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Vibrations of Plate Strips with Internal Periodic Structure

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In this note vibrations of thin periodic plate strips with periodically distributed systems of three concentrated masses are analysed. Results of the non-asymptotic tolerance model are compared to those by the exact discrete model. In an example, these models are used to calculate lower and higher frequencies of the travelling wave related to the internal periodic structure.

Key words: periodic plate strips, tolerance averaging technique, travelling wave.

1. Introduction

Objects under consideration are thin periodic plate strips with a span L. It is assumed that the plate strips are homogeneous, weightless and unbounded along the x_1-axis (hence, the height h, Young's modulus E and Poisson's ratio ν are constant). The internal periodic structure of the plate strips is related to a system of three periodically distributed concentrated masses M_1, M_2, M_3 along the x_1-axis. According to the given distribution of the masses, it is possible to distinguish a small, repeatable element, called the *periodicity cell*. Every cell has a span l along the x_1-axis, called *the microstructure parameter*, which is small compared to the span L of the plate.

Because governing equations of these plates have highly oscillating, periodic, non-continuous coefficients, they are not a good tool to investigate special problems. Thus, various averaged models are formulated to describe these plates by equations with constant coefficients.

There are many different approaches proposed to analyse periodic structures. Most of them is based on the homogenization method, e.g. for the perodic plates by Kohn and Vogelius [4]. However, these models neglect the effect of the size of the plate microstructure on a behaviour of these plates. In order to take into account this effect, the governing equations of the plate strip will be derived using the tolerance averaging technique.

The main aim of this paper is to obtain formulas for frequencies of travelling waves of the plate strip using the tolerance averaging technique, which was presented by Woźniak and Wierzbicki [6], Woźniak, Michalak and Jędrysiak [Eds.] [5]. Afterwards, similarly to the paper by Jędrysiak and Michalak [3], in order to verify obtained results, the "exact" solution will be derived, which is based on the approach shown in the book by Brillouin [1].

2. Modelling foundations

Our considerations are assumed to be treated independently of the x_2-coordinate. Let us introduce the denotations: $x = x_1$, $z = x_3$, $x \in [0, L]$, $z \in [-d/2, d/2]$, with d as a constant plate thickness. In the problem under consideration it is assumed that the plate strip is described in the interval $\Lambda = (0, L)$, with the basic cell $\Delta \equiv [-l/2, l/2]$ in the interval $\overline{\Lambda}$, where l is the length of the basic cell, called the microstructure parameter. The parameter l is assumed to satisfy the conditions: $l \ll L$ and $d \ll l$. A cell with a centre at $x \in \Lambda$ is denoted by $\Delta(x) \equiv (x - l/2, x + l/2)$. The plate strip can be made of two elastic isotropic materials, perfectly bonded across interfaces, characterised by Young's moduli E', E'', Poisson's ratios ν', ν'' and mass densities ρ', ρ'', respectively. We assume that $E(x)$, $\rho(x)$, $x \in \Lambda$, are periodic, highly oscillating functions in x, but that Poisson's ratio $\nu \equiv \nu' = \nu''$ is constant. Hence, under the condition $E' \neq E''$ and/or $\rho' \neq \rho''$ the plate material structure is periodic in the x-axis direction. By ∂ let us denote a derivative of x. Moreover, by $w(x,t)$ ($x \in \overline{\Lambda}$, $t \in (t_0, t_1)$) a plate strip deflection is denoted.

Periodic functions in x are introduced describing the plate strip properties: namely, the mass density per unit area of the midplane μ and the bending stiffness B, which can be defined by:

$$\mu(x) \equiv d\rho(x), \qquad B \equiv \frac{d^3}{12(1-\nu^2)}E(x). \tag{2.1}$$

Using the well known assumptions of the Kirchhoff-type plate theory for periodic plate strips the partial differential equation of the fourth order for deflection $w(x,t)$ is derived:

$$\partial\partial[B(x)\partial\partial w(x,t)] + \mu(x)\ddot{w}(x,t) = 0, \tag{2.2}$$

with coefficients being highly oscillating, non-continuous, periodic functions in x. Equation (2.2) describes free vibrations of the plate strips under consideration.

In order to derive the governing equations with constant coefficients a tolerance averaging technique will be applied.

3. Modelling approach

3.1. Introductory concepts

In the course of the tolerance modelling we use some introductory concepts: an *averaging operator*, a *slowly varying function*, a *tolerance-periodic function* and a *highly oscillating function*, which are recalled below.

Denote a cell at $x \in \Lambda_\Delta$ by $\Delta(x) \equiv x + \Delta$, $\Lambda_\Delta = \{x \in \Lambda \colon \Delta(x) \subset \Lambda\}$. The known *averaging operator* for an integrable function f is given by

$$\langle f \rangle(x) = \frac{1}{l} \int\limits_{\Delta(x)} f(y)\, dy, \qquad x \in \Lambda_\Delta\,. \tag{3.1}$$

For a periodic function f in x its averaged value calculated from this formula is constant.

Let $\partial^{(k)} f$ be the k-th derivative of function $f = f(x)$, $x \in \Lambda$, $k = 0, 1, ..., \alpha$ ($\alpha \geq 0$); $\partial^0 f \equiv f$. By $\tilde{f}^{(k)}(\cdot,\cdot)$ denote a function defined in $\overline{\Lambda} \times R^m$, and by δ – a tolerance parameter. Let us also introduce $\Lambda_x \equiv \Lambda \cap \bigcup\limits_{z \in \Delta(x)} \Delta(z)$, $x \in \overline{\Lambda}$.

Function $f \in H^\alpha(\Lambda)$ is called *the tolerance-periodic function*, $f \in TP^\alpha_\delta(\Lambda, \Delta)$, if for $k = 0, 1, ..., \alpha$, the following conditions are satisfied

$$(1^\circ) \qquad (\forall x \in \Lambda) \ \Big(\exists \tilde{f}^{(k)}(x,\cdot) \in H^0(\Delta)\Big) \ \Big[\|\partial^k f\big|_{\Lambda_x}(\cdot) - \tilde{f}^{(k)}(x,\cdot)\|_{H^0(\Lambda_x)} \leq \delta\Big],$$

$$(2^\circ) \qquad \int\limits_{\Delta(\cdot)} \tilde{f}^{(k)}(\cdot, z)\, dz \in C^0(\overline{\Lambda}).$$

Function $\tilde{f}^{(k)}(x,\cdot)$ is called *the periodic approximation of* $\partial^k f$ *in* $\Delta(x)$, $x \in \Lambda$, $k = 0, 1, ..., \alpha$.

