*ENGINEERING TRANSACTIONS* • *Engng. Trans.* • **64**, 2, 197–211, 2016
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

# Isogeometric Approximation for Dynamics of Infinite String Using Difference Equation Method

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An efficient method of vibration investigation of an infinite string using the isogeometric analysis (IGA) with B-spline basis functions is considered. The research objective is to compare IGA, finite element method (FEM) and exact formulation approaches. In the IGA approximation a system is divided into a set of regularly distributed coordinates assembled in a uniform knot vector. Transverse displacements are described by linear, quadratic, cubic and quartic B-spline basis functions. The geometrical and mass matrices are found for all types of approximations. The equilibrium conditions for an arbitrary interior element are expressed in the form of one difference equation equivalent to the infinite set of equations obtained by numerical IGA formulation for this dynamic problem. Assuming the wavy nature of a vibration propagation phenomenon the dispersive equations are obtained. The ranges of vibration frequencies for the dispersive and reactive cases are determined. The influences of the adopted discretization, mass distribution and initial axial force effects on the wave propagation phenomenon are examined.

**Key words:** FEM, FDM, NURBS, IGA.

## 1. Introduction

An isogeometric approach to analysis of structural systems is a recently developed new technique. The basic concept of this method is presented in papers by Hughes, Cottrell, Bazilevs and Reali [3–6]. IGA is a generalisation of the classical finite element method and, geometrically, it is based on the computer aided design (CAD) approach. Due to that IGA has a lot of common features with FEM and meshless methods. The essence of IGA method is to employ B-spline basis functions to describe the geometry and displacements of the analysed system. These common functions are linear combinations of polynomial curves described using a uniform knot vector which is a set of regularly distributed coordinates.

The goal of this study is to examine the efficiency of IG in the analysis of the propagation of a wave in an infinite string, which is accomplished by comparing the IG, FEM and analytical formulation approaches. The authors used in their investigations the very efficient difference equation methodology presented in their previous articles, e.g., Rakowski [12, 13] and Rakowski and Wielentejczyk [14, 15]. The fundamental work concerning dynamics of discrete systems, practically cited in all the publications in this field, is the monograph by Brillouin [1]. Brillioouin defined basic concepts of travelling wave propagations in periodic systems treated as the continuous ones with periodic inhomogeneities. Similar structures were analysed in many later works, and the most important were publications by Mead *et al.* [9, 10]. In these two articles many efficient methods for dynamic analysis of continuous systems were developed. Furthermore, the finite element method was applied for dispersive analysis of such systems in, e.g., Oris and Petyt [11], Belytschko and Mindle [2] and Rakowski [12]. In these works, based on equilibrium equations for every node, the authors defined the characteristic equations, whose roots (discrete waves numbers) allowed to determine the ranges of frequencies for which the system is dispersive or reactive (the numerical effect). The qualitative differences occurring in dynamic behaviour of one- and two-dimensional discrete models in comparison with continuous ones was explained in, e.g., Inhlenburg and Babuška [8]. In the paper by Hughes, Reali and Sangalli [7], investigations of smooth basis functions generated by IGA were initiated. The comparison of them with standard $C^o$ finite elements was done for the structural dynamics of a finite domain and for the wave propagation in an infinite system. In addition, the eigenvalue problem of free vibrations and the Helmholtz equation of the time-harmonic wave propagation were analysed.

