}{\mu}, \tag{4.11}$$

where

$$\mathbf{\Psi}' = \frac{1}{\mu} \Big[\Gamma^{(1)}(\Delta)\mathbf{H} - \nabla \big[(\lambda' + \mu)\Psi_1 - \sigma_i \Psi_{i+1} + \tau_1 \zeta_1 T_0 \Psi_5 + \tau^1 \zeta_2 \Psi_6 \big] \Big].$$

From Eqs. (4.10) and (4.11), we obtain

$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{U}(\mathbf{x}) = \widehat{\mathbf{\Psi}}(\mathbf{x}), \tag{4.12}$$

where $\widehat{\mathbf{\Psi}} = (\mathbf{\Psi}', \Psi_2, ..., \Psi_6)$ and

$$\mathbf{\Theta}^{(1)}(\Delta) = \Big(\Theta_{gh}^{(1)}(\Delta) \Big)_{8\times 8},$$

$$\Theta_{pp}^{(1)}(\Delta) = \Gamma^{(1)}(\Delta)(\Delta + \lambda_7^2) = \prod_{i=1}^{7} (\Delta + \lambda_i^2),$$

$$\Theta_{ll}^{(1)}(\Delta) = \Gamma^{(1)}(\Delta) = \prod_{i=1}^{6} (\Delta + \lambda_i^2), \qquad \Theta_{gh}^{(1)}(\Delta) = 0,$$

$$p = 1, 2, 3, \qquad g, h = 1, ..., 8, \qquad l = 4, ..., 8, \qquad g \neq h.$$

The expressions for $\mathbf{\Psi}'$ and Ψ_p, $p = 2, ..., 6$, can be rewritten in the form

$$\begin{aligned}\mathbf{\Psi}' &= \left[\frac{1}{\mu}\Gamma^{(1)}(\Delta)\mathbf{J} + w_{11}^{(1)}(\Delta)\,\nabla\,\mathrm{div}\right]\mathbf{H} + \sum_{i=2}^{6} w_{i1}^{(1)}(\Delta)\nabla\varphi_i,\\ \Psi_l &= w_{1l}^{(1)}(\Delta)\,\mathrm{div}\,\mathbf{H} + \sum_{i=2}^{6} w_{il}^{(1)}(\Delta)\varphi_i, \quad l = 2, ..., 6,\end{aligned} \tag{4.13}$$

where $\boldsymbol{J} = (\delta_{ij})_{3\times 3}$ and

$$w_{p1}^{(1)}(\Delta) = -\frac{1}{\widetilde{A}\mu}\left[(\lambda'+\mu)\widetilde{N}_{p1}^{(1)}(\Delta) - \sigma_k \widetilde{N}_{p;k+1}^{(1)}(\Delta) + \tau_1\zeta_1 T_0 \widetilde{N}_{p5}^{(1)}(\Delta) + \tau^1\zeta_2 \widetilde{N}_{p6}^{(1)}(\Delta)\right],$$

$$w_{pl}^{(1)}(\Delta) = \frac{\widetilde{N}_{pl}^{(1)}(\Delta)}{\widetilde{A}}, \quad p = 1, ..., 6, \quad l = 2, ..., 6.$$

From Eqs. (4.13), we have

$$\widehat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{(1)^{tr}}(\mathbf{D_x})\mathbf{Q}(\mathbf{x}), \tag{4.14}$$

where

$$\mathbf{R}^{(1)}(\mathbf{D_x}) = \left(R_{gh}^{(1)}(\mathbf{D_x})\right)_{8\times 8},$$

$$R_{ij}^{(1)}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{ij} + w_{11}^{(1)}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R_{i;p+2}^{(1)}(\mathbf{D_x}) = w_{1p}^{(1)}(\Delta)\frac{\partial}{\partial x_i}, \qquad R_{p+2;i}^{(1)}(\mathbf{D_x}) = w_{p1}^{(1)}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{p+2;l+2}^{(1)}(\mathbf{D_x}) = w_{pl}^{(1)}(\Delta), \qquad i, j = 1, 2, 3, \qquad p, l = 2, ..., 6.$$

From Eqs. (4.7), (4.12), and (4.14), we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}^{(1)}(\mathbf{D_x}) = \mathbf{\Theta}^{(1)}(\Delta). \tag{4.15}$$

We assume that

$$\lambda_p^2 \neq \lambda_l^2 \neq 0, \qquad p, l = 1, ..., 7, \qquad p \neq l.$$

Let

$$\mathbf{Y}^{(1)}(\mathbf{x})=\left(Y_{ij}^{(1)}(\mathbf{x})\right)_{8\times 8},\qquad Y_{pp}^{(1)}(\mathbf{x})=\sum_{g=1}^{7} r_{1g}^{(1)}\varsigma_g(\mathbf{x}),$$

$$Y_{ll}^{(1)}(\mathbf{x})=\sum_{g=1}^{6} r_{2g}^{(1)}\varsigma_g(\mathbf{x}),\qquad Y_{ij}^{(1)}(\mathbf{x})=0,$$

$$p=1,2,3,\qquad l=4,...,8,\qquad i,j=1,...,8,\qquad i\neq j,$$

where

$$\varsigma_g(\mathbf{x})=-\frac{e^{\iota\lambda_g|\mathbf{x}|}}{4\pi|\mathbf{x}|},$$

(4.16)
$$r_{1g}^{(1)}=\prod_{i=1,i\neq g}^{7}(\lambda_i^2-\lambda_g^2)^{-1},\qquad r_{2h}^{(1)}=\prod_{i=1,i\neq h}^{6}(\lambda_i^2-\lambda_h^2)^{-1},$$
$$g=1,...,7,\qquad h=1,...,6.$$

Lemma 1. The matrix $\mathbf{Y}^{(1)}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}^{(1)}(\Delta)$, i.e.,

(4.17)
$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x})=\delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Proof. To prove the lemma, it is sufficient to prove that

(4.18)
$$\mathit{\Gamma}^{(1)}(\Delta)(\Delta+\lambda_7^2)Y_{11}^{(1)}(\mathbf{x})\;=\;\delta(\mathbf{x}),$$

(4.19)
$$\mathit{\Gamma}^{(1)}(\Delta)Y_{44}^{(1)}(\mathbf{x})\;=\;\delta(\mathbf{x}).$$