Function $F \in H^\alpha(\Lambda)$ is called *the slowly-varying function*, $F \in SV^\alpha_\delta(\Lambda, \Delta)$, if

$$(1^\circ) \qquad F \in TP^\alpha_\delta(\Lambda, \Delta),$$

$$(2^\circ) \qquad (\forall x \in \Lambda) \ \Big[\widetilde{F}^{(k)}(x,\cdot)\big|_{\Delta(x)} = \partial^k F(x), \quad k = 0, \ldots, \alpha\Big].$$

Function $\phi \in H^\alpha(\Lambda)$ is called *the highly oscillating function*, $\phi \in HO^\alpha_\delta(\Lambda, \Delta)$, if

$$(1^\circ) \qquad \phi \in TP^\alpha_\delta(\Lambda, \Delta),$$

$$(2^\circ) \qquad (\forall x \in \Lambda) \ \Big[\tilde{\phi}^{(k)}(x,\cdot)\big|_{\Delta(x)} = \partial^k \tilde{\phi}(x), \quad k = 0, 1, ..., \alpha\Big],$$

$$(3^\circ) \qquad \forall F \in SV^\alpha_\delta(\Lambda, \Delta) \ \ \exists f \equiv \phi F \in TP^\alpha_\delta(\Lambda, \Delta)$$
$$\tilde{f}^{(k)}(x,\cdot)\big|_{\Delta(x)} = F(x) \partial^k \tilde{\phi}(x)\big|_{\Delta(x)}, \quad k = 1, \ldots, \alpha.$$

For $\alpha = 0$ let us denote $\tilde{f} \equiv \tilde{f}^{(0)}$.

Let $g(\cdot)$ be defined on $\overline{\Lambda}$ as a highly oscillating function, $g \in HO^2_\delta(\Lambda, \Delta)$, continuous together with its gradient $\partial^1 g$. However, the second derivative $\partial^2 g$ is a piecewise continuous and bounded. Function $g(\cdot)$ is *the fluctuation shape function* of the 2-nd kind, $FS^2_\delta(\Lambda, \Delta)$, if it depends on l as a parameter and holds following conditions:

$$\begin{array}{lll} (1^\circ) & \partial^k g \in O(l^{\alpha-k}) & \text{for} \ \ k = 0, 1, \ldots, \alpha, \quad \alpha = 2, \quad \partial^0 g \equiv g, \\ (2^\circ) & \langle \mu g \rangle(x) \approx 0 & \text{for every} \ \ x \in \Lambda_\Delta, \end{array}$$

where $\mu > 0$ is a certain periodic function; l is the microstructure parameter.

3.2. Tolerance modelling assumptions

The fundamental modelling assumptions of the tolerance averaging technique can be formulated similarly to those in the following books by Woźniak and Wierzbicki [6], Woźniak, Michalak and Jędrysiak [5], and Jędrysiak [2].

The micro-macro decomposition of the plate strip deflection w is the first assumption:

$$w(x,t) = W(x,t) + g^A(x)Q^A(x,t), \qquad A = 1, \ldots, N, \quad x \in \Lambda, \tag{3.2}$$

with $W(\cdot, t)$, $Q^A(\cdot, t) \in SV^2_\delta(\Lambda, \Delta)$ (for every t) being basic kinematic unknowns, called *the macrodeflection, the fluctuation amplitudes* respectively; and $g^A(\cdot) \in FS^2_\delta(\Lambda, \Delta)$, being a known fluctuation shape functions.

The tolerance averaging approximation is the second modelling assumption, where terms $O(\delta)$ are assumed to be negligibly small in the course of modelling, e.g. in formulas:

$$\begin{aligned} \langle \phi \rangle(x) &= \langle \overline{\phi} \rangle(x) + O(\delta), \\ \langle \phi F \rangle(x) &= \langle \phi \rangle(x) F(x) + O(\delta), \\ \langle \phi \partial_\alpha (g^A F) \rangle(x) &= \langle \phi \partial_\alpha g^A \rangle(x) F(x) + O(\delta), \end{aligned}$$
$$x \in \Lambda, \qquad \alpha = 1, 2, \qquad A = 1, \ldots, N, \qquad 0 < \delta \ll 1,$$
$$\phi \in TP^2_\delta(\Lambda, \Delta), \qquad F \in SV^2_\delta(\Lambda, \Delta), \qquad g^A \in FS^2_\delta(\Lambda, \Delta),$$

where δ is a tolerance parameter.

3.3. Modelling procedure

Following the aforementioned books the modelling procedure can be outlined as below.

The starting point is the formulation of the action functional

$$\mathcal{A}(w(\cdot)) = \int_{\Lambda} \int_{t_0}^{t_1} \mathcal{L}\left(y, \partial\partial w(y,t), \dot{w}(y,t)\right) dt\, dy, \tag{3.3}$$

where $\mathcal{L}$ is the lagrangean given by

$$\mathcal{L} = \frac{1}{2}\left(\mu \dot{w}\dot{w} - B\partial\partial w \partial\partial w\right). \tag{3.4}$$

Using the principle of stationary action to $\mathcal{A}$, after some manipulations, the known Eq. (2.2) of free vibrations for thin periodic plate strips is obtained. Then, micro-macro decomposition (3.2) is substituted to action functional (3.3). Applying averaging operator (3.1) to the action functional we arrive at the tolerance averaging of functional $\mathcal{A}(w(\cdot))$ in the form

$$\mathcal{A}_g(W(\cdot), Q^A(\cdot)) = \int_{\Lambda} \int_{t_0}^{t_1} \langle \mathcal{L}_g \rangle (y, \partial\partial W, \dot{W}, \dot{Q}^A, W, Q^A)\, dt\, dy, \tag{3.5}$$

with the averaged form $\langle \mathcal{L}_g \rangle$ of lagrangean (3.4)

$$\begin{aligned} \langle \mathcal{L}_g \rangle \; = \; -\frac{1}{2}\Big\{ & (\langle B \rangle \partial\partial W + 2\langle B\partial\partial g^B \rangle Q^B)\partial\partial W \\ & - \langle \mu \rangle \dot{W}\dot{W} + \langle B\partial\partial g^A \partial\partial g^B \rangle Q^A Q^B - \langle \mu g^A g^B \rangle \dot{Q}^A \dot{Q}^B \Big\}. \end{aligned} \tag{3.6}$$

From the principle of stationary action applied to $\mathcal{A}_g$ the system of Euler-Lagrange equations with constant coefficients is derived.