This paper is a continuation of the previous work of the authors presented in [16], where the dynamic responses of the infinite Rayleigh beam was examined using the difference equation methodology in IGA approach. The same method of analysis is adopted in this paper for the examination of vibration of an infinite string. Following this, the geometry and displacements of the considered linear string system are approximated by identical B-spline basis functions. These functions are the linear combination of polynomials created on the basis of the uniform knot vector defined as a set of regularly distributed coordinates. The quadratic, cubic and quartic B-spline functions are used in the calculation. The aim of this work is to assess the accuracy of this methodology in the dynamic analysis of infinite one-dimensional systems. The research is focused on a comparison between IGA, FEM and exact formulation. Additionally, the influences of parameters such as density of control points, mass distribution and initial axial force effects on wave propagation in the system are investigated. The analytical IGA problem is formulated using the finite difference equation (FDE) methodology. The geometrical and mass matrices for the interior element of string, located between two adjacent control points, are found. The equilibrium conditions are expressed in the form of one difference equation equivalent to the infinite set of equations defined in the numerical IGA formulation. This approximation enables us to obtain the analytical solution of the wave propagation problem for any regular distribution of control points. Using this approach, the convergence of the discrete system described by IGA and FEM methods (finite difference equation formulation) with the continuous system (differential equation formulation) is examined. This approach can clearly identify the causes of errors and explain the "parasitic" effects resulting from approximation methods. Assuming the wavy nature of string vibration, the dispersive equations of a propagation phenomenon are obtained. The analytical form of these equations makes it possible to efficiently find the influence of the required control point density (the length of the finite element in FEM formulation), mass distribution, order of the B-spline basis functions, initial axial force effects, and physical and geometrical parameters of a string on the results of calculations. The conclusions can be extended to an arbitrary one-dimensional finite system for any shape of wave propagation. The representations of dispersion equations are given in the form of diagrams of passing bands. They enable us to select correctly the discretization of the system in which both the required accuracy of results and time of numerical computations are taken into account. The passing bands are compared to the exact ones for continuous systems.

The paper has the following layout. In Sec. 2, the graphical and analytical forms of the B- spline basis functions for the uniform knot vector are presented. In the next step, they are employed to define the IGA equilibrium equations. From the strain and the kinematic energy expressions the geometrical and consistent mass matrices are obtained for any interior element. These matrices are formulated for linear, quadratic, cubic and quartic B-spline basis functions. Using the regular distribution of the control points (the identical functions between adjacent control points) the equilibrium conditions are derived in the form of one difference equation for various orders of the B-spline basis functions. These equations make it possible to easily identify the numerical errors in discrete systems with respect to continuous ones.

Subsequently in Sec. 3 the dispersion equations are discussed. These equations are obtained under assumption of the wave propagation function resulting from the imposed boundary conditions of the analysed system. The functions are derived for consistent and lumped mass models taking into account the axial force effects. Based on these equations, a detailed dispersive analysis is carried out. The passing band diagrams, determined for the entire range of frequencies, give the possibility to evaluate IGA applicability in the analysis of examined structures. The influence of the initial axial force effects, the mass distribution, density of control points and the order of the B-spline basis functions on passing bands are examined. The results are compared with FEM and analytical solutions.

In Sec. 4 the concluding remarks are presented.

## 2. Difference equation formulation

In this section, the analytical and graphical form of the B-spline basis functions is presented. Afterwards, the geometrical and consistent mass matrices are obtained for different orders of the basis functions assuming the regular distribution of control points. They are derived from the strain and the kinematic energy expressions. Taking into account the regular division of the system the equilibrium equations are expressed in the form of a single difference equation replacing the infinite number of equilibrium equations for an infinite system. The difference equilibrium equations are formulated for various orders of B-spline basis functions, that is, for the lumped and consistent mass model.

### *2.1. B-spline basis functions*

B-splines are polynomial curves consisting of linear combinations of B-spline basis functions. B-spline basis functions are described with the use of coordinates of a knot vector. In one-dimensional parametric space, the knot vector is written as $\{x_1, x_2, ..., x_i, x_{i+1}, ..., x_{n+p+1}\}$ where $x_i \in R$ is the $i$-th knot, $i$ is the knot index $I = 1, 2, ..., n+p+1$, $p$ is the polynomial order, and $n$ is the number of basis functions, see for instance Hughes *et al.* [6].

The B-spline basis functions are defined starting with constant functions for $p = 0$:

$$N_{i,0}(x) = \begin{cases} 1 & \text{if} \quad x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

and for $p = 1, 2, 3, \ldots$, they are defined as

$$N_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x). \tag{2.2}$$

For a uniform knot vector written as $\{0, 1, 2, 3, 4, \ldots\}$ the suitable B-spline basis functions of order $p = 0$ (constant), $p = 1$ (linear), $p = 2$ (quadratic), $p = 3$ (cubic), $p = 4$ (quartic), $p = 5$ (quintic) are presented in Fig. 1.