Consider

$$\sum_{i=1}^{7} r_{1i}^{(1)}=\frac{\sum\limits_{j=1}^{7}(-1)^{j+1}z_j}{z_8},$$

where

$$z_1=\prod_{i=3}^{7}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{7}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{7}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_5^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_2=\prod_{i=3}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{7}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{7}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_5^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_3 = \prod_{i=2, i\neq 3}^{7} (\lambda_1^2 - \lambda_i^2) \prod_{j=4}^{7} (\lambda_2^2 - \lambda_j^2) \prod_{l=5}^{7} (\lambda_4^2 - \lambda_l^2) \prod_{p=6}^{7} (\lambda_5^2 - \lambda_p^2)(\lambda_6^2 - \lambda_7^2),$$

$$z_4 = \prod_{i=2, i\neq 4}^{7} (\lambda_1^2 - \lambda_i^2) \prod_{j=3, j\neq 4}^{7} (\lambda_2^2 - \lambda_j^2) \prod_{l=5}^{7} (\lambda_3^2 - \lambda_l^2) \prod_{p=6}^{7} (\lambda_5^2 - \lambda_p^2)(\lambda_6^2 - \lambda_7^2),$$

$$z_5 = \prod_{i=2, i\neq 5}^{7} (\lambda_1^2 - \lambda_i^2) \prod_{j=3, j\neq 5}^{7} (\lambda_2^2 - \lambda_j^2) \prod_{l=4, l\neq 5}^{7} (\lambda_3^2 - \lambda_l^2) \prod_{p=6}^{7} (\lambda_4^2 - \lambda_p^2)(\lambda_6^2 - \lambda_7^2),$$

$$z_6 = \prod_{i=2, i\neq 6}^{7} (\lambda_1^2 - \lambda_i^2) \prod_{j=3, j\neq 6}^{7} (\lambda_2^2 - \lambda_j^2) \prod_{l=4, l\neq 6}^{7} (\lambda_3^2 - \lambda_l^2) \prod_{p=5, p\neq 6}^{7} (\lambda_4^2 - \lambda_p^2)(\lambda_5^2 - \lambda_7^2),$$

$$z_7 = \prod_{i=2}^{6} (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^{6} (\lambda_2^2 - \lambda_j^2) \prod_{l=4}^{6} (\lambda_3^2 - \lambda_l^2) \prod_{p=5}^{6} (\lambda_4^2 - \lambda_p^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_8 = \prod_{i=2}^{7} (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^{7} (\lambda_2^2 - \lambda_j^2) \prod_{l=4}^{7} (\lambda_3^2 - \lambda_l^2) \prod_{p=5}^{7} (\lambda_4^2 - \lambda_p^2) \prod_{k=6}^{7} (\lambda_5^2 - \lambda_k^2)(\lambda_6^2 - \lambda_7^2).$$

On simplifying the right-hand side of the above relation, we obtain

$$\sum_{i=1}^{7} r_{1i}^{(1)} = 0. \tag{4.20}$$

Similarly, we find that

$$\begin{aligned} &\sum_{i=2}^{7} r_{1i}^{(1)} (\lambda_1^2 - \lambda_i^2) = 0, && \sum_{i=3}^{7} r_{1i}^{(1)} \Big[\prod_{j=1}^{2} (\lambda_j^2 - \lambda_i^2) \Big] = 0, \\ &\sum_{i=4}^{7} r_{1i}^{(1)} \Big[\prod_{j=1}^{3} (\lambda_j^2 - \lambda_i^2) \Big] = 0, && \sum_{i=5}^{7} r_{1i}^{(1)} \Big[\prod_{j=1}^{4} (\lambda_j^2 - \lambda_i^2) \Big] = 0, \\ &\sum_{i=6}^{7} r_{1i}^{(1)} \Big[\prod_{j=1}^{5} (\lambda_j^2 - \lambda_i^2) \Big] = 0, && \prod_{j=1}^{6} r_{17}^{(1)} (\lambda_j^2 - \lambda_7^2) = 1. \end{aligned} \tag{4.21}$$

Also,

$$(\Delta + \lambda_p^2) \varsigma_g(\mathbf{x}) = \delta(\mathbf{x}) + (\lambda_p^2 - \lambda_g^2) \varsigma_g(\mathbf{x}), \qquad p, g = 1, ..., 7. \tag{4.22}$$

Now consider

$$\begin{aligned}
\Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2)Y_{11}^{(1)}(\mathbf{x}) &= \prod_{i=1}^{7}(\Delta+\lambda_i^2)\sum_{g=1}^{7} r_{1g}^{(1)}\varsigma_g(\mathbf{x}) \\
&= \prod_{i=2}^{7}(\Delta+\lambda_i^2)\sum_{g=1}^{7} r_{1g}^{(1)}\Big[\delta(\mathbf{x})+(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big] \\
&= \prod_{i=2}^{7}(\Delta+\lambda_i^2)\Big[\delta(\mathbf{x})\sum_{g=1}^{7} r_{1g}^{(1)}+\sum_{g=2}^{7} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big].
\end{aligned}$$

Using Eqs. (4.20)–(4.22) in the above relation, we obtain

$$\begin{aligned}
\Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2)Y_{11}^{(1)}(\mathbf{x}) &= \prod_{i=2}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=2}^{7} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big] \\
&= \prod_{i=3}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=2}^{7} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\Big[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] \\
&= \prod_{i=3}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] \\
&= \prod_{i=4}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] \\
&= \prod_{i=4}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] \\
&= \prod_{i=5}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] \\
&= \prod_{i=5}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] \\
&= \prod_{i=6}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] \\
&= \prod_{i=6}^{7}(\Delta+\lambda_i^2)\Big[\sum_{g=6}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] \\
&= (\Delta+\lambda_7^2)\Big[\sum_{g=6}^{7} r_{1g}^{(1)}\Big[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_6^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] \\
&= (\Delta+\lambda_7^2)\varsigma_7(\mathbf{x}) = \delta(\mathbf{x}).
\end{aligned}$$

Equation (4.19) can be proved in the similar way.