4. Model equations

After some manipulations from the system of Euler-Lagrange equations we obtain the following system of equations for $W(\cdot,t)$ and $Q^A(\cdot,t)$:

$$\begin{aligned} \langle B \rangle \partial\partial\partial\partial W + \langle B\partial\partial g^B \rangle \partial\partial Q^B + \langle \mu \rangle \ddot{W} &= 0, \\ \langle B\partial\partial g^A \rangle \partial\partial W + \langle B\partial\partial g^A \partial\partial g^B \rangle Q^B + \underline{\langle \mu g^A g^B \rangle} \ddot{Q}^B &= 0. \end{aligned} \tag{4.1}$$

The underlined term in these equations involves the microstructure parameter l. Coefficients of Eqs. (4.1) are constant. Equations (4.1) and micro-macro decomposition (3.2) constitute *the tolerance model of thin periodic strips*, which makes it possible to take into account the effect of the microstructure size on free vibrations of these plates. For the plate strip described in $\Lambda = (0, L)$ we have to formulate boundary conditions only for *the macrodeflection* W (on the edges $x = 0$, L), but not for *the fluctuation amplitudes* Q^A, $A = 1, \ldots, N$.

It can be observed that neglecting the term dependent on the microstructure paramter l in Eqs. (4.1), we arrive at the following equations of *the asymptotic model of thin periodic plate strips*:

$$\begin{aligned} \langle B\rangle\partial\partial\partial\partial W + \langle B\partial\partial g^B\rangle\partial\partial Q^B + \langle\mu\rangle\ddot{W} &= 0, \\ \langle B\partial\partial g^A\rangle\partial\partial W + \langle B\partial\partial g^A\partial\partial g^B\rangle Q^B &= 0. \end{aligned} \tag{4.2}$$

5. A special problem: a travelling wave in a weightless unbounded plate strip with a periodically distributed system of three concentrated masses

5.1. *The tolerance model*

5.1.1. Frequencies of a travelling wave

Let us consider a homogenous weightless and unbounded plate strip along the x axis, with a periodically distributed system of three concentrated masses M_1, M_2, M_3, cf. Fig. 1.

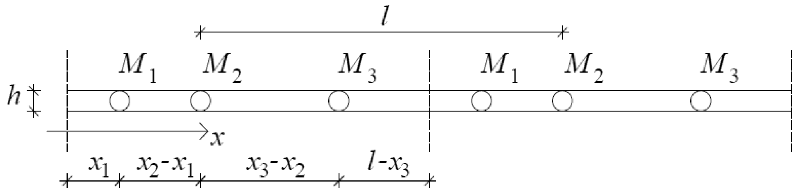

Fig. 1. The plate strip with a system of three periodically distributed concentrated masses.

Young's modulus E, Poisson's ratio ν and thickness d of the plate are assumed to be constant. Moreover, the plate mass is negligibly small when compared with concentrated masses M_1, M_2 and M_3. In our considerations two modeshape function g^A are assumed. Denote:

$$\begin{aligned} D &\equiv \langle B\rangle, & D^{11} &\equiv \langle B\partial\partial g^1\partial\partial g^1\rangle, & D^{22} &\equiv \langle B\partial\partial g^2\partial\partial g^2\rangle, \\ m &\equiv \langle\mu\rangle, & m^{11} &\equiv l^{-4}\langle\mu g^1 g^1\rangle, & m^{22} &\equiv l^{-4}\langle\mu g^2 g^2\rangle. \end{aligned} \tag{5.1}$$

Hence, equations (4.1) take the form:

$$\begin{aligned} D\partial\partial\partial\partial W + m\ddot{W} &= 0, \\ D^{11}Q^1 + l^4 m^{11}\ddot{Q}^1 &= 0, \\ D^{22}Q^2 + l^4 m^{22}\ddot{Q}^2 &= 0. \end{aligned} \tag{5.2}$$

Equations (5.2) stand for a system of independent equations for the macrodeflection and two fluctuation amplitudes. The first equation describes fundamental vibrations of the plate strip (e.g. lower frequencies of the travelling wave), while the second and the third refer to microstructural vibrations (related to higher frequencies of the travelling wave). Solutions to those equations can be assumed in the form:

$$\begin{aligned} W(x,t) &= A_W \exp[i(kx-\omega t)], \\ Q^1(x,t) &= A_{Q^1} \exp[i(kx-\omega t)], \\ Q^2(x,t) &= A_{Q^2} \exp[i(kx-\omega t)], \end{aligned} \tag{5.3}$$

where A_W, A_{Q^1}, A_{Q^2} are amplitudes, k is a wave number, t is a time coordinate and ω is a frequency. After some transformations formulas for the lower (ω_-) and higher $(\omega_{+:1};\omega_{+:2})$ frequencies can be obtained:

$$(\omega_-)^2 = \frac{Dk^4}{m}, \qquad (\omega_{+:1})^2 = \frac{D^{11}}{l^4 m^{11}}, \qquad (\omega_{+:2})^2 = \frac{D^{22}}{l^4 m^{22}}. \tag{5.4}$$

It can be observed, that only higher frequencies depend explicitly on the microstructure parameter l.