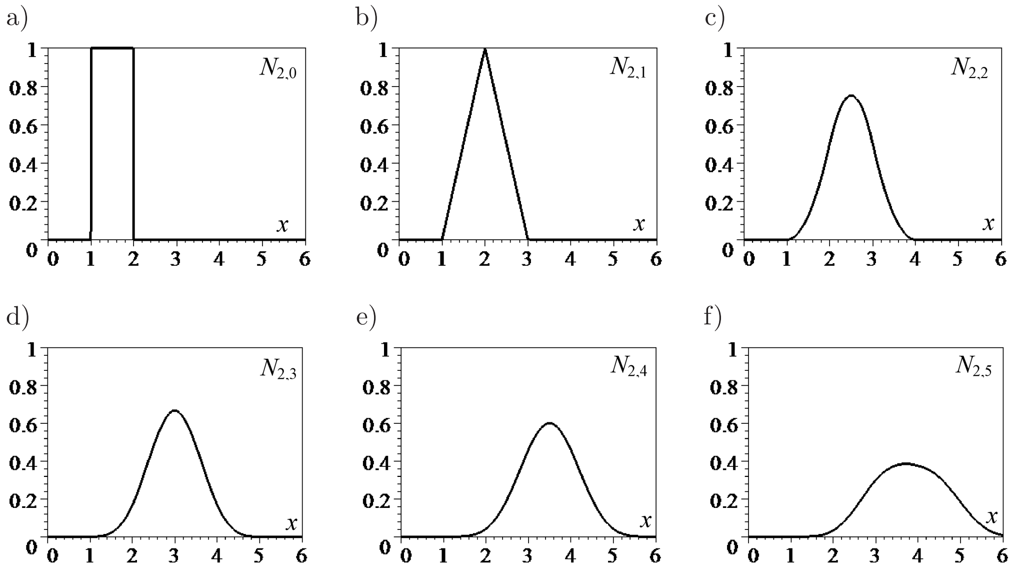


FIG. 1. Selected a) constant, b) linear, c) quadratic, d) cubic, e) quartic and f) quintic B-spline basis functions for the knot vector {0, 1, 2, 3, 4,. . . }.

The B-spline basis functions have an identical shape for an arbitrary one-dimensional interior element $(x_i,\ x_{i+1})$. The nonzero analytical B-spline equations described over this element $(N_{i,0} = 1)$ for various orders $p$ are

$$\text{for } p = 1 \ \text{(linear)} \quad \begin{cases} N_{i-1,1} = 1 - \eta, \\ N_{i,1} = \eta, \end{cases} \tag{2.3}$$

$$\text{for } p = 2 \ \text{(quadratic)} \quad \begin{cases} N_{i-2,2} = \dfrac{1}{2}\,(1-\eta)^2, \\[2mm] N_{i-1,2} = \dfrac{1}{2} + \eta - \eta^2, \\[2mm] N_{i,2} = \dfrac{1}{2}\eta^2, \end{cases} \tag{2.4}$$

$$\text{for } p = 3 \ \text{(cubic)} \quad \begin{cases} N_{i-3,3} = \dfrac{1}{6}\,(1-\eta)^3, \\[2mm] N_{i-2,3} = \dfrac{2}{3} - \eta^2 + \dfrac{1}{2}\eta^3, \\[2mm] N_{i-1,3} = \dfrac{1}{6} + \dfrac{1}{2}\eta + \dfrac{1}{2}\eta^2 - \dfrac{1}{2}\eta^3, \\[2mm] N_{i,3} = \dfrac{1}{6}\eta^3, \end{cases} \tag{2.5}$$

$$
\text{(2.6)} \qquad \text{for } p = 4 \text{ (quartic)} \quad \begin{cases} N_{i-4,4} = \dfrac{1}{24}\left(1-\eta\right)^4, \\ N_{i-3,4} = \dfrac{11}{24} - \dfrac{1}{2}\eta - \dfrac{1}{4}\eta^2 + \dfrac{1}{2}\eta^3 - \dfrac{1}{6}\eta^4, \\ N_{i-2,4} = \dfrac{11}{24} + \dfrac{1}{2}\eta - \tfrac{1}{4}\eta^2 - \dfrac{1}{2}\eta^3 + \dfrac{1}{4}\eta^4, \\ N_{i-1,4} = \dfrac{1}{24} + \dfrac{1}{6}\eta + \tfrac{1}{4}\eta^2 + \dfrac{1}{6}\eta^3 - \dfrac{1}{6}\eta^4, \\ N_{i,4} = \dfrac{1}{24}\eta^4, \end{cases}
$$

where $\eta = (x - x_i)/a$.

The graphical representation of these functions is shown in Fig. 2.

