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From Tolerance Modeling to Exact Description of Heat Conduction in Biperiodic Composites

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In the paper we study the temperature boundary effect behavior in the dividing wall made of a composite conductor in which every area parallel to the outside and the inside plane areas has identical biperiodic structure. The proposed modeling approach is restricted to the hexagonal case of bi-periodicity in which the hexagonal cell can be divided onto three rhombus parts with different thermal properties. Considerations deal with anisotropic conductors.

Key words: tolerance averaging, heat conduction, biperiodic conductors, boundary effect.

1. Introduction

In many problems of environmental engineering we deal with highly oscillating heat transfer through walls built with different types of materials. An example of this phenomena is the presence of disturbances of a constant temperature in the external air boundary layer localized close to walls constructed from materials with a dense microstructure. Unfortunately, to compare the conductor responses on the different types of oscillating boundary thermal loads, the approximated models are inadequate. That is why the investigation of exact description of a heat conduction in composite materials is justified. The tolerance modeling technique explained by prof. Cz. Woźniak and contributors in [5, 12–14] may lead to the feasibility of this type of comparison, provided that the used tolerance shape functions lead to situations in which the heat flux vector generated by the temperature approximation given by micro-macro hypothesis retain continuity of its component normal to the bonding surface of composite components. Modeling problems for heat conduction and for linear elasticity in hexagonal-type composites have been investigated in [1, 8, 11, 18].

Considerations will be focused on the damping effect of external temperature fluctuation by the wall built of the mentioned composite material. Such behavior is usually referred to as *the temperature effect behavior*.

2. Subject of considerations

Let us consider a special type of biperiodic hexagonal type structure in which every basic hexagonal cell $\lambda\Delta$ is divided into three rhombus parts $\lambda\diamondsuit_1$, $\lambda\diamondsuit_2$, $\lambda\diamondsuit_3$, cf. Fig. 1. Each of these three rhombus parts has different properties described below. The hexagonal cell is situated in Cartesian orthogonal coordinate system $Ox^1x^2x^3$ in which the plane Ox^1x^2 is a biperiodicity plane and $Ox^3 = Oz$ is normal to biperiodicity plane.

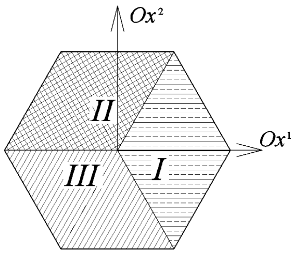

Fig. 1. Basic hexagonal cell.

Thermal properties of the composite is described by a heat conductivity tensor field represented in the introduced coordinate system by the symmetric matrix

$$K = K(x) = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \qquad \kappa \equiv [k_{13}, k_{23}]^T \tag{2.1}$$

taking constant values K^I, K^{II}, K^{III} in rhombus $\lambda\diamondsuit_1$, $\lambda\diamondsuit_2$, $\lambda\diamondsuit_3$, respectively, and by the scalar field $c(\cdot)$ of specific heat taking related constant values c^I, c^{II}, c^{III} in rhombus $\lambda\diamondsuit_1$, $\lambda\diamondsuit_2$, $\lambda\diamondsuit_3$.

We will assume jump discontinuities of $K(\cdot) \in R^{3\times 3}$ and c^I, c^{II}, c^{III} on interfaces between every neighboringrhombus parts as well as continuity of the heat flux vector in directions normal to the planes separating components.

Hence, composite components are perfectly bonded [9, 10].

This work presents two special cases of hexagonal-type structure. In the first case we introduce three rhombus parts with isotropic properties (Fig. 2) marked by heat conductivity tensor in form:

$$K_a = k^a I = k^a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad a = I, II, III. \tag{2.2}$$

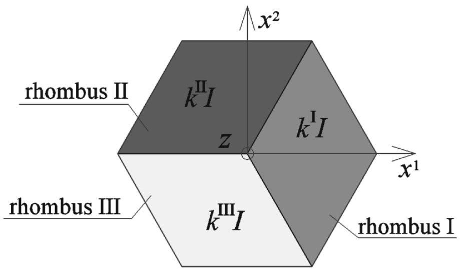

FIG. 2. Basic hexagonal cell with three isotropic rhombus parts.

In the second case the heat conductivity tensor of the first rhombus part of the basic hexagon will be assumed in the form

$$K = \begin{bmatrix} \dfrac{1}{\dfrac{s_I}{k^I} + \dfrac{s_{II}}{k^{II}}} & 0 & 0 \\ 0 & s_I k^I + s_{II} k^{II} & 0 \\ 0 & 0 & s_I k^I + s_{II} k^{II} \end{bmatrix} \tag{2.3}$$

in which $s_1 + s_2 = 1$, s_1, $s_2 > 0$ and k^I, k^{II} are two conductivity constants of some isotropic conductors.

