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SOME METHODS OF SOLVING PROBLEMS OF NON-LINEAR THERMO-VISCOELASTICITY

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The formulation of the problems of non-linear thermo-viscoelasticity, including also coupled problems is given. For that purpose application of thermodynamics of irreversible processes to the visco-elastic media is analysed. Certain methods of subsequent approximations are presented for the solution of problems of non-linear theory of thermo-viscoelasticity, among other "rapidly convergent" method. The rate of convergence of these methods is analysed. Numerical methods are also considered, mainly the method of nets and the method of finite elements.

1. The statement of the problem

In connection with the introduction into the industry of new materials, principally polymers, the theory of viscoelasticity has become of considerable importance for strength computation. This is a relatively general theory of continuum mechanics, in which the relationship between stress and strain is prescribed by means of certain operators in function of time. It comprises, as particular cases, the theory of creep and that of relexation [1].

As a consequence of two laws of continuum mechanics concerning the variation of the momentum and the moment of momentum under absence of couple-stresses, we have three differential equations for the symmetric stress tensor $\underset{\sim}{\sigma}$

$$\operatorname{Div} \underset{\sim}{\sigma} + \rho \mathbf{F} = \rho \mathbf{u}^{\cdot\cdot}, \tag{1.1}$$

where $\mathbf{F}$ — prescribed mass forces and ρ — density of the material. The displacement vector $\mathbf{u}$ is connected with the strain tensor $\underset{\sim}{\varepsilon}$ by the geometrical relations:

$$\underset{\sim}{\varepsilon} = \operatorname{Def} \mathbf{u}. \tag{1.2}$$

To particularize the medium, let us prescribe the operator of the stress-strain relationship $\check{\mathcal{F}}$ (the constitutive equations). If we consider non-isothermal processes, the state of stress of the medium is influenced by the temperature T and the constitutive equations take the form:

$$\underset{\sim}{\sigma} = \underset{\sim}{\check{\mathcal{F}}}(\underset{\sim}{\varepsilon}, T). \tag{1.3}$$

If the Duhamel-Neumann theory is considered to be valid the relation (1.3) may be written

$$\underset{\sim}{\sigma} = \underset{\sim}{\check{\mathcal{F}}}(\underset{\sim}{\varepsilon}^T), \qquad \underset{\sim}{\varepsilon}^T \equiv \underset{\sim}{\varepsilon} - \underset{\sim}{\alpha}\vartheta, \tag{1.4}$$

where $\underset{\sim}{\alpha}$ is the tensor of thermal expansion of the medium, ϑ — the decrease in temperature — that is the difference between the transient temperature T and the temperature of the natural state T_0. On substituting (1.2) and (1.4) into (1.1), we obtain a set of three equations for the components of the displacement vector:

$$\text{Div}\,\check{\underset{\sim}{\mathcal{F}}}(\text{Def}\,\mathbf{u}, T)+\rho\mathbf{F}=\rho\mathbf{u}^{\cdot\cdot}. \tag{1.5}$$

Let the following boundary conditions be prescribed on the surface Σ, bounding the volume V of the medium condsiered:

$$\check{\mathcal{L}}(\mathbf{u}, \check{\underset{\sim}{\mathcal{F}}}(\text{Def}\,\mathbf{u}, T), T)=\mathbf{S}, \tag{1.6}$$

where $\mathbf{S}$ denotes the prescribed external actions on Σ, and $\check{\mathcal{L}}$ is an operator. Let some initial conditions be prescribed in addition, such as, for instance:

$$\mathbf{u}=\mathbf{u}^0, \quad \mathbf{u}^{\cdot}=\mathbf{v}^0 \tag{1.7}$$

for $t=t_0$.

Then, with a prescribed temperature, the problem of mechanics of a continuum consists in integrating the Eqs. (1.5) with the boundary conditions (1.6) and the initial data (1.7).

A medium is geometrically linear or non-linear depending on whether the operator (1.2) is linear or non-linear. A medium is physically linear or non-linear depending on whether the operator (1.3) is linear or non-linear. For a geometrically and physically linear body a problem is said to be linear or non-linear depending on whether the boundary operator $\check{\mathcal{L}}$ (1.6) is linear or non-linear.

If the temperature is not prescribed, then, in order to complete the set of equations (1.5), the principles of phenomenological thermodynamics must be considered. Assuming the Fourier conduction law to be valid, the consequences of the laws of thermodynamics may be written in the form of the relationships [2]

$$d\psi+SdT+W^*\,dt=\check{\underset{\sim}{\mathcal{F}}}(\underset{\sim}{\varepsilon}^T)\,d\underset{\sim}{\varepsilon}, \tag{1.8}$$

$$T\frac{dS}{dt}=\text{div}\,(\underset{\sim}{\lambda}\,\text{grad}\,T)+\rho q+W^*, \tag{1.9}$$

where ψ is the free energy and S — entropy, which depend on certain thermodynamic parameters of state $\underset{\sim}{\mu}_i$, T; W^* — function of dissipation $\rho q(\mathbf{x}, t)$ — mass source of heat, $\underset{\sim}{\lambda}$ — the heat conduction tensor.

Let the following conditions be prescribed at the boundarv Σ of the body

$$\check{g}(T, \partial T/\partial n)=\vartheta_0 \tag{1.10}$$

and the initial data

$$T=T^0 \quad \text{for} \quad t=t_0. \tag{1.11}$$

Then the coupled problem of mechanics of a continuum consists in integrating the set of equations (1.5) and (1.9) with the boundary conditions (1.6) and (1.10) and the initial data (1.7) and (1.11). To solve this problem we must know the function of state $S(\underset{\sim}{\mu}_i, T)$, the dissipation function $W^*\geqslant 0$ and the particular form of the operatir $\check{\mathcal{F}}$. These quantities cannot be arbitrary, since they are interrelated by the Eq. (1.8).