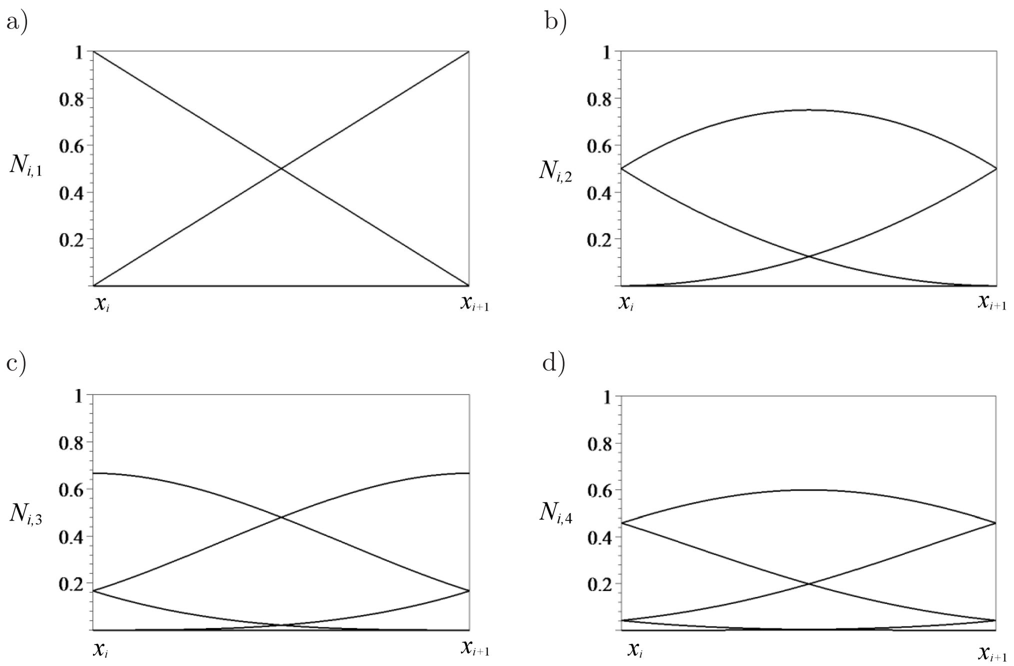


FIG. 2. a) Linear, b) quadratic, c) cubic and d) quartic B-spline basis functions for the interior element $\{x_i, x_{i+1}\}$.

### 2.2. *Geometrical and mass matrices*

The geometrical and mass matrices for the string can be derived from the expression of the strain $E_p$ and the kinematic energy $E_k$, see RAKOWSKI [13] as follows:

$$E_p = \frac{1}{2}\int H\left(\frac{\partial w}{\partial x}\right)^2 dx, \qquad E_k = \frac{1}{2}\int \rho A\dot{w}^2 dx. \tag{2.7}$$

Transverse displacements $w$ are connected with displacements of control points using IGA description by the following relation:

$$w(x) = \sum_{i=1}^{n} N_{i,p}\,\mathbf{u}_i. \tag{2.8}$$

The vector $\mathbf{u}_i$ includes $n$ discrete displacements of control points $w_r$.

Using (2.8) the expression for energies (2.7) can be transformed into matrix notation of the following form:

$$E_p = \frac{1}{2}\int H\frac{\partial \mathbf{B}^T}{\partial x}\frac{\partial \mathbf{B}}{\partial x}dx, \qquad E_k = \frac{1}{2}\int \rho A\dot{\mathbf{B}}^T\dot{\mathbf{B}}dx, \tag{2.9}$$

where $\mathbf{B}$ and $\mathbf{B}^T$ are the regular and transposed vectors corresponding to the control points and including the B-spline basis functions from Eqs. (2.3)–(2.6). $\dot{\mathbf{B}}$ denotes the first derivative of the vector $\mathbf{B}$ with respect to time $t$, $\dot{\mathbf{B}} = \frac{\partial}{\partial t}(\ )$.

Assuming regular division of one-dimensional system and based on the B-spline basis functions the geometrical $\mathbf{K}_G$ and the consistent mass $\mathbf{M}$ matrices corresponding to transverse displacements $w$ are derived below.