The subsequent two rhombuses, second and third, are assumed to be made of the same transversally anisotropic conductor determined by conductivity matrix (1.3) but rotated in every neighboring rhombus by an angle of $2\pi/3$ (Fig. 3) with $0z$ axis as the axis of rotation. Hence

$$K_a = \begin{cases} K & \text{for} \quad (x^1, x^2) \in \diamondsuit_I, \\ Q^T_{2\pi/3} K\, Q_{2\pi/3} & \text{for} \quad (x^1, x^2) \in \diamondsuit_{II}, \\ Q_{2\pi/3} K\, Q^T_{2\pi/3} & \text{for} \quad (x^1, x^2) \in \diamondsuit_{III}. \end{cases} \tag{2.4}$$

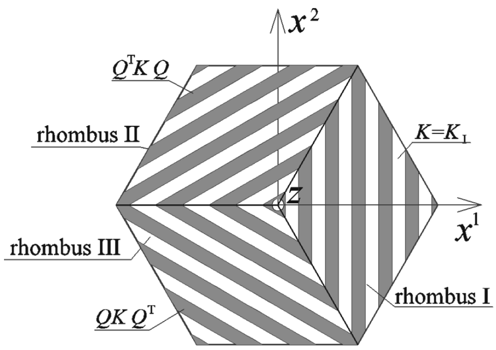

FIG. 3. Basic hexagonal cell with rotational symmetry.

Symbol $Q_{2\pi/3}$ denotes here the orthogonal matrix of rotation by $2\pi/3$ in R^2, namely

$$Q_\alpha \equiv \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{\alpha=\frac{2\pi}{3}}{=} \begin{bmatrix} -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} & 0 \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5}$$

A material, whose conductivity properties are described by the conductivity matrix (2.3), can be interpreted as a two-phased periodic layer with dense periodic structure. Under this interpretation $s_1 = l_1/(l_1 + l_2)$, $s_2 = l_2/(l_1 + l_2)$, where l_1, l_2 are the thicknesses of the first and the second layer, respectively.

The basic difficulty of the tolerance modeling procedure is a proper (for every considered physical problem) choice of the shape function system. This problem will be investigated in the subsequent sections.

3. Modeling procedure

The starting point of the tolerance modeling of heat conduction in periodic composites is the idea of the description of the temperature field by the finite slowly-varying fields u, ψ_A, $A = 1, ..., N$, and introducing the micro-macro approximation of the temperature field

$$\theta = \theta_M \equiv u + \lambda g^A \psi_A \tag{3.1}$$

into a parabolic heat transfer equation

$$c\dot{\theta} - (\nabla + \partial) \cdot [K(\nabla + \partial)]\theta = b. \tag{3.2}$$

In (3.1) and wherever we deal with a repeated index, a summation convention holds. The new fields above introduced, are the averaged temperature field u and fluctuation amplitude fields ψ_A and are new basic unknowns in the tolerance model equations. In (3.2) symbol b means the heat sources field and θ is temperature field. There is also denoted $\nabla \equiv \mathrm{grad} + \partial$ for $\mathrm{grad} \equiv [\partial_1, \partial_2, 0]^T$ and $\partial \equiv [0, 0, \partial_3]^T$ where $\partial_1 \equiv \partial/\partial x^1$, $\partial_2 \equiv \partial/\partial x^2$, $\partial_3 \equiv \partial/\partial x^3$. The tolerance modeling procedure yields to the model equations

$$\begin{aligned} &\langle c\rangle\dot{\theta} - (\nabla + \partial) \cdot [\langle K\rangle u + \langle K\,\mathrm{grad}\, g^A\rangle\psi_A] = \langle b\rangle, \\ &-\lambda^2(\langle g^A c g^B\rangle\dot{\psi}_B - \partial^T\langle g^A K g^B\rangle\partial\psi_B) \\ &\qquad + \lambda(\langle g^A \mathrm{grad}^T g^B K\rangle - \langle g^B \mathrm{grad}^T g^A K\rangle)\partial\psi_B \\ &\qquad + \langle \mathrm{grad}^T g^A K\,\mathrm{grad}\, g^B\rangle\psi_B + \langle \mathrm{grad}^T g^A K\rangle\nabla u = \lambda\langle g^A b\rangle \end{aligned} \tag{3.3}$$

in which the standard averaging $\langle\cdot\rangle(x)$ over cells $x+\lambda\Delta$ has been applied. For particulars the reader is referred to [6, 7, 12]. Fluctuation amplitudes in the tolerance model equations (3.3) describe the suppressions of the temperature perturbations together with the growth of the distance from the boundary of the region occupied by the composite.