2. THE CONSTITUTIVE EQUATIONS

A visco-elastic body is a body for which the postulate of macroscopic determinateness [3] is satisfied. This postulate states that in the absence of non-mechanical actions (energy supply) and with constant temperature, the stress tensor $\underset{\sim}{\sigma}(t)$ at a time t is completely and unambiguously determined by prescribing the strain tensor $\underset{\sim}{\varepsilon}(\tau)$ at every instant of time before t; $0 \leqslant \tau \leqslant t < t_{\infty}$.

Thus, the stress-strain relationship of the theory of visco-elasticity has the form of the operator tensor

$$\underset{\sim}{\sigma} = \underset{\sim}{\check{\mathcal{F}}}(\underset{\sim}{\varepsilon}), \tag{2.1}$$

which should be invariant under the group of transformations characterizing the class of anisotropy of the body considered. The operator $\underset{\sim}{\check{\mathcal{F}}}$ will be considered to be a mixed functional [4].

The symmetric tensor $\underset{\sim}{A}$ will be termed a mixed functional of a pair of quantities $\{\underset{\sim}{B}, t\}$, where $\underset{\sim}{B}(\tau)$ is a tensor function and t — a number

$$\underset{\sim}{A} = \underset{\sim}{\check{\mathcal{F}}}\{\underset{\sim}{B}(\tau), t\}, \tag{2.2}$$

if the law attaching a certain tensor $\underset{\sim}{A}$ (six numbers) to each pair $\{\underset{\sim}{B}, t\}$ in a certain region D is known. If t is fixed in (2.2), we shall obtain a tensor functional and if $\underset{\sim}{B}(\tau)$ is fixed — a tensor function. Thus, a mixed functional determines a functional operator — that is, an unambiguous correspondence between the tensor $\underset{\sim}{B}(\tau)$ which is considered to be an independent variable and the tensor $\underset{\sim}{A}(\tau)$. Conversely, if a functional operator is prescribed, it can be made to correspond with a definite mixed functional. The expression (2.2) for the relation $\underset{\sim}{\sigma} - \underset{\sim}{\varepsilon}$ will be replaced, for brevity, by (2.1). By locating the tensors $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ in certain function spaces, we can obtain (2.1), making use of theorems of representation of general functionals in various spaces. For the stress-strain relations, these laws were obtained in Refs. [5 to 7].

If the operator $\underset{\sim}{\check{\mathcal{F}}}$ is sufficiently smooth, differentiable a sufficient number of times in the Fréchet's sense, for instance, it can be represented in the following form [1]

$$\sigma_{ij} = \sum_{n=1}^{N} \int_{0}^{t} \dots \int_{0}^{t} \Gamma_{ij}^{(n)\, i_1 j_1 \dots i_n j_n}(t, \tau_1, \dots, \tau_n)\, \varepsilon_{i_1 j_1}(\tau_1) \dots \varepsilon_{i_n j_n}(\tau_n)\, d\tau_1 \dots d\tau_n . \tag{2.3}$$

The quantity N may be also infinity. The kernels $\Gamma^{(n)}$ represent a tensor of order $2(1+n)$ and are referred to as relaxation kernels of order n. These tensors are invariant under a certain group of transformations characterizing the type of mechanical anisotropy. For any type of anisotropy there is a symmetry in the indices i, j, i_k, j_k $(k=1, \dots, n)$ and also in the pairs of indices i_k, j_k and $i_l j_l$ $(k, l=1, \dots, n)$. If there is symmetry in the pairs of indices ij and $i_k j_k$, it is said that the reciprocity conditions are satisfied [7].

If the relaxation kernels of the first order have a singular additive component in the form of a delta function [1], the Eq. (2.3) can be inverted — that is, the strains can be expressed in terms of stresses and all the resolving kernels, which are termed creep kernels, are found by quadratures from the prescribed relaxation kernels [7]

$$\varepsilon_{ij} = \sum_{n=1}^{N} \int_0^t \dots \int_0^t \mathcal{K}_{ij}^{(n)\, i_1 j_1 \dots i_n j_n}(t, \tau_1, \dots, \tau_n)\, \sigma_{i_1 j_1}(\tau_1) \dots \sigma_{i_n j_n}(\tau_n)\, d\tau_1 \dots d\tau_n . \tag{2.4}$$

The model based on the relations (2.3) for $N \to \infty$ requires, in general, an infinite number of experiments for the determination of the relaxation kernels. Therefore, for simplification of these relations, we may confine ourselves to the first N terms of the expansion of (2.3) or (2.4). Such a theory is referred to as the N-fold theory of viscoelasticity. Further simplification of the theory may be achieved by requiring quasi-linearity for the general relations. In addition general theories of viscoelasticity [1] may bé considered. In such theories, only two principal terms of the expansion of the relaxation or creep kernels are preserved according to the degree of singularity. Tests show that for transient, sufficiently small loads, most materials behave as linearly elastic. This justifies the linear terms of physical relations, which are such responsible for transient elasticity, being the only preserved. Theories based on this simplification are fererred to as theories of transient linear elasticity.