Matrices for the first-order of B-splines ($p = 1$) are

$$\mathbf{K_G} = \frac{H}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{M} = \frac{\rho A a}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{2.10}$$

Matrices for the second-order of B-splines ($p = 2$) are

$$\mathbf{K_G} = \frac{H}{6a}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad \mathbf{M} = \frac{\rho A a}{120}\begin{bmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{bmatrix}. \tag{2.11}$$

Matrices for the third-order of B-splines ($p = 3$) are

$$\begin{gathered} \mathbf{K_G} = \frac{H}{120a}\begin{bmatrix} 6 & 7 & -12 & -1 \\ 7 & 34 & -29 & -12 \\ -12 & -29 & 34 & 7 \\ -1 & -12 & 7 & 6 \end{bmatrix}, \\ \mathbf{M} = \frac{\rho A a}{5040}\begin{bmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 60 & 933 & 1188 & 129 \\ 1 & 60 & 129 & 20 \end{bmatrix}. \end{gathered} \tag{2.12}$$

Matrices for the fourth-order of B-splines ($p = 4$) are

$$\mathbf{K_G} = \frac{H}{5040a}\begin{bmatrix} 20 & 109 & -69 & -59 & -1 \\ 109 & 950 & -186 & -814 & -59 \\ -69 & -186 & 510 & -186 & -69 \\ -59 & -814 & -186 & 950 & 109 \\ -1 & -59 & -69 & 109 & 20 \end{bmatrix},$$

(2.13)

$$\mathbf{M} = \frac{\rho A a}{362880}\begin{bmatrix} 70 & 1121 & 1581 & 251 & 1 \\ 1121 & 22810 & 42996 & 11446 & 251 \\ 1581 & 42996 & 110430 & 42996 & 1581 \\ 251 & 11446 & 42996 & 22810 & 1121 \\ 1 & 251 & 1581 & 1121 & 70 \end{bmatrix},$$

where $\rho$ is a density of material, $H$ is an initial axial force, $A$ is a cross-section area, $a$ is a regular spacing of control points, $w$ is the transverse displacement function of string.

### 2.3. *Difference equilibrium conditions*

If we take into account the suitable order of B-spline functions and regular division of the system we can express the equilibrium conditions for transverse vibrations of the infinite string in the form of finite difference equations, e.g., see Rakowski [13]. These recurrent equations replace infinite number of equilibrium equations with a finite system. The transverse displacements of selected control points for harmonic excitation of a string are described by the following function:

$$w(r) = w_r \exp(\mathrm{i}\omega t), \tag{2.14}$$

where $w_r$ is the transverse displacement amplitude of control points (coordinate $r$ in discrete notation), $\omega$ is the transverse vibration frequency, $t$ is the time, and i is the imaginary number, $\mathrm{i} = \sqrt{-1}$.

Based on the geometrical and mass matrices from Eqs. (2.10)–(2.13) the difference equilibrium equations for the lumped and consistent mass model and suitable order of the B-spline basis functions are formulated as follows:

- for $p = 1$
  - transverse vibrations for the lumped mass model

$$\Delta^2 w_r + \Omega^2 w_r = 0, \tag{2.15}$$

– transverse vibrations for the consistent mass model

$$\Delta^2 w_r + \frac{\Omega^2}{6}\left(\Delta^2 + 6\right) w_r = 0, \tag{2.16}$$

- for $p = 2$
  – transverse vibrations for the lumped mass model

$$\left(\Delta^4 + 6\Delta^2\right) w_r + 6\Omega^2 w_r = 0, \tag{2.17}$$

– transverse vibrations for the consistent mass model

$$\left(\Delta^4 + 6\Delta^2\right) w_r + \frac{\Omega^2}{20}\left(\Delta^4 + 30\Delta^2 + 120\right) w_r = 0, \tag{2.18}$$

- for $p = 3$
  – transverse vibrations for the lumped mass model

$$\left(\Delta^6 + 30\Delta^4 + 120\Delta^2\right) w_r + 120\Omega^2 w_r = 0, \tag{2.19}$$

– transverse vibrations for the consistent mass model

$$\left(\Delta^6 + 30\Delta^4 + 120\Delta^2\right) w_r + \frac{\Omega^2}{42}\left(\Delta^6 + 126\Delta^4 + 1680\Delta^2 + 5040\right) w_r = 0, \tag{2.20}$$

- for $p = 4$
  – transverse vibrations for the lumped mass model

$$\left(\Delta^8 + 126\Delta^6 + 1680\Delta^4 + 5040\Delta^2\right) w_r + 5040\Omega^2 w_r = 0, \tag{2.21}$$

– transverse vibrations for the consistent mass model

$$\begin{aligned}\left(\Delta^8 + 126\Delta^6 + 1680\Delta^4 + 5040\Delta^2\right) w_r \\ + \frac{\Omega^2}{72}\left(\Delta^8 + 510\Delta^6 + 17640\Delta^4 + 151\,200\Delta^2 + 362\,880\right) w_r = 0,\end{aligned} \tag{2.22}$$

where $\Omega^2 = \rho A \omega^2 a^2 / H$ denotes the frequency parameter.