We introduce the matrix

$$\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{R}^{(1)}(\mathbf{D_x})\mathbf{Y}^{(1)}(\mathbf{x}). \tag{4.23}$$

From Eqs. (4.15), (4.17), and (4.23), we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}^{(1)}(\mathbf{D_x})\mathbf{Y}^{(1)}(\mathbf{x}) = \boldsymbol{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution to Eq. (3.7) for $i = 1$.

Theorem 1. If the condition (3.6) is satisfied, then the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ defined by Eq. (4.23) is the fundamental solution of the system of Eqs. (3.2) and the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is represented in the following form:

$$\mathbf{G}^{(1)}(\mathbf{x}) = \left(G^{(1)}_{pk}(\mathbf{x})\right)_{8\times 8},$$

$$\mathbf{G}^{(1)}_{gh}(\mathbf{x}) = R^{(1)}_{gh}(\mathbf{D_x})Y^{(1)}_{11}(\mathbf{x}), \qquad \mathbf{G}^{(1)}_{gl}(\mathbf{x}) = R^{(1)}_{gl}(\mathbf{D_x})Y^{(1)}_{44}(\mathbf{x}),$$

$$g = 1, ..., 8, \qquad h = 1, 2, 3, \qquad l = 4, ..., 8.$$

5. Construction of matrices $\mathbf{G}^{(i)}(\mathbf{x})$, $i = 2, 3, 4$

5.1. Pseudo-oscillations

We introduce the matrix

$$\mathbf{G}^{(2)}(\mathbf{x}) = \mathbf{R}^{(2)}(\mathbf{D_x})\mathbf{Y}^{(2)}(\mathbf{x}), \tag{5.1}$$

where, the matrices $\mathbf{R}^{(2)}(\mathbf{D_x})$ and $\mathbf{Y}^{(2)}(\mathbf{x})$ can be obtained from matrices $\mathbf{R}^{(1)}(\mathbf{D_x})$ and $\mathbf{Y}^{(1)}(\mathbf{x})$, respectively, by taking $\omega = -\iota\tau$ and repeating the above procedure after Eq. (3.7).

Theorem 2. If the condition (3.6) is satisfied, then the matrix $\mathbf{G}^{(2)}(\mathbf{x})$ defined by Eq. (5.1) is the fundamental solution of the system of Eqs. (3.3).

5.2. Quasi-static oscillations

In this case, the matrix $\mathbf{N}^{(3)}(\Delta)$, operator $\Gamma^{(3)}(\Delta)$ and matrix operators $\boldsymbol{\Theta}^{(3)}(\Delta)$, $\mathbf{R}^{(3)}(\mathbf{D_x})$, $\mathbf{Y}^{(3)}(\mathbf{x})$, and $\mathbf{G}^{(3)}(\mathbf{x})$ are obtained as

(i) $$\widehat{\mathbf{N}}^{(3)}(\Delta) = \left(\widehat{N}^{(3)}_{gh}(\Delta)\right)_{6\times 6}$$

$$= \begin{pmatrix} \widetilde{\lambda} & -\sigma_1 & -\sigma_2 & -\sigma_3 & \tau_1\zeta_1 T_0 & \tau^1\zeta_2 \\ \sigma_1 & A_1\Delta-\beta_1 & A_4\Delta-\beta_4 & A_6\Delta-\beta_6 & \tau_1\xi_1 T_0 & \tau^1 v_1 \\ \sigma_2 & A_4\Delta-\beta_4 & A_2\Delta-\beta_2 & A_5\Delta-\beta_5 & \tau_1\xi_2 T_0 & \tau^1 v_2 \\ \sigma_3 & A_6\Delta-\beta_6 & A_5\Delta-\beta_5 & A_3\Delta-\beta_3 & \tau_1\xi_3 T_0 & \tau^1 v_3 \\ -\zeta_1 & \xi_1 & \xi_2 & \xi_3 & K\Delta+\tau_1 a\varsigma T_0 & \tau^1\varsigma \\ -\zeta_2 & v_1 & v_2 & v_3 & \tau_1\varsigma T_0 & D\Delta+\tau^1\varpi \end{pmatrix}_{6\times 6},$$

$$\mathbf{N}^{(3)}(\Delta) = \left(N^{(3)}_{gh}(\Delta)\right)_{6\times 6} = \Delta\left(\widehat{N}^{(3)}_{gh}(\Delta)\right)_{6\times 6}.$$

(ii) $$\Gamma^{(3)}(\Delta) = \Delta\prod_{i=1}^{5}(\Delta+\mu_i^2),$$

where μ_i^2, $i=1,...,5$, are the roots of the equation $|\widehat{\mathbf{N}}^{(3)}(-m)|=0$ (with respect to m).

(iii) $$\mathbf{\Theta}^{(3)}(\Delta) = \left(\Theta^{(3)}_{gh}(\Delta)\right)_{8\times 8},$$

$$\Theta^{(3)}_{pp}(\Delta) = \Gamma^{(3)}(\Delta)\Delta = \Delta^2\prod_{i=1}^{5}(\Delta+\mu_i^2),$$

$$\Theta^{(3)}_{ll}(\Delta) = \Gamma^{(3)}(\Delta) = \Delta\prod_{i=1}^{5}(\Delta+\mu_i^2), \qquad \Theta^{(3)}_{gh}(\Delta)=0,$$

$$p=1,2,3, \qquad g,h=1,...,8, \qquad l=4,...,8, \qquad g\neq h.$$

(iv) $$w^{(3)}_{p1}(\Delta) = -\frac{1}{\widetilde{A}\mu}\Big[(\lambda'+\mu)\widetilde{N}^{(3)}_{p1}(\Delta)-\sigma_k\widetilde{N}^{(3)}_{p;k+1}(\Delta)+\tau_1\zeta_1 T_0\widetilde{N}^{(3)}_{p5}(\Delta)+\tau^1\zeta_2\widetilde{N}^{(3)}_{p6}(\Delta)\Big],$$

$$w^{(3)}_{pl}(\Delta) = \frac{\widetilde{N}^{(3)}_{pl}(\Delta)}{\widetilde{A}}, \qquad p=1,...,6, \qquad l=2,...,6,$$

where $\widetilde{N}^{(3)}_{ij}$ is the cofactor of the element $N^{(3)}_{ij}$ of the matrix $\mathbf{N}^{(3)}$.