5.1.2. Eigenvalue problem on the periodicity cell

Mode-shape functions can be calculated as solutions to an eigenvalue problem on the periodicity cell. For the plate strip under consideration, the eigenvalue problem takes the following form:

$$B\partial\partial\partial\partial g(x) - \mu(x)\lambda^2 g(x) = 0, \tag{5.5}$$

with periodic boundary conditions on the cell edges; B is the stiffness defined by $(2.1)_2$; g are periodic functions related to eigenvalues $\lambda \equiv \alpha l$ (α is the wave number); and $\langle \mu g \rangle = 0$. Assuming that the plate mass is negligibly small when compared to concentrated masses, an exact form of eigenfunctions $g(x)$ can be found. These functions describe a shape of free vibrations of the cell.

The functions $g(x)$ can be found using some methods known from the structural mechanics. In the first step, the periodicity cell is divided into sections by the concentrated masses. Hence, we obtain four different sections: "4"–"1", "1"–"2", "2"–"3" and "3"–"5" (cf. Fig. 2). Equations of equilibrium for transversal forces and moments are written for points of concentrated masses.

Additionally, functions $g(x)$ have to satisfy the boundary conditions:

$$\begin{aligned} g(0) &= g(l) = v_4 = v_5, & \partial g(0) &= \partial g(l) = \varphi_4 = \varphi_5, \\ \partial\partial g(0) &= \partial\partial g(l), & \partial\partial\partial g(0) &= \partial\partial\partial g(l) \end{aligned} \tag{5.6}$$

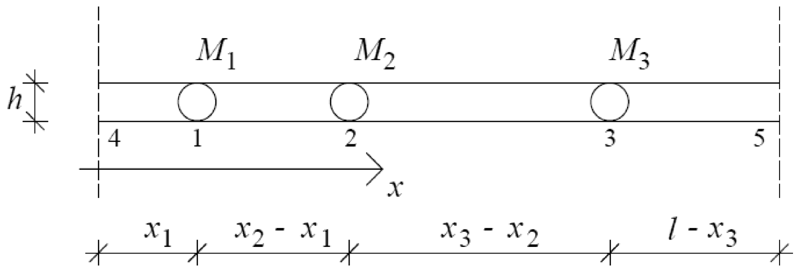

FIG. 2. A periodicity cell.

and a normalizing condition $<\mu g>=0$. Hence, we obtain a system of six equations with six unknown displacements and rotations of joints. After some transformations, we arrive at a characteristic equation in the form of determinant:

$$\det \mathbf{L}_{pr} = 0, \qquad p, r = 1, ..., 6. \tag{5.7}$$

Expressions of the matrix $\mathbf{L}_{pr}$ are presented in Appendix. As a result, a second order equation is obtained, hence it is possible to derive two different eigenvalues ω^2. Both of them are related to different mode-shape functions g^A, $A = 1,\ 2$. Introducing notations:

$$\begin{aligned}
\boldsymbol{\psi}_1 &= \begin{bmatrix} L_{23} & L_{22} & L_{24} & L_{25} & L_{26} \\ L_{33} & L_{32} & L_{34} & L_{35} & L_{36} \\ L_{43} & L_{42} & L_{44} & L_{45} & L_{46} \\ L_{53} & L_{52} & L_{54} & L_{55} & L_{56} \\ L_{63} & L_{62} & L_{64} & L_{65} & L_{66} \end{bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{bmatrix} L_{21} & L_{23} & L_{24} & L_{25} & L_{26} \\ L_{31} & L_{33} & L_{34} & L_{35} & L_{36} \\ L_{41} & L_{43} & L_{44} & L_{45} & L_{46} \\ L_{51} & L_{53} & L_{54} & L_{55} & L_{56} \\ L_{61} & L_{63} & L_{64} & L_{65} & L_{66} \end{bmatrix}, \\
\boldsymbol{\psi}_3 &= \begin{bmatrix} L_{21} & L_{22} & L_{23} & L_{25} & L_{26} \\ L_{31} & L_{32} & L_{33} & L_{35} & L_{36} \\ L_{41} & L_{42} & L_{43} & L_{45} & L_{46} \\ L_{51} & L_{52} & L_{53} & L_{55} & L_{56} \\ L_{61} & L_{62} & L_{63} & L_{65} & L_{66} \end{bmatrix}, \quad \boldsymbol{\psi}_4 = \begin{bmatrix} L_{21} & L_{22} & L_{24} & L_{23} & L_{26} \\ L_{31} & L_{32} & L_{34} & L_{33} & L_{36} \\ L_{41} & L_{42} & L_{44} & L_{43} & L_{46} \\ L_{51} & L_{52} & L_{54} & L_{53} & L_{56} \\ L_{61} & L_{62} & L_{64} & L_{63} & L_{66} \end{bmatrix}, \\
\boldsymbol{\psi}_5 &= \begin{bmatrix} L_{21} & L_{22} & L_{24} & L_{25} & L_{23} \\ L_{31} & L_{32} & L_{34} & L_{35} & L_{33} \\ L_{41} & L_{42} & L_{44} & L_{45} & L_{43} \\ L_{51} & L_{52} & L_{54} & L_{55} & L_{53} \\ L_{61} & L_{62} & L_{64} & L_{65} & L_{63} \end{bmatrix}, \quad \boldsymbol{\Xi} = \begin{bmatrix} L_{21} & L_{22} & L_{24} & L_{25} & L_{26} \\ L_{31} & L_{32} & L_{34} & L_{35} & L_{36} \\ L_{41} & L_{42} & L_{44} & L_{45} & L_{46} \\ L_{51} & L_{52} & L_{54} & L_{55} & L_{56} \\ L_{61} & L_{62} & L_{64} & L_{65} & L_{66} \end{bmatrix},
\end{aligned} \tag{5.8}$$

unknown deflections and rotations are given by:

$$
\begin{aligned}
v_1 &= \frac{l\det \boldsymbol{\psi}_1}{\det \boldsymbol{\Xi}}, \quad v_2 = \frac{l\det \boldsymbol{\psi}_2}{\det \boldsymbol{\Xi}}, \quad v_4 = v_5 = -l, \quad v_3 = -\frac{M_1}{M_3}v_1 - \frac{M_2}{M_3}v_2, \\
\varphi_1 &= \frac{l\det \boldsymbol{\psi}_3}{\det \boldsymbol{\Xi}}, \quad \varphi_2 = \frac{l\det \boldsymbol{\psi}_4}{\det \boldsymbol{\Xi}}, \quad \varphi_3 = \frac{l\det \boldsymbol{\psi}_5}{\det \boldsymbol{\Xi}}, \\
\varphi_4 &= \varphi_5 = \frac{\omega^2}{12lB}[M_1 v_1[x_1(x_1-l)(2x_1-l)] + M_2 v_2[x_2(x_2-l)(2x_2-l)] \\
&\qquad + M_3 v_3[x_3(x_3-l)(2x_3-l)]].
\end{aligned}
\tag{5.9}
$$