Let us write the physical relations of the principal quasi-linear theory of viscoelasticity:

$$\begin{aligned} s_{ij} &= \int_0^t \Gamma(t, \tau)\, e_{ij}(\tau)\, d\tau - \int_0^t \Gamma_n(t, \tau, \theta, e)\, e_{ij}(\tau)\, d\tau , \\ \sigma &= \int_0^t \Gamma_1(t, \tau)\, \theta(\tau)\, d\tau - \int_0^t \Gamma_m(t, \tau, \theta, e)\, \theta(\tau)\, d\tau . \end{aligned} \tag{2.5}$$

The linear and non-linear relaxation kernels contain a singular additive component:

$$\begin{aligned} \Gamma(t, \tau) &= 2G\, \delta(t-\tau) - \tilde{\Gamma}(t, \tau) , \\ \Gamma_1(t, \tau) &= K \delta(t-\tau) - \tilde{\Gamma}_1(t, \tau) , \\ \Gamma_n(t, \tau, \theta, e) &= \varphi(\theta, e)\, \delta(t-\tau) - \tilde{\Gamma}_n(t, \tau, \theta, e) , \\ \Gamma_m(t, \tau, \theta, e) &= \psi(\theta, e)\, \delta(t-\tau) - \tilde{\Gamma}_m(t, \tau, \theta, e) , \end{aligned} \tag{2.6}$$

where G is the shear modulus and K — the bulk moduls of elasticity:

$$\begin{aligned} &\theta \equiv \varepsilon_{ij}(\tau)\, \delta^{ij}, \quad e_{ij}(\tau) \equiv \varepsilon_{ij}(\tau) - \frac{1}{3}\theta(\tau)\, \delta_{ij} , \\ &e(\tau) \equiv e_i{}^j(\tau)\, e^i{}_j(\tau) , \\ &\sigma \equiv \frac{1}{3}\sigma_{ij}(\tau)\, \delta^{ij}, \quad s_{ij}(\tau) \equiv \sigma_{ij}(\tau) - \sigma(\tau)\, \delta_{ij} , \\ &s(\tau) \equiv s_i{}^j(\tau)\, s^i{}_j(\tau) . \end{aligned} \tag{2.7}$$

A particular case of the theory (2.5) is the following quasi-linear theory of visco-elasticity [8]:

(2.8) $$s_{ij}-\int_0^t \Gamma(t-\tau)\,e_{ij}(\tau)\,d\tau-\int_0^t \Gamma_\varphi(t-\tau)\,\varphi(e,\theta)\,e_{ij}(\tau)\,d\tau,$$

(2.9) $$\sigma=\int_0^t \Gamma_1(t-\tau)\,\theta(\tau)\,d\tau-\int_0^t \Gamma_\psi(t-\tau)\,\psi(e,\theta)\,\theta(\tau)\,d\tau.$$

It is assumed that linear and non-linear relaxation kernels can be separated into a singular and regular component:

(2.10) $$\begin{aligned}&\Gamma(t)=2G\,\delta(t)-\tilde{\Gamma}(t), && \Gamma_1(t)=K\delta(t)-\tilde{\Gamma}_1(t),\\ &\Gamma_\varphi(t)=\mathring{\Gamma}_\varphi\,\delta(t)-\tilde{\Gamma}_\varphi(t), && \Gamma_\varphi(t)=\mathring{\Gamma}_\psi\,\delta(t)-\tilde{\Gamma}_\psi(t).\end{aligned}$$

If $\mathring{\Gamma}_\varphi=\mathring{\Gamma}_\psi=0$, the relevant theory is referred to as a quasi-linear theory of transient linear elasticity. If the volume of the body varies in an elastic manner, the relations (2.9) take the form:

(2.11) $$\sigma=K\theta.$$

If we consider an incompressible body, the physical relations (2.8) and (2.9) take the form:

(2.12) $$s_{ij}=\int_0^t \Gamma(t-\tau)\,e_{ij}(\tau)\,d\tau-\int_0^t \Gamma_\varphi(t-\tau)\,\varphi(e)\,e_{ij}(\tau)\,d\tau.$$

If, in the theories (2.8), (2.9) and (2.12), we assume that

(2.13) $$\tilde{\Gamma}(t)=\tilde{\Gamma}_1(t)=\tilde{\Gamma}_\varphi(t)=\tilde{\Gamma}_\psi(t)=0,$$

we obtain from (2.12) the theory of small elastic-plastic strain for active Ilyushin loading [9] and from (2.8) and (2.9) — a generalization of that theory [10]. If $\varphi(e)$ is a linear function of e, the relation (2.12) describes the principal Ilyushin-Ogibalov cubic theory of viscoelasticity [11].

Another particular case of the theory (2.5) is the principal theory of viscoelasticity quadratic in the deviators, which is considered in Ref. [12]:

(2.14) $$\begin{aligned}s_{ij}&=\int_0^t \Gamma(t-\tau)\,e_{ij}(\tau)\,d\tau+\int_0^t Q(t-\tau,\theta)\,\theta(\tau)\,e_{ij}(\tau)\,d\tau,\\ \sigma&=\int_0^t \Gamma_1(t-\tau)\,\theta(\tau)\,d\tau+\int_0^t Q_1(t-\tau,\theta)\,\theta^2(\tau)\,d\tau+\int_0^t Q_2(t-\tau,\theta)\,e(\tau)\,d\tau.\end{aligned}$$

If the reciprocity conditions are satisfied, the non-linear relaxation kernels Q and Q_2 are interdependent:

(2.15) $$Q(t,\theta)=\theta\,\frac{\partial Q(t,\theta)}{\partial\theta}=\frac{2}{3}\,Q_2(t,\theta).$$

It is important to observe that the principal non-linear theories of relaxation and creep are not interrelated by inversion [1]. However, if the relaxation function

$\tilde{\Gamma}(t)$ varies little, we can show two cases, in which they are inverse with reference to each other, within a certain degree of accuracy [1]. In general, however, relations of the principal non-linear theory of relaxation, (2.8) and (2.9), for instance, may be inverted and represented in the form of the principal non-linear theory of creep:

$$(2.16)\qquad \begin{aligned} e_{ij} &= \int_0^t \mathcal{K}(t-\tau)s_{ij}(\tau)\,d\tau + \int_0^t \mathcal{K}_\xi(t-\tau)\xi(\sigma, s)s_{ij}(\tau)\,d\tau\,, \\ \theta &= \int_0^t \mathcal{K}_1(t-\tau)\sigma(\tau)\,d\tau + \int_0^t \mathcal{K}_\eta(t-\tau)\eta(\sigma,\rho)\sigma(\tau)\,d\tau, \end{aligned}$$

the non-linear creep kernels being considered to be functionals of the stress tensor. If a particular loading process is considered, we can find non-linear creep kernels in terms of the known non-linear relaxation kernels by the method of iteration [1].