The above central difference operators are as follows:

$$\Delta^2 = E_r^{-1} - 2 + E_r^1,$$

$$\Delta^4 = E_r^{-2} - 4E_r^{-1} + 6 - 4E_r^1 + E_r^2,$$

$$\Delta^6 = E_r^{-3} - 6E_r^{-2} + 15E_r^{-1} - 20 + 15E_r^1 - 6E_r^2 + E_r^3,$$

$$\Delta^8 = E_r^{-4} - 8E_r^{-3} + 28E_r^{-2} - 56E_r^{-1} + 70 - 56E_r^1 + 28E_r^2 - 8E_r^3 + E_r^4,$$

where $E_r^n$ is Boole's shifting operator, $E_r^n = E^n(f_r) = f_{r+n}$.

### *2.4. Limits of the difference equilibrium equations*

The convergence of difference Eqs. (2.15)–(2.22) to the exact differential equations of continuous systems is examined below.

Taking for example Eqs. (2.19) and (2.20) and dividing both sides by $a^4$, we can rewrite them in the following form:

$$H\left(a^4\frac{\Delta^6}{a^6}+30a^2\frac{\Delta^4}{a^4}+120\frac{\Delta^2}{a^2}\right)w_r+120\rho A\omega^2 w_r=0, \tag{2.23}$$

$$H\left(a^4\frac{\Delta^6}{a^6}+30a^2\frac{\Delta^4}{a^4}+120\frac{\Delta^2}{a^2}\right)w_r + \frac{\rho A\omega^2}{42}\left(a^6\frac{\Delta^6}{a^6}+126a^4\frac{\Delta^4}{a^4}+1680a^2\frac{\Delta^2}{a^2}+5040\right)w_r=0. \tag{2.24}$$

Knowing that

$$\lim_{a\to 0}\left(\frac{\Delta^j}{a^j}w_r\right)=\frac{\partial^j w}{\partial x^j}, \qquad \lim_{a\to 0}\left(a^n\frac{\Delta^j}{a^j}w_r\right)=0, \tag{2.25}$$

where $j$ is the derivative order ($j, n = 1, 2, 3\ldots$), the limit for $a$ tending to 0 in the above difference equations attains identical form for all types of approximations:

$$H\frac{\partial^2 w(x)}{\partial x^2}+\rho A\omega^2 w(x)=0, \tag{2.26}$$

where $w = w(x)$ is the vertical displacement amplitude of the continuous string.

This procedure gives the same final differential equation of motion for various $p$.

## 3. Dispersive analysis

In this section, the effective method of results' evaluation obtained from FEM and IGA approximations is proposed.

The presented method enables us to make the qualitative analysis of the accuracy and efficiency of numerical calculations for the entire range of vibration frequencies taking into account the discretization, the order of the B-splines basis functions $p$ and the mass distribution.

All examinations are performed on the basis of the analytical dispersion equations derived from the difference equilibrium equations which include the relationships between the wave number $k$ and the frequency parameter of vibration $\Omega$.

The image of the function $\Omega(k)$ shows the so-called passing bands in the dispersion system. The comparison of the passing band convergence to the exact results provides the complete verification of the accuracy of the adopted approximation method.

### *3.1. Dispersive equation – exact solution for continuous system*

The differential equilibrium equation for transverse vibrations of the infinite string is of the form (2.26). Assuming that displacements decay in infinity and the travelling wave occurs in the system, the exact solution for amplitudes of the differential equation can be given as follows:

$$w(x) = C\exp(\mathrm{i}k_w x). \tag{3.1}$$

In the above formulas $x$ is a coordinate, $k_w = k/a$, $\mathrm{i} = \sqrt{-1}$, $C$ is an arbitrary constant, $k_w$ is an imaginary exact wave number of transverse wave motion, i is the imaginary number, and $a$ is regular spacing of control points.