Using these deflections and rotations, mode-shape functions g^A can be described similarly to deflections of a beam. From the structural mechanics some functions can be introduced as follows:

$$
\begin{aligned}
r(\xi) &= 1 - 3\xi^2 + 2\xi^3, & u(\xi) &= \xi - 2\xi^2 + \xi^3, \\
\overline{r}(\xi) &= 3\xi^2 - 2\xi^3, & \overline{u}(\xi) &= \xi^2 - \xi^3,
\end{aligned}
\tag{5.10}
$$

where $\xi \in [0,\, 1]$. Hence, mode-shape functions g^A can be written in the form:

$$
g^A(x) = \begin{cases} AB_1 & \text{for } x \in [0,\, x_1], \\ AB_2 & \text{for } x \in [x_1,\, x_2], \\ AB_3 & \text{for } x \in [x_2,\, x_3], \\ AB_4 & \text{for } x \in [x_3,\, l], \end{cases}
\tag{5.11}
$$

where A is a constant amplitude and:

$$
\begin{aligned}
B_1 &= r\left(\frac{x}{x_1}\right) v_4 + \overline{r}\left(\frac{x}{x_1}\right) v_1 + \left[u\left(\frac{x}{x_1}\right)\varphi_4 - \overline{u}\left(\frac{x}{x_1}\right)\varphi_1\right] x_1, \\
B_2 &= r\left(\frac{x-x_1}{x_2-x_1}\right) v_1 + \overline{r}\left(\frac{x-x_1}{x_2-x_1}\right) v_2 \\
&\qquad + \left[u\left(\frac{x-x_1}{x_2-x_1}\right)\varphi_1 - \overline{u}\left(\frac{x-x_1}{x_2-x_1}\right)\varphi_2\right](x_2-x_1), \\
B_3 &= r\left(\frac{x-x_2}{x_3-x_2}\right) v_2 + \overline{r}\left(\frac{x-x_2}{x_3-x_2}\right) v_3 \\
&\qquad + \left[u\left(\frac{x-x_2}{x_3-x_2}\right)\varphi_2 - \overline{u}\left(\frac{x-x_2}{x_3-x_2}\right)\varphi_3\right](x_3-x_2), \\
B_4 &= r\left(\frac{x-x_3}{l-x_3}\right) v_3 + \overline{r}\left(\frac{x-x_3}{l-x_3}\right) v_5 \\
&\qquad + \left[u\left(\frac{x-x_3}{l-x_3}\right)\varphi_3 - \overline{u}\left(\frac{x-x_3}{l-x_3}\right)\varphi_5\right](l-x_3).
\end{aligned}
\tag{5.12}
$$

5.2. "Exact" solution

In order to verify the results obtained, the exact solution (or the discrete model) is also used to describe the behaviour of the plate strip under consideration.

Let us consider three following cells $j-1$, j, $j+1$. Denote distances between the concentrated masses by a_1, a_2 and a_3, respectively (cf. Fig. 2). Let us describe the deflections of the points with concentrated masses M_1, M_2 and M_3 in the j-th cell by $v_j^{M_1}$, $v_j^{M_2}$ and $v_j^{M_3}$ and the rotations of these points by $\phi_j^{M_1}$, $\phi_j^{M_2}$ and $\phi_j^{M_3}$.

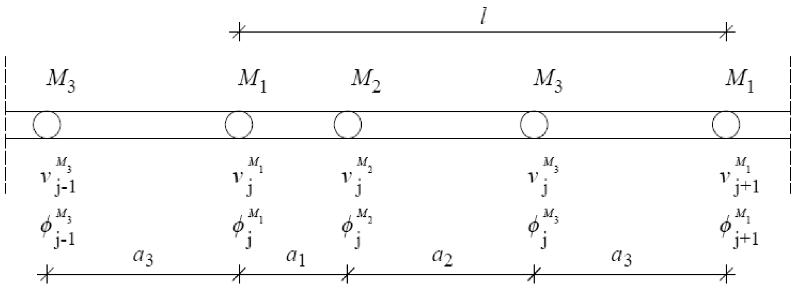

Fig. 3. The periodicity cell with denotations used in the "exact" solution.

Assuming that the plate stiffness is constant for the whole structure and given by B, it is possible to apply the equations of equilibrium for transversal forces and moments for points with concentrated masses, known from structural mechanics. The solution to this system of equations will be found in the form:

$$v_j^{M_p} = A_{w_{M_p}} \exp[i(kjl - \omega t)], \qquad \phi_j^{M_p} = A_{\phi_{M_p}} \exp[i(kjl - \omega t)], \tag{5.13}$$

where $A_{w_{M_p}}$ and $A_{\phi_{M_p}}$ are amplitudes, k is a wave number and ω is a frequency. After some transformations we obtain a system of equations presented below:

$$\begin{aligned} T_{1j3j-1} - T_{1j2j} - \omega^2 M_1 v_j^{M_1} &= 0, & M_{1j3j-1} + M_{1j2j} &= 0, \\ T_{2j1j} - T_{2j3j} - \omega^2 M_2 v_j^{M_2} &= 0, & M_{2j1j} + M_{2j3j} &= 0, \\ T_{3j2j} - T_{3j1j+1} - \omega^2 M_3 v_j^{M_3} &= 0, & M_{3j2j} + M_{3j1j+1} &= 0, \end{aligned} \tag{5.14}$$