In the subsequent considerations, the heat flux plays an important role

$$q_M \equiv K\nabla(u+\lambda g^A\psi_A) \tag{3.4}$$

generated by the micro-macro approximation (3.1) of the temperature field. This special kind of heat flux is referred to as *the tolerance heat flux vector*. If the component of the tolerance heat flux vector, normal to the surfaces Γ separating ingredients of the composite, is continuous on these surfaces then we say that *the tolerance heat flux continuity condition* is satisfied.

4. Boundary effect equation

The theory of linear ordinary differential equations suggests to investigate the fluctuation amplitude ψ_A in the form of the decomposition $\psi_A=\overline{\psi}_A+\widetilde{\psi}_A$ onto two terms $\overline{\psi}_A$ and $\widetilde{\psi}_A$. The first part $\overline{\psi}_A$ should satisfy equation

$$\begin{aligned}-\lambda^2\Big(\langle g^Acg^B\rangle\dot{\overline{\psi}}_B-\partial^T\langle g^AKg^B\rangle\partial\overline{\psi}_B\Big)\\ +\lambda\left(\langle g^A\mathrm{grad}^T g^BK\rangle-\langle g^B\mathrm{grad}^T g^AK\rangle\right)\partial\overline{\psi}_B\\ +\langle \mathrm{grad}^T g^AK\,\mathrm{grad}\,g^B\rangle\overline{\psi}_B=0\end{aligned} \tag{4.1}$$

while the second part should be an arbitrary solutions to

$$\begin{aligned}-\lambda^2\Big(\langle g^Acg^B\rangle\dot{\widetilde{\psi}}_B-\partial^T\langle g^AKg^B\rangle\partial\widetilde{\psi}_B\Big)\\ +\lambda\left(\langle g^A\mathrm{grad}^T g^BK\rangle-\langle g^B\mathrm{grad}^T g^AK\rangle\right)\partial\varphi_B\\ +\langle \mathrm{grad}^T g^AK\,\mathrm{grad}\,g^B\rangle\varphi_B+\langle\mathrm{grad}^T g^AK\rangle\nabla u=\lambda\langle g^Ab\rangle.\end{aligned} \tag{4.2}$$

Equation (4.1) will be referred to as *the nonstationary boundary effect equation*. At the same time equation

$$\begin{aligned}\lambda^2\partial^T\langle g^AKg^B\rangle\partial\overline{\psi}_B+\lambda\left(\langle g^A\mathrm{grad}^T g^BK\rangle-\langle g^B\mathrm{grad}^T g^AK\rangle\right)\partial\overline{\psi}_B\\ +\langle \mathrm{grad}^T g^AK\,\mathrm{grad}\,g^B\rangle\overline{\psi}_B=0\end{aligned} \tag{4.3}$$

will be referred to as *the stationary boundary effect equation* or simply *the boundary effect equation*, cf. [2–4].

5. Shape functions

In the tolerance modeling technique a proper indication of the finite sequence $h^1(x)$, $h^2(x)$, ..., $h^N(x)$ of tolerance shape functions is the most important modeling problem. In accordance with the principles of the tolerance modeling, this indication is verified only by *a posteriori* evaluation of the accuracy of the obtained solutions. In addition to the formal conditions required to be met by string tolerance shape functions, there are no known guidelines for constructing them.

In this paper we are to investigate the tolerance shape functions leading to *the tolerance heat flux continuity condition* formulated at the end of the previous section. For the anisotropic conductors the existence of such tolerance shape functions is still an open mathematical problem. Such shape functions play a crucial role in the *the extended tolerance model*, [7], in the framework of which the exact description of the heat transfer in composite materials can be obtained.

6. Passage to the limit model

Under the assumption that *the tolerance heat flux continuity condition* for the tolerance shape functions $h^1(x)$, $h^2(x)$, $\ldots, h^N(x)$ is satisfied we shall pass from the tolerance model represented by (3.3) to the extended tolerance model. Such procedure has been realized for two-phased periodic laminated composites in [17].