For the determination of linear and non-linear relaxation and creep kernels, the simplest experiments are used [1, 8, 13]. The most general of the existing expressions of the operator of the stress-strain relationship is (2.3). This, is of course, a very narrow class of operators. (It may be compared with a function represented at zero by Taylor's series only). All the other existing theories, including the theory of creep, are particular cases of the theory (2.3), (2.4).

3. The coupled problem of thermo-viscoelasticity

Rigorous statement of the coupled problem of thermo-viscoelasticity (outlined in Section 1) requires additional thermodynamic assumptions. As a fundamental assumption, we suppose that the parameters of state $\underset{\sim}{\mu}_i$ are operators of the tensor $\underset{\sim}{\varepsilon}^T$, of the same nature as the operator $\check{\underset{\sim}{\mathcal{F}}}$ discussed in the foregoing section:

$$(3.1)\qquad \underset{\sim}{\mu}_i = \underset{\sim}{\mu}_i(\underset{\sim}{\varepsilon}^T) = \underset{\sim}{\mu}_i(\underset{\sim}{\varepsilon} - \underset{\sim}{\alpha}\vartheta)\,.$$

Then, we have the following relations for a deformable solid [2]:

$$(3.2)\qquad \begin{aligned} &\frac{\partial\psi}{\partial T} - \frac{\partial\psi}{\partial\underset{\sim}{\mu}_i}\underset{\sim}{\mu}_i^\varepsilon\underset{\sim}{\alpha} = -S, \\ &\frac{\partial\psi}{\partial\underset{\sim}{\mu}_i}\underset{\sim}{\mu}_i^\varepsilon = \check{\underset{\sim}{\mathcal{F}}}(\underset{\sim}{\varepsilon}^T), \\ &\frac{\partial\psi}{\partial\underset{\sim}{\mu}_i}\underset{\sim}{\mu}_i^t = -W^*, \qquad i = 1, 2, \ldots, N, \end{aligned}$$

where

$$\delta\underset{\sim}{\mu}_i = \underset{\sim}{\mu}_i^\varepsilon\,d\underset{\sim}{\varepsilon}^T + \underset{\sim}{\mu}_i^t\,dt$$

is the complete variation of the parameter of state $\underset{\sim}{\mu}_i$. The dissipation function has the form:

$$(3.3)\qquad W^* = \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}^T)^{\cdot} - \Big(\sum_{i=1}^N \underset{\sim}{\mu}_i^2\Big)^{\cdot}.$$

The equation of heat supply (1.9) reduces to the form:

$$\rho C_p T^{\cdot} - \operatorname{div}(\underset{\sim}{\lambda} \operatorname{grad} T) = -T_0 [\underset{\sim}{\alpha} \check{\underset{\sim}{\mathcal{F}}}(\underset{\sim}{\varepsilon}^T)]^{\cdot} + \rho q + W^*, \tag{3.4}$$

where T_0 is the "mean" temperature [1]. It is clear that the expression (3.3) with implicit parameters of state $\underset{\sim}{\mu}_i$ is not convenient for actual computation; therefore, an additional assumption is introduced concerning the structure of the parameters $\underset{\sim}{\mu}_i$ — for example, the assumption that the number of such parameters is $N=1$. Then, for a quasi-linear operator $\check{\underset{\sim}{\mathcal{F}}}$ in an isotropic medium:

$$W^* = s^{ij} e_{ij} + \sigma \theta^T - \frac{1}{2G} (s_{ij} s^{ij})^{\cdot} - \frac{1}{K} (\sigma^2)^{\cdot}. \tag{3.5}$$

In particular, it follows from (3.5) that $W^*=0$ for an elastic body and

$$W^* = \sigma_n [e_n \omega(e_n)]^{\cdot} \tag{3.6}$$

for an elastic-plastic body, where

$$\sigma_n = (s^{ij} s_{ij})^{1/2}, \qquad e_n = (e^{ij} e_{ij})^{1/2},$$

and $\omega(e_n)$ is the Ilyushin plasticity function.

For a quasi-linear isotropic viscoelastic body $\alpha_{ij} = \alpha\delta_{ij}$, $\lambda_{ij} = \lambda\delta_{ij}$, and the equation of heat supply may be written in the form:

$$\rho C_p T^{\cdot} - \lambda \Delta T = -3\alpha T_0 \sigma^{\cdot} + \rho q + W^*. \tag{3.7}$$

If the volume does not undergo relaxation, that is if the relation (2.11) holds, the Eq. (3.7) is equivalent to the equation:

$$\rho C_v T^{\cdot} - \lambda \Delta T = -3\alpha K T_0 \theta^{\cdot} + \rho q + W^*, \tag{3.8}$$

where

$$C_v = C_p - 9\alpha K T_0, \tag{3.9}$$

where Δ is the Laplacian operator.

The theorem of existence is for a certain class of quasi-static coupled problems of non-linear thermo-viscoelasticity, demonstrated in Ref. [2], and the uniqueness theorem for dynamic and quasi-static problems of the linear theory of thermo-viscoelasticity — in Ref. [14].

4. Iteration methods

Let us consider the non-coupled quasi-static problem of viscoelasticity. The equations of equilibrium in a certain system of coordinates have the form:

$$\sigma_{ij,j} + \rho F_i = 0. \tag{4.1}$$

The deformations will be considered to be small:

$$\varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{4.2}$$

Let the zero displacements be prescribed in a part Σ_1 of the boundary Σ of the body, the volume of which is V, and let a load S_i^0 be prescribed on the other part of the boundary, Σ_2:

$$u_i|_{\Sigma_1}=0, \quad \sigma_{ij} l_j|_{\Sigma_2}=S_i^0. \tag{4.3}$$

On substituting the relations (1.3) into (4.1) and making use of (4.2), we obtain a system of three integro-differential equations for the displacement vector $\mathbf{u}$

$$\check{\sigma}_{ij,j}(\mathbf{u}, T)+\rho F_i=0 \tag{4.4}$$

with the boundary conditions (4.3).