Substituting the solution for the continuous system (3.1) into equation (2.26), we obtain the exact string dispersive equation for continuous system described by discrete variables:

$$\Omega_e^2 = k^2, \tag{3.2}$$

where $\Omega_e^2 = \rho A\omega^2 a^2/H$ denotes the frequency parameter.

Equation (3.2) shows that the vibrations of a continuous string have the form of a travelling wave in the entire examined frequency range. The frequency parameter $\Omega$ is real for every value of the wave number $k$. The continuous system is always dispersive and passing bands are unlimited contrary to those in the approximation methods (FEM, IGA).

### *3.2. Dispersive equations – FEM analysis*

The difference equilibrium equations and suitable dispersive equations using FEM methodology for the infinite string were derived by Rakowski in [13]. In the case when a travelling wave occurs in the system the dispersive equations for different mass distribution are as follows:

- transverse vibrations for the lumped mass model

$$\Delta^2 w_r + \Omega^2 w_r = 0, \tag{3.3}$$

- transverse vibrations for the consistent mass model

$$\Delta^2 w_r + \frac{\Omega^2}{6}\left(\Delta^2 + 6\right) w_r = 0. \tag{3.4}$$

Assuming that the displacements decay in infinity and the travelling wave occurs in the system the solutions are as follows:

$$w_r = C'\exp(\mathrm{i}kr), \tag{3.5}$$

where $C'$ is an arbitrary constant, i is the imaginary number, $k$ is the discrete wave number and $r$ is node coordinate.

In the case when a travelling wave occurs in the system the dispersive equations for different mass distribution are as follows:

- transverse vibrations for the lumped mass model

$$\Omega^2 - 2f = 0, \tag{3.6}$$

- transverse vibrations for the consistent mass model

$$(3 - f)\Omega^2 - 6f = 0, \tag{3.7}$$

where $f = 1 - \cos k$, $\Omega^2 = \rho A \omega^2 a^2 / H$.

Equations (3.6) and (3.7) are obtained for finite element approximation with linear shape functions.

### *3.3. Dispersive equations – IGA analysis*

Assuming that the displacements decay in infinity and the travelling wave occurs in the system the solutions of Eqs. (2.15)–(2.22) are the same as those in Eq. (3.5).

Substituting Eq. (3.5) into the difference Eqs. (2.15)–(2.22) we can obtain characteristic equations of the form:

- for $p = 1$
  - transverse vibrations for the lumped mass model

$$\Omega^2 - 2f = 0,$$

  - transverse vibrations for the consistent mass model

$$(3 - f)\Omega^2 - 6f = 0,$$

- for $p = 2$
  - transverse vibrations for the lumped mass model

$$3\Omega^2 - 2f(3 - f) = 0,$$

  - transverse vibrations for the consistent mass model

$$(f^2 - 15f + 30)\Omega^2 - 20f(3 - f) = 0,$$

- for $p = 3$
  - transverse vibrations for the lumped mass model

$$15\Omega^2 - f(f^2 - 15f + 30) = 0,$$

  - transverse vibrations for the consistent mass model

$$(f^3 - 63f^2 + 420f - 630)\Omega^2 - 42f(-30 + 15f - f^2) = 0,$$

- for $p = 4$
  - transverse vibrations for the lumped mass model

$$315\Omega^2 - f(630 - 420f + 63f^2 - f^3) = 0,$$

  - transverse vibrations for the consistent mass model

$$(f^4 - 255f^3 + 4410f^2 - 18900f + 22680)\Omega^2 - 72f(630 - 420f + 63f^2 - f^3) = 0,$$

where $f = 1 - \cos k$, $\Omega^2 = \rho A \omega^2 a^2 / H$.

## 4. Dispersive analysis – passing bands

Based on the previously derived dispersion equations for various approximation methods (FEM, IGA) the diagrams of the passing bands of the travelling wave are found. They present the relationship between the frequency parameter $\Omega$ and the discrete wave number $k \in (0, \pi)$.

The proposed diagrams clearly show the influence of approximation methods on propagation wave phenomenon in the system. The progress of them depends on the control density points (mesh density in FEM), types of approximation functions, mass distributions and axial force effects and they can indicate errors of these methods in relation to the exact approach.