(v) $$\mathbf{R}^{(3)}(\mathbf{D_x}) = \left(R^{(3)}_{gh}(\mathbf{D_x})\right)_{8\times 8},$$

$$R^{(3)}_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma^{(3)}(\Delta)\delta_{ij} + w^{(3)}_{11}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(3)}_{i;p+2}(\mathbf{D}_\mathbf{x}) = w^{(3)}_{1p}(\Delta)\frac{\partial}{\partial x_i}, \qquad R^{(3)}_{p+2;i}(\mathbf{D}_\mathbf{x}) = w^{(3)}_{p1}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(3)}_{p+2;l+2}(\mathbf{D}_\mathbf{x}) = w^{(3)}_{pl}(\Delta), \qquad i,j = 1,2,3, \qquad p,l = 2,...,6.$$

(vi)
$$\mathbf{Y}^{(3)}(\mathbf{x}) = \left(Y^{(3)}_{ij}(\mathbf{x})\right)_{8\times 8},$$

$$Y^{(3)}_{pp}(\mathbf{x}) = r^{(3)}_{11}\varsigma^*_1(\mathbf{x}) + r^{(3)}_{12}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{5} r^{(3)}_{1;g+2}\widetilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{ll}(\mathbf{x}) = r^{(3)}_{21}\varsigma^*_1(\mathbf{x}) + \sum_{g=1}^{5} r^{(3)}_{2;g+1}\widetilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{ij}(\mathbf{x}) = 0, \qquad p = 1,2,3, \qquad l = 4,...,8, \qquad i,j = 1,...,8, \qquad i \neq j,$$

where

$$\varsigma^*_1(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \qquad \varsigma^*_2(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \qquad \widetilde{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\mu_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \qquad g = 1,...,5,$$

$$r^{(3)}_{11} = -\sum_{p=1}^{5}\left(\prod_{j=1,j\neq p}^{5}\mu_j^2\right)\prod_{i=1}^{5}\mu_i^{-4},$$

$$r^{(3)}_{12} = r^{(3)}_{21} = \prod_{i=1}^{5}\mu_i^{-2}, \qquad r^{(3)}_{1;l+2} = \mu_l^{-4}\prod_{i=1,i\neq l}^{5}(\mu_i^2-\mu_l^2)^{-1},$$

$$r^{(3)}_{2;l+1} = -\mu_l^{-2}\prod_{i=1,i\neq l}^{5}(\mu_i^2-\mu_l^2)^{-1}, \qquad l = 1,...,5.$$

On introducing the matrix

$$\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{R}^{(3)}(\mathbf{D}_\mathbf{x})\mathbf{Y}^{(3)}(\mathbf{x}), \tag{5.2}$$

we obtain

$$\mathbf{F}^{(3)}(\mathbf{D}_\mathbf{x})\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{F}^{(3)}(\mathbf{D}_\mathbf{x})\mathbf{R}^{(3)}(\mathbf{D}_\mathbf{x})\mathbf{Y}^{(3)}(\mathbf{x}) = \mathbf{\Theta}^{(3)}(\Delta)\mathbf{Y}^{(3)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(3)}(\mathbf{x})$ is a fundamental solution to Eq. (3.7) for $i = 3$.

Theorem 3. If the condition (3.6) is satisfied, then the matrix $\mathbf{G}^{(3)}(\mathbf{x})$ defined by Eq. (5.2) is the fundamental solution of the system of Eqs. (3.4).

5.3. Equilibrium theory

In this case, the matrix $\mathbf{N}^{(4)}(\Delta)$, operator $\Gamma^{(4)}(\Delta)$ and matrix operators $\mathbf{\Theta}^{(4)}(\Delta)$, $\mathbf{R}^{(4)}(\mathbf{D_x})$, $\mathbf{Y}^{(4)}(\mathbf{x})$, and $\mathbf{G}^{(4)}(\mathbf{x})$ are obtained as

(i) $$\widehat{\mathbf{N}}^{(4)}(\Delta) = \left(\widehat{N}^{(4)}_{gh}(\Delta)\right)_{4\times 4} = \begin{pmatrix} \widetilde{\lambda} & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \sigma_1 & A_1\Delta-\beta_1 & A_4\Delta-\beta_4 & A_6\Delta-\beta_6 \\ \sigma_2 & A_4\Delta-\beta_4 & A_2\Delta-\beta_2 & A_5\Delta-\beta_5 \\ \sigma_3 & A_6\Delta-\beta_6 & A_5\Delta-\beta_5 & A_3\Delta-\beta_3 \end{pmatrix}_{4\times 4},$$

$$\mathbf{N}^{(4)}(\Delta) = \left(N^{(4)}_{gh}(\Delta)\right)_{4\times 4} = \Delta\left(\widehat{N}^{(4)}_{gh}(\Delta)\right)_{4\times 4}.$$

(ii) $$\Gamma^{(4)}(\Delta) = \Delta\prod_{i=1}^{3}(\Delta+\omega_i^2),$$

where ω_i^2, $i=1,2,3$, are the roots of the equation $|\widehat{\mathbf{N}}^{(4)}(-m)|=0$ (with respect to m).

(iii) $$\mathbf{\Theta}^{(4)}(\Delta) = \left(\Theta^{(4)}_{gh}(\Delta)\right)_{8\times 8},$$

$$\Theta^{(4)}_{pp}(\Delta) = \Gamma^{(4)}(\Delta)\Delta = \Delta^2\prod_{i=1}^{3}(\Delta+\omega_i^2),$$

$$\Theta^{(4)}_{ll}(\Delta) = \Gamma^{(4)}(\Delta) = \Delta\prod_{i=1}^{3}(\Delta+\omega_i^2),\, \Theta^{(4)}_{gh}(\Delta) = 0,$$

$$p = 1,2,3,7,8, \qquad g,h = 1,...,8, \qquad l = 4,5,6, \qquad g\neq h.$$

(iv) $$w^{(4)}_{p1}(\Delta) = -\frac{1}{\varrho\widetilde{\lambda}\mu}\left[(\lambda'+\mu)\widetilde{N}^{(4)}_{p1}(\Delta)-\sigma_k\widetilde{N}^{(4)}_{p;k+1}(\Delta)\right],$$