In the first step of this procedure we shall introduce instead of micro-macro hypothesis (3.4) a new hypothesis in the form

$$\theta(x,z,t) = \theta_M(x,z,t) + a_{m,n}(z,t)\phi^{m,n}(x), \tag{6.1}$$

for

$$\phi^{m,n}(x) = \frac{\lambda^2}{4}\varphi^m(\lambda x_+)\varphi^n(\lambda x_-)$$

and for

$$\begin{aligned} &\varphi_+^m(x_+) = \cos\frac{2n\pi}{\sqrt{3}}x_\pm, \qquad x_+ \in \left\langle 0, \frac{\sqrt{3}}{2}\right\rangle, \\ &\varphi_-^n(x_-) = \cos\frac{2n\pi}{\sqrt{3}}x_\pm, \qquad x_- \in \left\langle 0, \frac{\sqrt{3}}{2}\right\rangle, \\ &x_\pm \equiv \frac{1}{2}x_2 \pm \frac{\sqrt{3}}{2}(x_1 - 1), \end{aligned} \tag{6.2}$$

where in long wave term $\theta_M \equiv u + h^A(x)\psi_A$ we shall assume that fluctuation amplitudes $\psi_A = \psi_A(z,t)$ do not depend on periodic coordinates $x \equiv (x_1, x_2)$ and field $u = u_(x,z,t)$ is not restricted by the slowly-varying condition, cf. [1]. At the same time $\theta_{\rm res} = \theta - \theta_M(x,z,t)$ is, under *the tolerance heat flux continuity condition* a certain continuously differentiable scalar field and $\theta_{\rm res}(x,z,t) = a_p(z,t)\phi^p(x)$, $p \equiv (m,n)$, is the Fourier expansion with coefficients $a_{m,n} = a_{m,n}(z,t)$ and with respect to the orthogonal basis

$$\phi^{m,n}(x) = \frac{\lambda^2}{4}\varphi^m\left(\frac{x_+}{\lambda}\right)\varphi^n\left(\frac{x_-}{\lambda}\right).$$

In the second step of the mentioned procedure we shall assume that the orthogonal system of functions $\phi^{m,n}(x)$ is independent of the thermal and geometrical properties of the conductor. Hence, $m,n > 0$ in (6.1) and $\nabla_\Xi \phi^p = 0$ almost everywhere on Γ. That is why

$$\begin{aligned} &\langle c\phi^p\rangle = 0, \quad \langle K\phi^p\rangle = 0, \quad p = 1,2,...,\\ &\langle ch^A\rangle = 0, \quad \langle Kh^A\rangle = 0, \quad A = 1,...,N. \end{aligned} \tag{6.3}$$

After introducing new representations $h^A(x) \equiv \lambda g^A(\lambda^{-1}x)$ for tolerance shape functions $h^A(x)$ and $\phi^p(x) \equiv \lambda\varphi^p(\lambda^{-1}x)$, $p \equiv (m,n)$, aforementioned assumptions lead to the following extended model equations

$$\begin{aligned}
&\langle c\dot{u}\rangle - \nabla^T[\langle K\nabla u\rangle - \langle K\nabla_\Xi g\rangle\psi_A - \langle K\nabla_\Xi\phi^p\rangle a_p] = -\langle b\rangle,\\
&\lambda^2\left\{\begin{bmatrix} \langle g^A_\varepsilon c g^B\rangle & \langle g^A c\varphi^p\rangle \\ \langle\varphi^p c g^B\rangle & \langle\varphi^p c\varphi^q\rangle\end{bmatrix}\begin{bmatrix}\dot\psi_B \\ \dot a_q\end{bmatrix} - \nabla^T_\Phi\left(\begin{bmatrix} \langle g^A_\varepsilon c g^B\rangle & \langle g^A c\varphi^p\rangle \\ \langle\varphi^p K g^B\rangle & \langle\varphi^p c\varphi^q\rangle\end{bmatrix}\begin{bmatrix}\partial\psi_B \\ \partial a_q\end{bmatrix}\right)\right\}\\
&\qquad + \lambda\left(\begin{bmatrix} \langle {\rm grad}^T g^A K g^B\rangle & \langle {\rm grad}^T g^A K\varphi^p\rangle \\ \langle {\rm grad}^T \varphi^p K g^B\rangle & \langle {\rm grad}^T \phi^p k\varphi^q\rangle\end{bmatrix}\right.\\
&\qquad \left. - \begin{bmatrix} \langle {\rm grad}^T g^B K g^A\rangle & \langle {\rm grad}^T \varphi^q K g^A\rangle \\ \langle {\rm grad}^T g^B K\varphi^p\rangle & \langle {\rm grad}^T \varphi^q K\varphi^p\rangle\end{bmatrix}\right)\begin{bmatrix}\partial\psi_B \\ \partial a_q\end{bmatrix}\\
&\qquad + \begin{bmatrix} \langle {\rm grad}^T g^A K {\rm grad}\, g^B\rangle & \langle {\rm grad}^T g^A K {\rm grad}\,\varphi^q\rangle \\ \langle {\rm grad}^T \varphi^p K {\rm grad}\, g^B\rangle & \langle {\rm grad}^T \varphi^p k {\rm grad}\,\varphi^q\rangle\end{bmatrix}\begin{bmatrix}\psi_B \\ a_q\end{bmatrix}\\
&\qquad + \begin{bmatrix} \langle {\rm grad}^T g^A K\nabla u\rangle \\ \langle {\rm grad}^T \varphi^p K\nabla u\rangle\end{bmatrix} = \lambda\begin{bmatrix}\langle g^A b\rangle \\ \langle\varphi^p b\rangle\end{bmatrix}.
\end{aligned} \tag{6.4}$$