The iteration methods for solving the problem (4.4), (4 3) consist in what follows. The tensor operator (2.3) is represented in the form of a sum of two operators — a linear operator $\check{\sigma}_{ij}^0$, of any form, and a non-linear operator σ_{ij}^1:

$$\check{\sigma}_{ij}(\varepsilon_{kl}^T)=\check{\sigma}_{ij}^0(\varepsilon_{kl}^T)+\check{\sigma}_{ij}^1(\varepsilon_{kl}^T). \tag{4.5}$$

We assume that the linear problem

$$\check{\sigma}_{ij,j}^0(\mathbf{u}, T)+\rho F_i=0, \tag{4.6}$$

$$u_i|_{\Sigma_1}=0, \quad \check{\sigma}_{ij}^0(\mathbf{u}, T)\, l_j=S_i^0, \tag{4.7}$$

has a unique solution, which is considered to be a zero approximation $\mathbf{u}_{(0)}$ to the non-linear problem.

Next we build up subsequent approximations by solving, at each iteration step, the linear problem:

$$\check{\sigma}_{ij,j}^0(\mathbf{u}_{(n+1)}, T)=\check{\sigma}_{ij,j}^0(\mathbf{u}_{(n)}, T)-\beta_{(n)}[\check{\sigma}_{ij,j}\mathbf{u}_{(n)}, T)+\rho F_i]; \tag{4.8}$$

$$\begin{gathered} u_{i\,(n+1)}|_{\Sigma_1}=0, \\ \check{\sigma}_{ij}^0(\mathbf{u}_{(n+1)}, T)\, l_j|_{\Sigma_2}=\check{\sigma}_{ij}^0(\mathbf{u}_{(n)}, T)\, l_j|_{\Sigma_2}-\beta_{(n)}[\check{\sigma}_{ij}(\mathbf{u}_{(n)}, T)\, l_j|_{\Sigma_2}-S_i^0], \end{gathered} \tag{4.9}$$

where $\beta_{(n)}$ is a sequence of positive numbers (iteration parameters), by selection of which we can improve the convergence. The separation (4.5) may be effected at each iteration step so that the operators $\check{\sigma}_{ij}^0$ and σ_{ij}^1 depend, in general, on the order of the iteration step.

In the theory of plasticity, the method just described is for $\beta_{(n)}=1$, known under the name of the method of elastic solutions, put forward by ILYUSHIN [9]. Its convergence was demonstrated in Ref. [15]. For the particular case of the quasi--linear theory of viscoelasticity with transient linear elasticity (2.12) the convergence of the method was demonstrated in Ref. [16]. Problems of existence were considered in Ref. [17], in which the convergence of the iteration method was also shown for general quasi-linear systems. In Ref. [8] these methods were generalized to quasi-linear operator systems. It was assumed that if the tensor operator $\check{\sigma}_{ij}$ (1.3) is sufficiently regular (it is differentiable in the sense of Fréchet, for example, in a certain functional space H), the operator $\check{\sigma}_{ij}^0$ may be assumed in the form of the first differential of the operator $\check{\sigma}_{ij}$. In such a case the weak differential is identical with the strong (Fréchet) differential. The first differential $D\check{\sigma}_{ij}$ and the functional derivatives $\partial\sigma_{ij}/\partial\varepsilon_{kl}$ can therefore be found from the formulae:

$$D\check{\sigma}_{ij}\{\varepsilon_{kl}, h_{kl}\}=\left[\frac{\partial\check{\sigma}_{ij}}{\partial\varepsilon_{kl}}h_{kl}\right]=\frac{d}{d\xi}\check{\sigma}_{ij}\{\varepsilon_{kl}+\xi h_{kl}\}|_{\xi=0}\,. \tag{4.10}$$

Similarly, for the second differential $D^2\check{\sigma}_{ij}$ and the second functional derivatives we have:

$$D^2\sigma_{ij}\{\varepsilon_{kl}, h_{kl}\}=\left[\frac{\partial^2\sigma_{ij}}{\partial\varepsilon_{kl}\partial\varepsilon_{mn}}h_{kl}h_{mn}\right]=\frac{d^2}{d\xi^2}\check{\sigma}_{ij}\{\varepsilon_{kl}+\xi h_{kl}\}|_{\xi=0}\,, \tag{4.11}$$

where h_{kl} denotes the tensor of the strain increment. The operator $D\check{\sigma}_{ij}$ is linear in h_{kl}. Let us introduce the notations

$$\begin{aligned}\int_{\Omega} f(\mathbf{x},\tau)\,d\Omega&=\int_0^t d\tau\int_V^t f(\mathbf{x},\tau)\,dV,\\ \int_{\Xi} f(\mathbf{x},\tau)\,d\Xi&=\int_0^t d\tau\int_{\Sigma} f(\mathbf{x},\tau)\,d\Sigma.\end{aligned} \tag{4.12}$$

By the term weak solution [18] of the problem of the theory of viscoelasticity (4.4) and (4.3), we shall understand the function vector $\mathbf{u}$ satisfying, for any continuous differentiable function vector $\mathbf{v}$, the integral identity:

$$\int_{\Omega}\check{\sigma}_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,d\Omega=\int_{\Omega}\rho F_i v_i\,dV+\int_{\Xi_2}S_i^0 v_i\,d\Xi. \tag{4.13}$$

Let us consider the following functional spaces $H_1(\Omega)$, $H_2(\Omega)$, $H_1^-(V)$, $H_2^-(V)$, into which scalar products have been introduced according to the equations:

$$(\mathbf{u},\mathbf{v})_{H_1}=\int_{\Omega}\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,d\Omega\,, \tag{4.14}$$

$$(\mathbf{u},\mathbf{v})_{H_2}=\int_{\Omega}[\check{C}_{ijkl}\varepsilon_{kl}(\mathbf{u})]\varepsilon_{ij}(\mathbf{v})\,d\Omega\,, \tag{4.15}$$

$$(\mathbf{u},\mathbf{v})_{H_1^-}=\int_V\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,dV, \tag{4.16}$$

$$(\mathbf{u},\mathbf{v})_{H_2^-}=\int_V C_{ijkl}\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,dV, \tag{4.17}$$

where the tensor C_{ijkl} and the operator tensor $\check{C}_{ijkl}$ are positively definite and symmetric in the indices i, j; k, l and the pairs of indices ij, kl. Then, the following theorems are valid [8].