$$w^{(4)}_{pl}(\Delta) = \frac{\widetilde{N}^{(4)}_{pl}(\Delta)}{\varrho\widetilde{\lambda}}, \qquad w^{(4)}_{qp}(\Delta) = 0,$$

$$w^{(4)}_{p5}(\Delta) = \frac{1}{\varrho\widetilde{\lambda}K}\left[\zeta_1\widetilde{N}^{(4)}_{p1}(\Delta)-\xi_k\widetilde{N}^{(4)}_{p;k+1}(\Delta)\right],$$

$$w^{(4)}_{p6}(\Delta) = \frac{1}{\varrho\widetilde{\lambda}D}\left[\zeta_2\widetilde{N}^{(4)}_{p1}(\Delta)-v_k\widetilde{N}^{(4)}_{p;k+1}(\Delta)\right],$$

$$w_{55}^{(4)}(\Delta) = \Delta \prod_{i=1}^{3} (\Delta + \omega_i^2) K^{-1}, \qquad w_{66}^{(4)}(\Delta) = \Delta \prod_{i=1}^{3} (\Delta + \omega_i^2) D^{-1},$$

$$w_{56}^{(4)}(\Delta) = w_{65}^{(4)}(\Delta) = 0, \qquad p = 1,2,3,4, \qquad l = 2,3,4, \qquad q = 5,6,$$

where $\widetilde{N}_{ij}^{(4)}$ is the cofactor of the element $N_{ij}^{(4)}$ of the matrix $\mathbf{N}^{(4)}$.

(v)
$$\mathbf{R}^{(4)}(\mathbf{D_x}) = \left(R_{gh}^{(4)}(\mathbf{D_x}) \right)_{8\times 8},$$

$$R_{ij}^{(4)}(\mathbf{D_x}) = \frac{1}{\mu} \Gamma^{(4)}(\Delta)\delta_{ij} + w_{11}^{(4)}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j},$$

$$R_{i;p+2}^{(4)}(\mathbf{D_x}) = w_{1p}^{(4)}(\Delta) \frac{\partial}{\partial x_i}, \qquad R_{l+2;i}^{(4)}(\mathbf{D_x}) = w_{l1}^{(4)}(\Delta) \frac{\partial}{\partial x_i},$$

$$R_{l+2;p+2}^{(4)}(\mathbf{D_x}) = w_{lp}^{(4)}(\Delta), \qquad R_{ki}^{(4)}(\mathbf{D_x}) = R_{k;i+3}^{(4)}(\mathbf{D_x}) = 0,$$

$$R_{77}^{(4)}(\mathbf{D_x}) = w_{55}^{(4)}(\Delta), \qquad R_{78}^{(4)}(\mathbf{D_x}) = R_{87}^{(4)}(\mathbf{D_x}) = 0,$$

$$R_{88}^{(4)}(\mathbf{D_x}) = w_{66}^{(4)}(\Delta), \quad i,j = 1,2,3, \quad p = 2,...,6, \quad k = 7,8, \quad l = 2,3,4.$$

(vi)
$$\mathbf{Y}^{(4)}(\mathbf{x}) = \left(Y_{ij}^{(4)}(\mathbf{x}) \right)_{8\times 8},$$

$$Y_{pp}^{(4)}(\mathbf{x}) = r_{11}^{(4)} \varsigma_1^*(\mathbf{x}) + r_{12}^{(4)} \varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{3} r_{1;g+2}^{(4)} \widehat{\varsigma}_g(\mathbf{x}),$$

$$Y_{kk}^{(4)}(\mathbf{x}) = r_{21}^{(4)} \varsigma_1^*(\mathbf{x}) + \sum_{g=1}^{3} r_{2;g+1}^{(4)} \widehat{\varsigma}_g(\mathbf{x}),$$

$$Y_{ij}^{(4)}(\mathbf{x}) = 0, \qquad p = 1,2,3,7,8, \qquad k = 4,5,6, \qquad i,j = 1,...,8, \qquad i \neq j,$$

where

$$\widehat{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\omega_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \quad g = 1,2,3,$$

$$r_{11}^{(4)} = -\frac{\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2}{\omega_1^4\omega_2^4\omega_3^4}, \quad r_{12}^{(4)} = r_{21}^{(4)} = \prod_{i=1}^{3} \omega_i^{-2},$$

$$r_{1;l+2}^{(4)} = \omega_l^{-4} \prod_{i=1, i\neq l}^{3} (\omega_i^2 - \omega_l^2)^{-1}, \quad r_{2;l+1}^{(4)} = -\omega_l^{-2} \prod_{i=1, i\neq l}^{3} (\omega_i^2 - \omega_l^2)^{-1}, \quad l = 1,2,3.$$

If we introduce the matrix

(5.3)
$$\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}).$$

then, we obtain

$$\mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}) = \boldsymbol{\Theta}^{(4)}(\Delta)\mathbf{Y}^{(4)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(4)}(\mathbf{x})$ is a solution to Eq. (3.7) for $i = 4$.

Theorem 4. If the condition (3.6) is satisfied, then the matrix $\mathbf{G}^{(4)}(\mathbf{x})$ defined by Eq. (5.3) is the fundamental solution of the system of Eqs. (3.5).

6. Basic properties of $\mathbf{G}^{(1)}(\mathbf{x})$

Theorem 5. Each column of the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution of the system of Eqs. (3.2) at every point $\mathbf{x} \in \mathbb{E}^3$ except the origin.