which is the system of six homogenous equations for amplitudes. In order to find nonzero solution of the problem, the determinant of the system of equations has to be equal to zero. As a result we arrive at a characteristic equation in the form:

$$\alpha M_3^3 l^9 \omega^6 - \beta M_3^2 l^6 \omega^4 + \gamma M_3 l^3 \omega^2 - B^3 \delta = 0, \tag{5.15}$$

where the dimensionless coefficients are given by:

$$
\begin{aligned}
\alpha \equiv\ & z_1 z_2[3(\gamma_1+\gamma_2)(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)+\gamma_1\gamma_2\gamma_3(2+\cos(kl))],\\
\beta \equiv\ & 3\{z_1 z_2\gamma_1^2(\gamma_2+\gamma_3)^2[1+2\gamma_1(\gamma_2+\gamma_3)(1-\cos(kl))]\\
&\qquad + z_1\gamma_3^2(\gamma_1+\gamma_2)^2[1+2\gamma_3(\gamma_1+\gamma_2)(1-\cos(kl))]\\
&\qquad + z_2\gamma_2^2(\gamma_1+\gamma_3)^2[1+2\gamma_2(\gamma_1+\gamma_3)(1-\cos(kl))],\\
\gamma \equiv\ & 36(z_1+z_2+1)(2+\cos(kl)),\\
\delta \equiv\ & 432(\cos(kl)-1)^2
\end{aligned}
\tag{5.16}
$$

and

$$
\gamma_i \equiv \frac{a_i}{l}, \qquad z_1 \equiv \frac{M_1}{M_3}, \qquad z_2 \equiv \frac{M_2}{M_3}.
$$

Introducing the denotations:

$$
\tilde{\alpha} \equiv 27\alpha^2\delta + 2\beta^3 - 9\alpha\beta\gamma, \qquad \tilde{\beta} \equiv 3\alpha\gamma - \beta^2,
\tag{5.17}
$$

solutions to the characteristic equation can be written as:

$$
\begin{aligned}
\omega_-^2 \equiv\ & \frac{1}{3\alpha}\left[\beta - \frac{1}{\sqrt[3]{2}}\left(\mathrm{Re}\sqrt[3]{\tilde{\alpha}+i\sqrt{-\tilde{\alpha}^2-4\tilde{\beta}^3}}\right.\right.\\
&\qquad\left.\left. - \sqrt{3}\,\mathrm{Im}\sqrt[3]{\tilde{\alpha}+i\sqrt{-\tilde{\alpha}^2-4\tilde{\beta}^3}}\right)\right]\frac{B}{M_3 l^3},\\
\omega_{+:1}^2 \equiv\ & \frac{1}{3\alpha}\left[\beta - \frac{1}{\sqrt[3]{2}}\left(\mathrm{Re}\sqrt[3]{\tilde{\alpha}+i\sqrt{-\tilde{\alpha}^2-4\tilde{\beta}^3}}\right.\right.\\
&\qquad\left.\left. + \sqrt{3}\,\mathrm{Im}\sqrt[3]{\tilde{\alpha}+i\sqrt{-\tilde{\alpha}^2-4\tilde{\beta}^3}}\right)\right]\frac{B}{M_3 l^3},\\
\omega_{+:2}^2 \equiv\ & \frac{1}{3\alpha}\left[\beta - \sqrt[3]{4}\,\mathrm{Re}\sqrt[3]{\tilde{\alpha}+i\sqrt{-\tilde{\alpha}^2-4\tilde{\beta}^3}}\right]\frac{B}{M_3 l^3},
\end{aligned}
\tag{5.18}
$$

where ω_- is the lower frequency, while $\omega_{+:1}$ and $\omega_{+:2}$ are the first and second higher frequency, respectively.

6. Calculation results

Frequencies of the travelling wave can be obtained using the tolerance model and the exact solution. In order to compare results, some numerical examples will be calculated using both the models.

In calculations it is assumed that the plate strip thickness h is constant and equal to $0.1l$. Concentrated mass M_3 is assumed to be a mass of reference, to which masses M_1 and M_2 will be compared. Calculation examples are performed for different mass distribution (coordinates x_1, x_2 and x_3 in the tolerance model and a_1, a_2 and a_3 in the exact solution) and for different mass ratios. There are three calculation cases:

(1°) with symmetric distribution of masses ($x_1 = 0.1l$, $x_2 = 0.5l$, $x_3 = 0.9l$) and with symmetric mass ratios ($M_1/M_3 = 1$, $M_2/M_3 = 3$);

(2°) with uniform, symmetric distribution of masses ($x_1 = (1/6)l$, $x_2 = (1/2)l$, $x_3 = (5/6)l$), but with not symmetric mass ratios ($M_1/M_3 = 3$, $M_2/M_3 = 1$);

(3°) with the same mass proportions as in the (2°) case ($M_1/M_3 = 3$, $M_2/M_3 = 1$), but with not-symmetric distribution of masses ($x_1 = 0.1l$, $x_2 = 0.7l$, $x_3 = 0.9l$).

Frequencies can be presented in the dimensionless form, obtained by the transformation below:

$$w_- = \sqrt{\frac{M_3 l^3}{B}}\omega_-, \qquad w_{+:1} = \sqrt{\frac{M_3 l^3}{B}}\omega_{+:1}, \qquad w_{+:2} = \sqrt{\frac{M_3 l^3}{B}}\omega_{+:2}. \tag{6.1}$$

Results are shown in the form of dispersion curves describing frequencies versus the dimensionless wave number $q \equiv kl \in [-\pi; \pi]$. Curves ES_-, $\mathrm{ES}_{+:1}$,

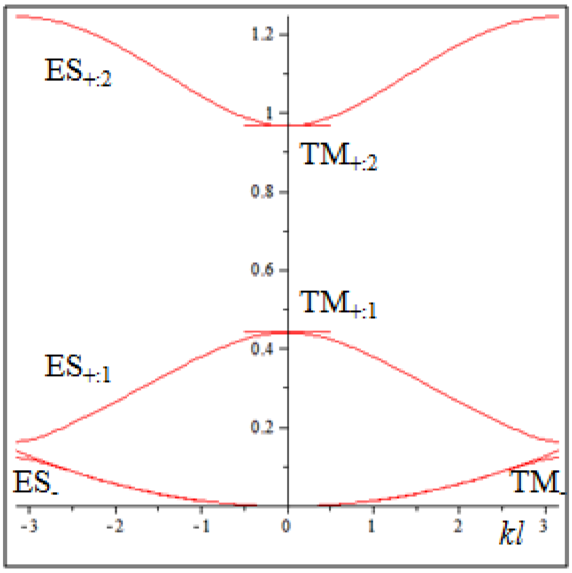

FIG. 4. Dispersion curves of the frequency parameter in the (1°) calculation case.