THEOREM 1. *Let* $\mathbf{u}_{(0)}\in H_2(\Omega)$ *exist such that the linear problem* (4.6), (5.7) *has a unique solution and the inequalities*

$$\check{\sigma}^1_{ij}(\mathbf{u}_{(0)})h_{ij}\leqslant m\,[\check{C}_{ijkl}h_{kl}]h_{ij}\,, \tag{4.18}$$

$$m\,[\check{C}_{ijkl}h_{kl}]h_{ij}\leqslant\left[\frac{\partial\check{\sigma}_{ij}}{\partial\varepsilon_{kl}}h_{kl}\right]h_{ij}\leqslant M\,[\check{C}_{ijkl}h_{kl}]h_{ij} \tag{4.19}$$

are satisfied for any symmetric tensor h_{ij}, *where* $0<m\leqslant M<\infty$. *Let*

$$\rho F_i \in L_p(\Omega), \quad p>\frac{4}{3}, \qquad S_i^0 \in L_p(\Xi), \quad p>\frac{3}{2}. \tag{4.20}$$

Then, there exists in some neighbourhood $\|\mathbf{u}-\mathbf{u}_{(0)}\|\leqslant r$ *a weak solution* u^* *of the problem* (4.4), (4.3), *which is unique in that neighbourhood. To this solution converges, for any* $\beta\in(0, 2/(m+M)]$, *an iteration process beginning from* $\mathbf{u}_{(0)}$ (4.8), (4.9), *and*

$$\|\mathbf{u}_{(n)}-\mathbf{u}^*\|_2 \leqslant q^n \|\mathbf{u}_{(0)}-\mathbf{u}^*\|_2 \tag{4.21}$$

where

$$q=1-\beta m. \tag{4.22}$$

From this theorem it follows that if we have found the first approximation $\mathbf{u}_{(1)}$ and if the solution is to be found with an accuracy δ — that is $\|\mathbf{u}_{(n)}-\mathbf{u}^*\|_2<\delta$ — the number n of iterations necessary for the achievement of such an accuracy is found from the inequality:

$$n>\frac{1}{\ln q}\ln\frac{\delta(1-q)}{\|\mathbf{u}_{(1)}-\mathbf{u}_{(0)}\|_2}. \tag{4.23}$$

The best convergence will be achieved for $\beta=2/(m+M)$. The value of β may be changed at each iteration step, so that $\beta_{(n)}\in(0, 2/(m+M)]$. Let us observed also that if there exists a "linearity region" of the operator $\check{\sigma}_{ij}(\varepsilon_{kl})$, and if the zero approximation belongs to that region, the condition (4.18) is satisfied automatically.

We define as a generalized solution of the theory of viscoelasticity (4.4), (4.3) the function vector $\mathbf{u}$ satisfying for any continuously differentiable function vector $\mathbf{v}$ the integral identity:

$$\int_V \check{\sigma}_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,dV=\int_V \rho F_i v_i\,dV+\int_V S_i^0 v_i\,d\Sigma. \tag{4.24}$$

Then, a generalized solution of the problem (4.4), (4.3) can be found by the method of elastic solutions.

THEOREM 2. *Let* $\mathbf{u}_{(0)}\in H_{\bar{2}}(V)$ *exist such that the elastic problem corresponding to the linear problem of viscoelasticity* (4.6), (4.7) *has a unique solution. In addition let the conditions* (4.18), (4.19) *be satisfied and*

$$\rho F_i\in L_p(V), \quad p>\frac{6}{5}, \qquad S_i^0\in L_p(\Sigma), \quad p>\frac{4}{3}.$$

Then, there exists in a certain neighbourhood

$$\|\mathbf{u}-\mathbf{u}_{(0)}\|_{\bar{2}}\leqslant r \tag{4.25}$$

a generalized solution $\mathbf{u}^{**}$ *of the problem* (4.4), (4.3), *unique in that neighbourhood and such that the iteration process* (4.8), (4.9) *converges to it for any* $\beta\in(0, 2/(m+M)]$ *beginning from* $\mathbf{u}_{(0)}$, *and*

$$\|\mathbf{u}_{(n)}-\mathbf{u}^{**}\|_{\bar{2}}\leqslant q^n\|\mathbf{u}_{(0)}-\mathbf{u}^{**}\|_{\bar{2}}, \tag{4.26}$$

where q *is determined by the Eq.* (4.22).

The following conclusion is drawn from the above theorem. If Green's tensor is known for a linear elastic problem in a given region under prescribed boundary conditions, the solution of the corresponding problem of the non-linear theory of viscoelasticity is found by quadratures in the form of recurrence relations.

The iteration processes considered converge at a rate of geometrical progression. If the factor q of that progession is near unity the convergence is relatively slow.

If for a linear problem of the anisotropic theory of viscoelasticity a solution can be found by any means, thus expressing the relaxation kernels in function of the coordinates, the rapidly converging iteration method explained in Ref. [19] can be used for the solution of the non-linear problem.