Theorem 6. If the condition (3.6) is satisfied, then the fundamental solution of the system $\widetilde{\mathbf{F}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ is the matrix

$$\mathbf{W}(\mathbf{x}) = \Big(W_{gh}(\mathbf{x})\Big)_{8\times 8},$$

$$W_{pq}(\mathbf{x}) = \left[\frac{1}{\widetilde{\lambda}}\frac{\partial^2}{\partial x_p \partial x_q} - \frac{1}{\mu}\widetilde{R}_{pq}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{44}(\mathbf{x}) = \frac{A_2A_3 - A_5^2}{\varrho}\varsigma_1^*(\mathbf{x}), \qquad W_{45}(\mathbf{x}) = W_{54}(\mathbf{x}) = \frac{A_5A_6 - A_4A_3}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{46}(\mathbf{x}) = W_{64}(\mathbf{x}) = \frac{A_4A_5 - A_2A_6}{\varrho}\varsigma_1^*(\mathbf{x}), \qquad W_{55}(\mathbf{x}) = \frac{A_1A_3 - A_6^2}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{56}(\mathbf{x}) = W_{65}(\mathbf{x}) = \frac{A_4A_6 - A_1A_5}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{66}(\mathbf{x}) = \frac{A_1A_2 - A_4^2}{\varrho}\varsigma_1^*(\mathbf{x}), \qquad W_{77}(\mathbf{x}) = \frac{\varsigma_1^*(\mathbf{x})}{K}, \qquad W_{88}(\mathbf{x}) = \frac{\varsigma_1^*(\mathbf{x})}{D},$$

$$W_{p;q+3}(\mathbf{x}) = W_{p+3;q}(\mathbf{x}) = W_{lk}(\mathbf{x}) = W_{kl}(\mathbf{x}) = W_{78}(\mathbf{x}) = W_{87}(\mathbf{x}) = 0,$$

$$\widetilde{R}_{pq} = \frac{\partial^2}{\partial x_p \partial x_q} - \Delta\delta_{pq}, \qquad p,q = 1,2,3, \qquad k = 1,...,6, \qquad l = 7,8.$$

Lemma 2. If condition (3.6) is satisfied, then

$$\Delta w_{p1}^{(1)}(\Delta) = \frac{1}{\widetilde{A}}(\Delta + \lambda_7^2)\widetilde{N}_{p1}^{(1)}(\Delta) - \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{p1}, \qquad p = 1,...,6. \tag{6.1}$$

Proof. Consider

$$w_{p1}^{(1)}(\Delta) = -\frac{1}{\widetilde{A}\mu}\Big\{(\lambda' + \mu)\widetilde{N}_{p1}^{(1)}(\Delta) - \sigma_k \widetilde{N}_{p;k+1}^{(1)}(\Delta) + \tau_1\zeta_1 T_0 \widetilde{N}_{p5}^{(1)}(\Delta) + \tau^1\zeta_2\widetilde{N}_{p6}^{(1)}(\Delta)\Big\}.$$

Now

$$\Gamma^{(1)}(\Delta)\delta_{p1} = \frac{1}{\widetilde{A}}\,|\mathbf{N}^{(1)}(\Delta)|\delta_{p1} = \frac{1}{\widetilde{A}}\Big\{[\widetilde{\lambda}\Delta + \rho\omega^2]\widetilde{N}_{p1}^{(1)}(\Delta) - \sigma_k\Delta\widetilde{N}_{p;k+1}^{(1)}(\Delta) + \tau_1\zeta_1 T_0\Delta\widetilde{N}_{p5}^{(1)}(\Delta) + \tau^1\zeta_2\Delta\widetilde{N}_{p6}^{(1)}(\Delta)\Big\}.$$

Therefore,

$$\begin{aligned}\Delta w_{p1}^{(1)}(\Delta) &= -\frac{1}{\widetilde{A}\mu}\Big\{(\lambda' + \mu)\Delta\widetilde{N}_{p1}^{(1)}(\Delta) - \sigma_k\Delta\widetilde{N}_{p;k+1}^{(1)}(\Delta) + \tau_1\zeta_1 T_0\Delta\widetilde{N}_{p5}^{(1)}(\Delta) + \tau^1\zeta_2\Delta\widetilde{N}_{p6}^{(1)}(\Delta)\Big\}\\ &= -\frac{1}{\widetilde{A}\mu}\Big[\widetilde{A}\,\Gamma^{(1)}(\Delta)\delta_{p1} - (\mu\Delta + \rho\omega^2)\widetilde{N}_{p1}^{(1)}(\Delta)\Big]\\ &= \frac{1}{\widetilde{A}}(\Delta + \lambda_7^2)\widetilde{N}_{p1}^{(1)}(\Delta) - \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{p1}.\end{aligned}$$

Theorem 7. If condition (3.6) is satisfied and $\mathbf{x} \in \mathbb{E}^3 - \{\mathbf{0}\}$, then

$$G_{gh}^{(1)}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{j=1}^{6} x_{11j}\varsigma_j(\mathbf{x}) + \widetilde{R}_{gh}\,x_{117}\,\varsigma_7(\mathbf{x}),$$

$$G_{g;l+2}^{(1)}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{j=1}^{6} x_{1lj}\varsigma_j(\mathbf{x}), \qquad G_{l+2;g}^{(1)}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{j=1}^{6} x_{l1j}\varsigma_j(\mathbf{x}),$$

$$G_{l+2;k+2}^{(1)}(\mathbf{x}) = \sum_{j=1}^{6} x_{lkj}\varsigma_j(\mathbf{x}), \qquad g,h = 1,2,3, \qquad l,k = 2,...,6,$$

where

$$x_{l1j} = -\frac{r_{2j}^{(1)}}{\widetilde{A}\lambda_j^2}\widetilde{N}_{l1}^{(1)}(-\lambda_j^2), \qquad x_{lpj} = \frac{r_{2j}^{(1)}}{\widetilde{A}}\widetilde{N}_{lp}^{(1)}(-\lambda_j^2),$$

$$x_{117} = \frac{1}{\rho\omega^2} = \frac{1}{\mu\lambda_7^2}, \qquad j,l = 1,..,6, \qquad p = 2,...,6. \tag{6.2}$$

Proof. From Eq. (4.22),

$$\Delta\varsigma_j(\mathbf{x}) = -\lambda_j^2\varsigma_j(\mathbf{x}), \qquad j = 1, ..., 7.$$

Thus, we have

$$-\frac{1}{\lambda_j^2}\left(\frac{\partial^2}{\partial x_g \partial x_h} - \widetilde{R}_{gh}\right)\varsigma_j(\mathbf{x}) = \delta_{gh}\,\varsigma_j(\mathbf{x}), \qquad \mathbf{x} \neq \mathbf{0}.$$