A formal procedure of rescalling tolerance shape functions leads to *the limit model equations*

$$
\langle c\dot{u}\rangle - \nabla^T[\langle K\nabla u\rangle - \langle K\nabla_{\Xi} g\rangle\psi_A - \langle K\nabla_{\Xi}\varphi^p\rangle a_p] = -\langle b\rangle,
$$

$$
\begin{aligned}
&\lambda^2\left\{\begin{bmatrix} 0 & 0 \\ 0 & \langle\varphi^p c\varphi^q\rangle\end{bmatrix}\begin{bmatrix}\dot{\psi}_B \\ \dot{a}_q\end{bmatrix} - \nabla_{\Phi}^T\left(\begin{bmatrix} 0 & 0 \\ 0 & \langle\varphi^p c\varphi^q\rangle\end{bmatrix}\begin{bmatrix}\partial\psi_B \\ \partial a_q\end{bmatrix}\right)\right\} \\
&+\lambda\left(\begin{bmatrix} 0 & \langle \mathrm{grad}^T g^A K\varphi^q\rangle \\ 0 & \langle \mathrm{grad}^T \varphi^p k\varphi^q\rangle\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \langle \mathrm{grad}^T g^B K\varphi^p\rangle & \langle \mathrm{grad}^T \varphi^q K\varphi^p\rangle\end{bmatrix}\right)\begin{bmatrix}\partial\psi_B \\ \partial a_q\end{bmatrix} \\
&+\begin{bmatrix} \langle \mathrm{grad}^T g^A K \mathrm{grad}\, g^B\rangle & \langle \mathrm{grad}^T g^A K \mathrm{grad}\, \varphi^q\rangle \\ \langle \mathrm{grad}^T \varphi^p K \mathrm{grad}\, g^B\rangle & \langle \mathrm{grad}^T \varphi^p k \mathrm{grad}\, \varphi^q\rangle\end{bmatrix}\begin{bmatrix}\psi_B \\ a_q\end{bmatrix} \\
&+\begin{bmatrix} \langle \mathrm{grad}^T g^A K\nabla u\rangle \\ \langle \mathrm{grad}^T \varphi^p K\nabla u\rangle\end{bmatrix} = \lambda\begin{bmatrix}\langle g^A b\rangle \\ \langle\varphi^p b\rangle\end{bmatrix}.
\end{aligned} \tag{6.5}
$$

The limit model of the heat conduction in the considered isotropic hexagonal-type composite represented by Eqs. (6.5) is valid provided that the system of tolerance shape functions h^1, h^2,..., h^N leads to *the tolerance heat flux continuity condition*.

Let $\mathrm{rot}_\alpha(x) = Q_\alpha(x - x_0)^T + x_0$, for $x \in R^2$, $\alpha \in R$, and for an arbitrary cell center x_0 as a center of rotation. In the subsequent considerations we shall assume that three shape functions

$$
\begin{aligned}
h^1(x) &= \lambda g(\lambda^{-1}x), \\
h^2(x) &= h^1(\mathrm{rot}_{2\pi/3}(x)), \\
h^3(x) &= h^2(\mathrm{rot}_{2\pi/3}(x))
\end{aligned} \tag{6.6}
$$

determined by a single basis function $g(\cdot)$ is taken as the system of tolerance shape functions. In [16] the basis function $g(\cdot)$ satisfying condition (6.6) and *the tolerance heat flux continuity condition*. Hence, the limit model of the heat conduction in the considered anisotropic hexagonal-type composite represented by Eqs. (6.5) is valid.

7. Final remarks

The answer to the question of whether amplitude fluctuations ψ_A can be eliminated from the limit model equations (6.5) is still not known. It means that the answer to the question whether the short-term and long-term temperature boundary disturbances are transmitted through the hexagonal-type conductor independently is also an open problem. It must be emphasized that for two-phased laminated periodic conductors this problem has been solved in [17] and the exact description of the boundary effect phenomena in this case has been obtained.

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