With this method, $\check{\sigma}_{ij}^0$ can be selected in the form of a Fréchet differential of the operator tensor $\check{\sigma}_{ij}(\varepsilon_{kl})$ based on an approximation obtained by requiring that the following conditions may be satisfied for each tensor h_{ij}

(4.27) $$m_1 h_{ij} h_{ij} \leqslant \left[\frac{\partial \check{\sigma}_{ij}}{\partial \varepsilon_{kl}} h_{kl}\right] h_{ij} \leqslant M_1 h_{ij} h_{ij},$$

(4.28) $$\left|\left[\frac{\partial^2 \check{\sigma}_{ij}}{\partial \varepsilon_{kl}\, \partial \varepsilon_{mn}} h_{kl} h_{mn}\right] h_{ij}\right| \leqslant L (h_{ij} h_{ij})^{3/2},$$

where m_1, M_1, L are positive constants with a dimension of stress.

THEOREM 3. *Let a* $\mathbf{u}_{(0)} \in H$ *exist such that the linear problem* (4.6), (4.7) *has a unique solution and the conditions*, (4.20), (4.27) *and* (4.28) *are satisfied. In addition let be a positive number such that*

(4.29) $$\int_\Omega \check{\sigma}_{ij}^1(\mathbf{u}_{(0)}) \varepsilon_{ij}(\mathbf{u}_{(0)})\, d\Omega \leqslant m_1 a \int_\Omega \varepsilon_{ij}(\mathbf{u}_{(0)}) \varepsilon_{ij}(\mathbf{u}_{(0)})\, d\Omega,$$

where $\check{\sigma}_{ij}^0(\mathbf{u}_{(0)})$ *is found from the Eq.* (4.5). *Then number* α, $0 < \alpha \leqslant 1$, *can be found such that the problem* (4.4), (4.3) *has a unique weak solution in the neighbourhood* $\|\mathbf{u}_{(0)} - \mathbf{u}^*\|_1 \leqslant r_0$, *if the inequality* $q \leqslant a^{-\alpha} C$ *is satisfied, where* $q \equiv \frac{3}{2} \frac{L}{m_1} \Omega^{-\frac{\alpha}{2}}$, $C \equiv \alpha(1+\alpha)^{-\frac{1+\alpha}{\alpha}}$ *and* r_0, r_1 *are the smaller and the greater root of the equation* $qr^{1+\alpha} - r + a = 0$, *respectively.*

For $\beta = 1$, *the iteration process* (4.8), (4.9) *tends to the above solution, and*

(4.30) $$\|\mathbf{u}_{(n)} - \mathbf{u}^*\|_1 \leqslant q^{\frac{(1+\alpha)^n - 1}{\alpha}} \|\mathbf{u} - \mathbf{u}^*\|_1^{(1+\alpha)^n}.$$

The latter inequality can be expressed in the form:

(4.31) $$\|\mathbf{u}_{(n)} - \mathbf{u}_{(0)}\|_1 \leqslant C_1 \delta^{(1+\alpha)^n},$$

where the quantity C_1 *can be made arbitrarily small by selecting an appropriate zero approximation. Then, making use of the Stirling formula, we obtain, for large* n,

(4.32) $$\|\mathbf{u}_{(n)} - \mathbf{u}^*\|_1 \leqslant (n!)^{-(1+\alpha)^n n^{-2}}.$$

Similarly to the Theorem 2, which results from the Theorem 1 — that is, similarly to the reduction to an elastic solution — we can formulate a theorem resulting from Theorem 3, making use of the space $H_{\overline{1}}(V)$ with the scalar product (4.16) [8].

The coupled problem of thermo-viscoelasticity can be solved by the method of a small parameter. Let us suppose that the operator $\check{\mathcal{F}}(\underset{\sim}{\varepsilon}^T)$ can be expanded in some manner in a series of the parameter $\kappa \equiv \alpha T_0$:

$$\check{\underset{\sim}{\mathcal{F}}}(\underset{\sim}{\varepsilon}^T)=\check{\underset{\sim}{\mathcal{F}}}_0(\underset{\sim}{\varepsilon})+\kappa\check{\underset{\sim}{\mathcal{F}}}_1(\underset{\sim}{\varepsilon}^T)+\kappa^2\check{\underset{\sim}{\mathcal{F}}}_2(\underset{\sim}{\varepsilon}^T)+\dots, \tag{4.33}$$

where $\check{\underset{\sim}{\mathcal{F}}}_0(\underset{\sim}{\varepsilon})$ is an operator independent of α. Then, the solution of the problem stated is sought for in the form of power series:

$$\mathbf{u}=\sum_{n=0}^{\infty}\mathbf{u}_{(n)}\kappa^n, \qquad T=\sum_{n=0}^{\infty}T_{(n)}\kappa^n. \tag{4.34}$$

To obtain a zero approximation we solve the non-coupled problem:

$$\operatorname{Div}\check{\underset{\sim}{\mathcal{F}}}_0(\operatorname{Def}\mathbf{u})+\rho\mathbf{F}=\rho\mathbf{u}^{\cdot\cdot}, \tag{4.35}$$

$$\rho C_p T^{\cdot}-\operatorname{div}(\underset{\sim}{\lambda}\operatorname{grad}T)=\rho q+W_0^*, \tag{4.36}$$

with the boundary conditions

$$\check{\mathcal{L}}(\mathbf{u},\check{\mathcal{F}}(\operatorname{Def}\mathbf{u}))=\mathbf{S}, \tag{4.37}$$

$$\check{g}\left(T,\frac{\partial T}{\partial n}\right)=\vartheta_0, \tag{4.38}$$

and the initial data (1.7), (1.11).