Consider

$$\begin{aligned}(6.3)\qquad G_{gh}^{(1)}(\mathbf{x}) &= R_{gh}^{(1)}(\mathbf{D}_{\mathbf{x}})Y_{11}^{(1)}(\mathbf{x})\\ &= \left[\frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{gh} + w_{11}^{(1)}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}\right]\sum_{j=1}^{7} r_{1j}^{(1)}\varsigma_j(\mathbf{x})\\ &= \sum_{j=1}^{7} r_{1j}^{(1)}\left\{\left[-\frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2) + w_{11}^{(1)}(-\lambda_j^2)\right]\frac{\partial^2}{\partial x_g \partial x_h}\right.\\ &\qquad\left. + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)\widetilde{R}_{gh}\right\}\varsigma_j(\mathbf{x}).\end{aligned}$$

From Eq. (6.1), we have

$$(6.4)\qquad w_{11}^{(1)}(-\lambda_j^2) = -\frac{1}{\widetilde{A}\lambda_j^2}(-\lambda_j^2 + \lambda_7^2)\widetilde{N}_{11}^{(1)}(-\lambda_j^2) + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2).$$

Using Eq. (6.4) in Eq. (6.3), we obtain

$$\begin{aligned}(6.5)\qquad G_{gh}^{(1)}(\mathbf{x}) &= \sum_{j=1}^{7} r_{1j}^{(1)}\left\{\left[-\frac{1}{\widetilde{A}\lambda_j^2}(-\lambda_j^2 + \lambda_7^2)\widetilde{N}_{11}^{(1)}(-\lambda_j^2)\right]\frac{\partial^2}{\partial x_g \partial x_h}\right.\\ &\qquad\left. + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)\widetilde{R}_{gh}\right\}\varsigma_j(\mathbf{x}).\end{aligned}$$

Now,

$$\Gamma^{(1)}(-\lambda_j^2)r_{1j}^{(1)} = 0, \qquad j = 1, ..., 6,$$

$$\Gamma^{(1)}(-\lambda_j^2)r_{1j}^{(1)} = 1, \qquad j = 7,$$

and

$$(6.6)\qquad \begin{aligned}(-\lambda_j^2 + \lambda_7^2)r_{1j}^{(1)} &= r_{2j}^{(1)}, \qquad j = 1, ..., 6,\\ (-\lambda_j^2 + \lambda_7^2)r_{1j}^{(1)} &= 0, \qquad j = 7.\end{aligned}$$

By virtue of Eq. (6.6), Eq. (6.5) becomes

$$G_{gh}^{(1)}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h} \sum_{j=1}^{6} \Big[-\frac{1}{\widetilde{A}\lambda_j^2} r_{2j}^{(1)} \widetilde{N}_{11}^{(1)}(-\lambda_j^2) \Big] \varsigma_j(\mathbf{x}) + \widetilde{R}_{gh} \frac{1}{\mu \lambda_7^2} \varsigma_7(\mathbf{x})$$

$$= \frac{\partial^2}{\partial x_g \partial x_h} \sum_{j=1}^{6} x_{11j} \varsigma_j(\mathbf{x}) + \widetilde{R}_{gh}\, x_{117}\, \varsigma_7(\mathbf{x}).$$

The remaining formulae of the above theorem can be proved in the similar way.

Lemma 3. If the condition (3.6) is satisfied, then

$$\begin{gathered} \sum_{j=1}^{6} r_{2j}^{(1)} = \sum_{j=1}^{6} r_{2j}^{(1)} \lambda_j^2 = \sum_{j=1}^{6} r_{2j}^{(1)} \lambda_j^4 = \sum_{j=1}^{6} r_{2j}^{(1)} \lambda_j^6 = \sum_{j=1}^{6} r_{2j}^{(1)} \lambda_j^8 = 0, \\ \sum_{j=1}^{6} r_{2j}^{(1)} \lambda_j^{10} = -1, \qquad \sum_{j=1}^{6} \frac{r_{2j}^{(1)}}{\lambda_j^2} = \prod_{i=1}^{6} \lambda_i^{-2} = \frac{\widetilde{A}}{\rho \omega^2 \widetilde{N}_{11}^{(1)}(0)}, \end{gathered} \tag{6.7}$$

and

$$\sum_{j=1}^{6} x_{11j} = -(\rho\omega^2)^{-1}, \qquad \sum_{j=1}^{6} x_{11j} \lambda_j^2 = -\widetilde{\lambda}^{-1}. \tag{6.8}$$

Proof. Consider

$$\widetilde{N}_{11}^{(1)}(-\lambda_j^2) = -KD\varrho \lambda_j^{10} + B_1 \lambda_j^8 + B_2 \lambda_j^6 + B_3 \lambda_j^4 + B_4 \lambda_j^2 + \widetilde{N}_{11}^{(1)}(0), \tag{6.9}$$

where B_p, $p = 1, ..., 4$, are coefficients, independent of λ_j and skipped due to lengthy calculations.

It is easier to prove the relations (6.7) using Eq. (4.16).

From Eqs. (6.7) and (6.9), we obtain

$$\sum_{j=1}^{6} \frac{r_{2j}^{(1)}}{\lambda_j^2} \widetilde{N}_{11}^{(1)}(-\lambda_j^2) = \sum_{j=1}^{6} r_{2j}^{(1)} [-KD\varrho\lambda_j^8 + B_1\lambda_j^6 + B_2\lambda_j^4 + B_3\lambda_j^2 + B_4 + \widetilde{N}_{11}^{(1)}(0)\lambda_j^{-2}]$$

$$= \widetilde{N}_{11}^{(1)}(0) \sum_{j=1}^{6} \frac{r_{2j}^{(1)}}{\lambda_j^2} = \frac{\widetilde{A}}{\rho\omega^2},$$

and

$$\sum_{j=1}^{6} r_{2j}^{(1)} \widetilde{N}_{11}^{(1)}(-\lambda_j^2)$$

$$= \sum_{j=1}^{6} r_{2j}^{(1)} \big[- KD\varrho\lambda_j^{10} + B_1\lambda_j^8 + B_2\lambda_j^6 + B_3\lambda_j^4 + B_4\lambda_j^2 + \widetilde{N}_{11}^{(1)}(0)\big] = KD\varrho.$$

Therefore, from Eq. (6.2), we have

$$\sum_{j=1}^{6} x_{11j} = -\sum_{j=1}^{6} \frac{r_{2j}^{(1)}}{\widetilde{A}\lambda_j^2} \widetilde{N}_{11}^{(1)}(-\lambda_j^2) = -(\rho\omega^2)^{-1},$$

$$\sum_{j=1}^{6} x_{11j}\lambda_j^2 = -\sum_{j=1}^{6} \frac{r_{2j}^{(1)}}{\widetilde{A}} \widetilde{N}_{11}^{(1)}(-\lambda_j^2) = -\frac{KD\varrho}{\widetilde{A}} = -\widetilde{\lambda}^{-1}.$$

Theorem 8. The relations

$$G_{pl}^{(1)}(\mathbf{x}) - W_{pl}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|), \qquad p,l = 1,...,8, \tag{6.10}$$

hold in the neighborhood of the origin.