$ES_{+:2}$ stand for the exact solution, while the plots TM_{-}, $TM_{+:1}$, $TM_{+:2}$ are the results of the tolerance model obtained using the exact form of the mode-shape functions g^A.

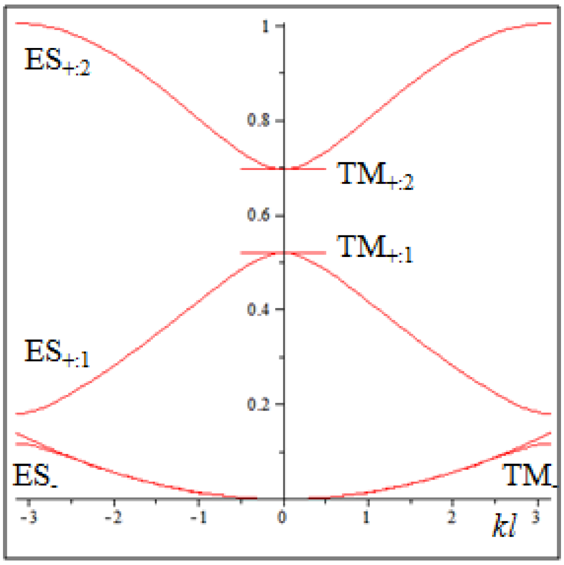

FIG. 5. Dispersion curves of the frequency parameter in the (2°) calculation case.

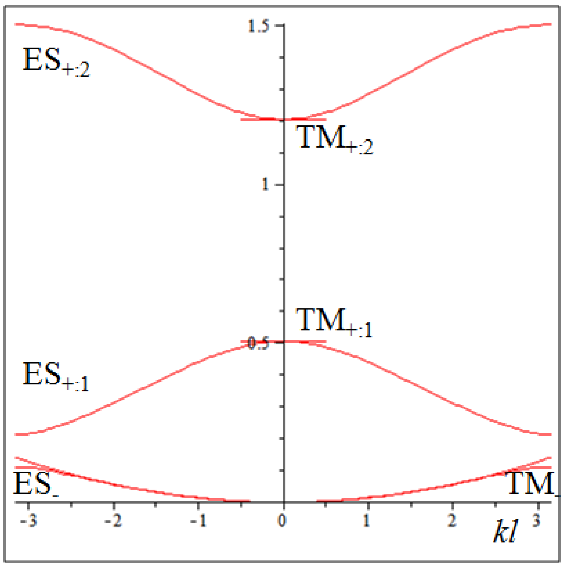

FIG. 6. Dispersion curves of the frequency parameter in the (3°) calculation case.

7. Final remarks

In this paper *the tolerance averaging technique* has been used to obtain governing equations with constant coefficients for thin plate strips. Additionally, the exact solution has been presented in order to evaluate the obtained results.

By analyzing results in Fig. 4–6, one can conclude that:

- the tolerance model can be used to analyze not only the lower frequencies of the travelling wave but also the higher frequencies, which are related to the internal periodic structure and dependent on the microstructure parameter l;
- the lower frequency dispersion curve in the tolerance model is equal to its equivalent obtained in the exact solution in the wide range of dimensionless wave number q;
- differences of results of the higher frequencies calculated using the tolerance model and the exact solution are very small and do not exceed 1% for so-called "long-wave propagation problems", which occur when the dimensionless wave number q is small (e.g. $q \in [-0.1\pi;\ 0.1\pi]$).

Appendix A.

Let us denote coordinates of concentrated masses M_1, M_2, M_3 as x_1, x_2 and x_3, respectively. Bearing in mind that B is a constant plate strip stiffness and l is the *microstructure parameter*, expressions in $\mathbf{L}_{pr}$ matrix can be defined as:

$$
\begin{aligned}
L_{11} &= -\frac{1}{2}\frac{M_1(l^2 - x_1 l + 2x_1^2)}{x_1 l}\omega^2 \\
&\quad - \frac{12[(x_2^3 - 3x_1x_2^2 + 3x_1^2x_2)(x_1 + 3l - 3x_3) + x_1(l - x_3)^2(x_1 + 3l - x_3)]B}{x_1(x_1 - x_2)^3(l - x_3 + x_1)^3}, \\
L_{12} &= -\frac{1}{2}\frac{M_2x_2(l - x_2)(l - 2x_2)}{x_1^2 l}\omega^2 + \frac{12B}{(x_1 - x_2)^3}, \\
L_{13} &= -\frac{1}{2}\frac{M_3x_3(l - x_3)(l - 2x_3)}{x_1^2 l}\omega^2 - \frac{12(x_1 + 3l - 3x_3)B}{x_1(l - x_3 + x_1)^3}, \\
L_{14} &= \frac{6[(-x_1^3 - 3(l - x_3)x_1^2 - 3(l - x_3)^2x_1 + (l - x_3)^3)x_2^2]B}{(l - x_3 + x_1)^3(x_1 - x_2)^2x_1^2} \\
&\quad + \frac{6[2x_1(x_1^3 + 3(l - x_3)x_1^2 + 3(l - x_3)^2x_1 - (l - x_3)^3)x_2 + 2x_1^2(l - x_3)^3]B}{(l - x_3 + x_1)^3(x_1 - x_2)^2x_1^2}, \\
L_{15} &= \frac{6B}{(x_1 - x_2)^2}, \qquad L_{16} = \frac{12(l - x_3)^2B}{x_1(l - x_3 + x_1)^3},
\end{aligned}
$$