In Fig. 3a the passing bands $\Omega(k)$ of an infinite string approximated by IGA for various orders of the B-spline basis functions $p$ and the lumped (lower branches) and consistent (upper branches) mass model are presented. The IGA

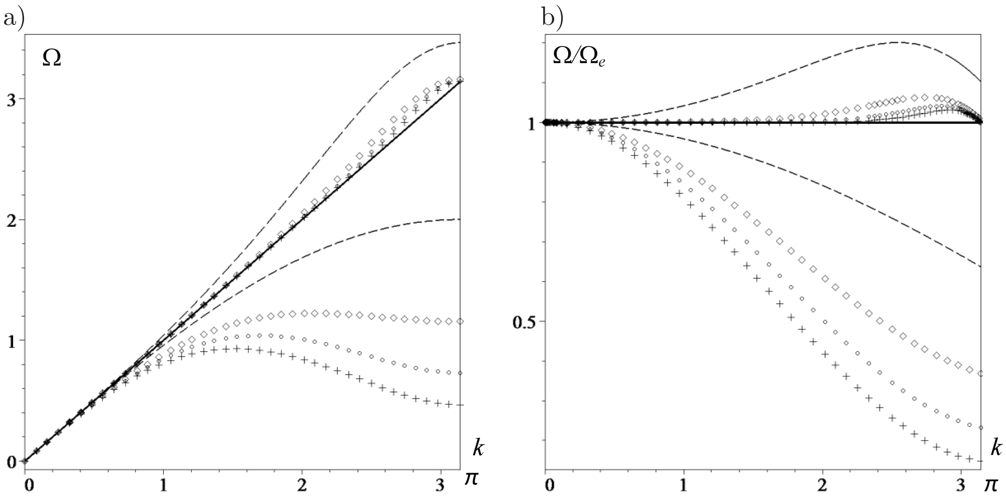


Fig. 3. a) The IGA passing bands of the infinite string and b) the comparison of the IGA passing bands to the exact solution $\Omega_e$ for the following order of B-spline basis functions: $p = 1$ (- - -), $p = 2$ (◇ ◇ ◇), $p = 3$ (○ ○ ○), $p = 4$ (+ + +), exact (—).

passing bands are compared with exact ones (the solid line). The passing bands for FEM approach with linear shape function are the same as for the first-order B-spline basis function ($p = 1$).

Figure 3b greatly simplifies evaluation of numerical calculations' accuracy. It shows the deviation of the IGA frequency parameter $\Omega$ with respect to the exact solution $\Omega_e$ for continuous system. For orders $p = 3$ and $p = 4$ of the basis functions and consistent mass model, the errors of the IGA do not exceed 10%. For a wide range of the passing bands the errors are less than 5%. The lumped mass model for increasing order of the B-spline basis functions gives results varying significantly from the exact solutions. The error rises with increasing values of the discrete wave number $k$. Such a dynamic behaviour, due to the adoption of the lumped mass model, indicates that the IGA coincides only in a very narrow range, i.e., for very low values of the frequency parameter $\Omega$ (high density of mesh control points $a$) with the exact solution. The reason of this phenomenon is difficulty in formulating the difference equilibrium equation for the lumped mass model taking into account the influence of a mass acting at adjoining control points. The main problem is to obtain the diagonal mass matrix for IGA approximation, see Cottrell, Hughes and Bazilevs [4].

## 5. Conclusions

The efficiency of IGA approach using the vibration analysis of one-dimensional system is presented. The difference equation method is adopted for the analysis of transverse vibrations of the discrete infinite string. The geometrical and mass matrices are formulated. The results are obtained in analytical closed form and compared with the FEM and exact ones. The convergence of the difference equilibrium equations of motions to the exact ones is investigated. Assuming regular structure of the system and the difference formulation of the problem, and despite infinite numbers of unknowns, it is shown that the solutions can be obtained in analytical closed form what gives the possibility of qualitative and quantitative evaluation of dynamic behaviour of one-dimensional infinite systems.

The influence of adopted discretization model, mass distributions and initial axial force effects on wave propagation phenomenon are examined. The frequency ranges of passing bands are determined. It can be noticed that the lumped mass model limits very much the usefulness of the IGA for the dynamic analysis of infinite discrete strings.

## Acknowledgment

This work was supported by the internal grant 11-080/12 (DS).

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*Received March 11, 2015; accepted version January 3, 2016.*