Proof. For $p, l = 1, 2, 3$, consider

$$G_{pl}^{(1)}(\mathbf{x}) - W_{pl}(\mathbf{x}) = \frac{\partial^2}{\partial x_p \partial x_l}\overline{Y}_{11}(\mathbf{x}) + \widetilde{R}_{pl}\,\overline{Y}_{22}(\mathbf{x}), \tag{6.11}$$

where

$$\begin{aligned} \overline{Y}_{11}(\mathbf{x}) &= \sum_{j=1}^{6} x_{11j}\varsigma_j(\mathbf{x}) - \frac{\varsigma_2^*(\mathbf{x})}{\widetilde{\lambda}}, \\ \overline{Y}_{22}(\mathbf{x}) &= x_{117}\,\varsigma_7(\mathbf{x}) + \frac{\varsigma_2^*(\mathbf{x})}{\mu}. \end{aligned} \tag{6.12}$$

From Eq. (6.12), we have

$$\begin{aligned} \overline{Y}_{11}(\mathbf{x}) &= \sum_{j=1}^{6} \frac{-x_{11j}}{4\pi} \sum_{l=0}^{\infty} \frac{\iota^l \lambda_j^l}{l!} |\mathbf{x}|^{l-1} + \frac{|\mathbf{x}|}{8\pi\widetilde{\lambda}} \\ &= -\frac{1}{8\pi}\left[2\sum_{j=1}^{6} x_{11j} \sum_{l=0}^{\infty} \frac{\iota^l \lambda_j^l}{l!} |\mathbf{x}|^{l-1} - \frac{|\mathbf{x}|}{\widetilde{\lambda}}\right] \\ &= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{j=1}^{6} x_{11j} - |\mathbf{x}|\left(\sum_{j=1}^{6} x_{11j}\lambda_j^2 + \frac{1}{\widetilde{\lambda}}\right)\right] - \frac{\iota}{4\pi}\sum_{j=1}^{6} x_{11j}\lambda_j + \overline{Y}_{33}(\mathbf{x}). \end{aligned} \tag{6.13}$$

Similarly,

$$\overline{Y}_{22}(\mathbf{x}) = -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}x_{117} - |\mathbf{x}|\left(x_{117}\lambda_7^2 - \frac{1}{\mu}\right)\right] - \frac{\iota}{4\pi}x_{117}\lambda_7 + \overline{Y}_{44}(\mathbf{x}), \tag{6.14}$$

where

$$\begin{aligned} \overline{Y}_{33}(\mathbf{x}) &= -\frac{1}{4\pi}\sum_{j=1}^{6} x_{11j}\sum_{l=3}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1}, \\ \overline{Y}_{44}(\mathbf{x}) &= -\frac{1}{4\pi}x_{117}\sum_{l=3}^{\infty}\frac{\iota^l\lambda_7^l}{l!}|\mathbf{x}|^{l-1}. \end{aligned} \tag{6.15}$$

Clearly

$$\begin{aligned} &\overline{Y}_{hh}(\mathbf{x}) = O(|\mathbf{x}|^2), \qquad \frac{\partial}{\partial x_k}\overline{Y}_{hh}(\mathbf{x}) = O(|\mathbf{x}|), \\ &\frac{\partial^2}{\partial x_k \partial x_i}\overline{Y}_{hh}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|), \qquad k,i = 1,2,3, \qquad h = 3,4. \end{aligned} \tag{6.16}$$

Consider

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{x_i}{|\mathbf{x}|^3}, \qquad \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = \left[\frac{3x_i^2}{|\mathbf{x}|^5} - \frac{1}{|\mathbf{x}|^3}\right].$$

Hence,

$$\Delta\frac{1}{|\mathbf{x}|} = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = 0.$$

Therefore,

$$\left(\frac{\partial^2}{\partial x_p \partial x_l} - \widetilde{R}_{pl}\right)\frac{1}{|\mathbf{x}|} = \delta_{pl}\Delta\frac{1}{|\mathbf{x}|} = \mathbf{0}. \tag{6.17}$$

Equation (6.11) with the aid of Eqs. (6.8), (6.13)–(6.17) becomes

$$G_{pl}^{(1)}(\mathbf{x}) - W_{pl}(\mathbf{x}) = \frac{\partial^2}{\partial x_p \partial x_l}\overline{Y}_{33}(\mathbf{x}) + \widetilde{R}_{pl}\,\overline{Y}_{44}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|).$$

Similarly other formulae of Eq. (6.10) can be proved.

Therefore, matrix $\mathbf{W}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{G}^{(1)}(\mathbf{x})$ in the neighborhood of the origin.

7. Conclusions

The linear theory of thermoelastic diffusion with a triple porosity was derived without utilizing Darcy's law in the current paper. After reducing the governing equations in an isotropic medium, the fundamental solution $\mathbf{G}^{(1)}(\mathbf{x})$ of system of Eqs. (3.2) for the case of steady oscillations was obtained. Additionally, the fundamental solutions $\mathbf{G}^{(i)}(\mathbf{x})$, $i = 2, 3, 4$, of system of Eqs. (3.3)–(3.5) in the cases of pseudo-, quasi-static oscillations and equilibrium were obtained. The fundamental solution $\mathbf{G}^{(1)}(\mathbf{x})$ of system of Eqs. (3.2) makes it possible to investigate three-dimensional boundary value problems in the theory of triple porosity thermoelastic diffusion elastic solids using the potential method. Also by this method, it is possible to construct fundamental solutions of the system of equations for the linear theory of isotropic thermoelastic materials with triple porosity.

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