$$
\begin{aligned}
L_{21} &= \frac{1}{6}\frac{M_1(l-x_1)(l-2x_1)}{l}\omega^2 \\
&\qquad - \frac{6[x_2(x_2-2x_1)(x_1+3l-3x_3)+(l-x_3)^2(l+3x_1-x_3)]B}{(x_1-x_2)^2(l-x_3+x_1)^3}, \\
L_{22} &= \frac{1}{6}\frac{M_2x_2(l-x_2)(l-2x_2)}{x_1 l}\omega^2 - \frac{6B}{(x_1-x_2)^2}, \\
L_{23} &= \frac{1}{6}\frac{M_3x_3(l-x_3)(l-2x_3)}{x_1 l}\omega^2 + \frac{6(x_1+3l-3x_3)B}{x_1(l-x_3+x_1)^3}, \\
L_{24} &= \frac{2[(2x_1^3+6(l-x_3)x_1^2+6(l-x_3)^2x_1-(l-x_3)^3)x_2+3x_1(l-x_3)^3]B}{(l-x_3+x_1)^3(x_1-x_2)x_1}, \\
L_{25} &= -\frac{2B}{(x_1-x_2)}, \qquad L_{26} = \frac{6(l-x_3)^2B}{x_1(l-x_3+x_1)^3}, \qquad L_{31} = \frac{12B}{(x_1-x_2)^3}, \\
L_{32} &= M_2\omega^2 - \frac{12(x_1-x_3)[(x_1-2x_2+x_3)^2+(x_1-x_2)(x_2-x_3)]B}{(x_1-x_2)^3(x_2-x_3)^3}, \\
L_{33} &= \frac{12B}{(x_2-x_3)^3}, \qquad L_{34} = -\frac{6B}{(x_1-x_2)^2}, \qquad L_{36} = \frac{6B}{(x_2-x_3)^2}, \\
L_{35} &= \frac{6(x_1-x_3)(x_1-2x_2+x_3)B}{(x_1-x_2)^2(x_2-x_3)^2}, \qquad L_{42} = \frac{6(x_1-x_3)(x_1-2x_2+x_3)B}{(x_1-x_2)^2(x_2-x_3)^2}, \\
L_{41} &= \frac{6B}{(x_1-x_2)^2}, \qquad L_{43} = -\frac{6B}{(x_2-x_3)^2}, \qquad L_{44} = -\frac{2B}{(x_1-x_2)}, \\
L_{45} &= \frac{4(x_1-x_3)B}{(x_1-x_2)(x_2-x_3)}, \qquad L_{46} = -\frac{2B}{(x_2-x_3)}, \\
L_{51} &= \frac{1}{2}\frac{M_1x_1(l-x_1)(l-2x_1)}{(l-x_3)^2 l}\omega^2 - \frac{12(l+3x_1-x_3)B}{(l-x_3)(l-x_3+x_1)^3}, \\
L_{52} &= \frac{1}{2}\frac{M_2x_2(l-x_2)(l-2x_2)}{(l-x_3)^2 l}\omega^2 - \frac{12B}{(x_2-x_3)^3}, \\
L_{53} &= -\frac{1}{2}\frac{M_3(2x_3^2-3x_3l+2l^2)}{(l-x_3)l}\omega^2 \\
&\qquad - \frac{12[(l-x_2)(x_3-l-3x_1)(-3x_3^2+3(l+x_2)x_3-(l^2+lx_2+x_2^2))]B}{(x_2-x_3)^3(l-x_3)(l-x_3+x_1)^3} \\
&\qquad - \frac{12x_1^2(x_1+3(l-x_3))B}{(x_2-x_3)^3(l-x_3+x_1)^3}, \\
L_{54} &= \frac{12x_1B}{(l-x_3+x_1)^3}, \qquad L_{55} = -\frac{6B}{(x_2-x_3)^2},
\end{aligned}
$$

$$
\begin{aligned}
L_{56} &= \frac{6[(l-x_2)(l-2x_3+x_2)x_1^3+(l-x_2)[(3l-x_3)^2+(x_2+x_3)^2-2x_2^2]x_1^2]B}{(x_2-x_3)^2(l-x_3)^2(l-x_3+x_1)^3} \\
&\quad + \frac{6[3(l-x_3)^2(l-x_2)(l-2x_3+x_2)x_1+(l-x_3)^3(l-x_2)(l-2x_3+x_2)]B}{(x_2-x_3)^2(l-x_3)^2(l-x_3+x_1)^3}, \\
L_{61} &= \frac{1}{6}\frac{M_1x_1(l-x_1)(l-2x_1)}{(l-x_3)l}\omega^2-\frac{6(l+3x_1-x_3)B}{(l+3x_1-x_3)^3}, \\
L_{62} &= \frac{1}{6}\frac{M_2x_2(l-x_2)(l-2x_2)}{(l-x_3)l}\omega^2+\frac{6B}{(x_2-x_3)^2}, \\
L_{63} &= \frac{1}{6}\frac{M_3x_3(l-2x_3)}{l}\omega^2 \\
&\qquad -\frac{6[x_1^2(x_1+3l-3x_3)+(l-x_2)(l-2x_3+x_2)(3x_1+l-x_3)]B}{(x_2-x_3)^2(l-x_3+x_1)^3}, \\
L_{64} &= \frac{6x_1(l-x_3)B}{(l-x_3+x_1)^3}, \qquad L_{65}=-\frac{2B}{(x_2-x_3)}, \\
L_{66} &= -\frac{4(l-x_2)[x_1^3+(l-x_3)^3]B}{(x_2-x_3)(l-x_3+x_1)^3(l-x_3)} \\
&\qquad -\frac{6x_1[x_1(2l-x_3-x_2)+2(l-x_3)(l-x_2)]B}{(x_2-x_3)(l-x_3+x_1)^3}.
\end{aligned}
$$

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