We first solve the Eqs. (4.35) with the boundary and initial conditions (4.37) and (1.7), thus finding the zero approximation $\mathbf{u}_{(0)}$. We substitute this into the expression of W_0^* and solve the equation of heat conduction (4.36) with a heat source $\rho q+W^*(\mathbf{u}_{(0)})$. For the subsequent approximations, $n\geqslant 1$, we have

$$\operatorname{Div}\check{\underset{\sim}{\mathcal{F}}}_{(n)}(\operatorname{Def}\mathbf{u}_{(n)}-\underset{\sim}{\alpha}\vartheta_{(n-1)})=\rho\mathbf{u}_{(n)}^{\cdot\cdot}, \tag{4.39}$$

$$\rho C_p T^{\cdot}-\operatorname{div}(\underset{\sim}{\lambda}\operatorname{grad}T_{(n)})=T_0[\check{\underset{\sim}{\mathcal{F}}}(\underset{\sim}{\varepsilon}_{(n-1)}-\underset{\sim}{\alpha}\vartheta_{(n-1)})]^{\cdot}+W_n^*(\mathbf{u}_{(n)}), \tag{4.40}$$

with the boundary conditions

$$\check{\mathcal{L}}(\mathbf{u},\check{\underset{\sim}{\mathcal{F}}}(\operatorname{Def}\mathbf{u}_{(n)}))=0, \tag{4.41}$$

$$\check{g}\left(T,\frac{\partial T}{\partial n}\right)=0, \tag{4.42}$$

and the homogeneous initial conditions. We solve the Eqs. (4.39) with the boundary conditions (4.41). Then, by substituting the solution $\mathbf{u}_{(n)}$ into W_n^*, we solve the equation of heat conduction with a heat source $T_0\check{\underset{\sim}{\mathcal{F}}}^{\cdot}+W_n^*$.

The convergence of the series (4.34) must be verified in each particular case.

If there exists Green's function $D(\mathbf{x},t)$ for the equations of heat conduction in the region considered. The assumption which was used for the proof of the existence theorem [2] may be used as a method for solving the coupled problem.

Let the operator tensor $\check{\sigma}_{ij}=\check{\mathcal{F}}_{ij}(\varepsilon_{kl})$ be potential — that is, let an operator $\check{W}$ exist such that

$$\frac{\partial \check{W}}{\partial \varepsilon_{ij}}=\check{\sigma}_{ij}(\varepsilon_{kl})\,. \tag{4.43}$$

Let us denote

$$\check{\varphi}\equiv\int_{\Omega}\check{W}d\Omega\,,\qquad A\{\mathbf{u}\}\equiv\int_{\Omega}\rho F_i u_i d\Omega+\int_{\Xi_2}S_i^0 u_i\,d\Xi. \tag{4.44}$$

Then, the relations (4.13) may be expressed in the form:

$$D\check{\varphi}\{\varepsilon_{ij}(\mathbf{u}),\varepsilon_{ij}(\mathbf{v})\}\equiv D\check{\varphi}\{\mathbf{u},\mathbf{v}\}=A\{\mathbf{v}\}, \tag{4.45}$$

where $D\check{\varphi}$ is determined by the formula (4.10).

Then, the following theorem of minimum of a functional is valid.

THEOREM 4. *Let the conditions* (4.19) *be satisfied in the region*:

$$\|\mathbf{u}-\mathbf{u}_{(0)}\|\leqslant r\,. \tag{4.46}$$

Then, if a weak solution of the problem $\mathbf{u}^*$ *constitutes an internal point of that region, the functional*

$$\Phi\equiv\varphi\{\mathbf{u}\}-A\{\mathbf{u}\} \tag{4.47}$$

has a minimum at that point, unique in the region (4.46).

5. NUMERICAL METHODS IN THE NON-LINEAR THEORY OF THERMO-VISCOELASTICITY

We have considered, above all, analytic methods for solving boundary-value problems of the non-linear theory of viscoelasticity. However, in most cases important to practict it is extremely difficult, and usually impossible to find an analytic solution of a linear elastic problem. Therefore approximate methods have found application in modern theoretical physics, among which a special role is played, owing to the possibility of electronic computers being used, by difference methods [20 to 26], which are also successfully used for solving problems of elasticity [27, 28] and those of the theory of viscoelasticity [29, 30]. The difference methods are closely connected with what is referred to as "direct" methods-that is, variational methods, Bubnov-Galerkin methods those of least squares etc. [31]. To variational methods is also related the method of finite elements, which has recently found wide application in works [32, 33].

Let us consider the Ritz method for solving quasi-static boundary-value problems of the non-linear theory of viscoelasticity (4.4), (4.3). This method consists in obtaining, in a class B of functions, an approximate value of the function vector $\mathbf{u}^*$ for which the functional φ (4.47) reaches its minimum in B. On the other hand, the functional φ reaches in the space H_2 its minimum for the wealk solution of the boundary-value problem (4.4), (4.3). As a class B, let us consider the function vectors $\mathbf{u}_{(N)}$, which can be expressed in the form:

$$\mathbf{u}_{(N)}=\sum_{k=1}^{N} C_k \boldsymbol{\psi}_{(k)}(\mathbf{x}, t)\,, \tag{5.1}$$

where $\boldsymbol{\psi}_{(k)}(\mathbf{x}, t)$ are functions of the coordinates satisfying the kinematic boundary conditions (4.2) [31]. The coefficients C_k $(k=1, 2, ..., N)$ are found from the conditions of minimum of the function

$$\Phi(C)\equiv\Phi(C_1, ..., C_N)=\check{\Phi}\{\mathbf{u}_{(N)}\}=\int_{\Omega}\varphi\{\mathbf{u}_{(N)}\}\,d\Omega-A\{\mathbf{u}_{(N)}\} \tag{5.2}$$

— that is, from the set of equations

$$\frac{\partial\Phi}{\partial C_m}=\chi_m(C)\equiv\int_{\Omega}\check{\sigma}_{ij}(\mathbf{u}_{(N)})\,\varepsilon_{ij}(\boldsymbol{\psi}_{(m)})\,d\Omega-\int_{\Omega}\rho F_i\,\psi_{i(m)}\,d\Omega-\int_{\Xi_2}S_i^0\psi_{i\,(m)}\,d\Xi\,, \quad m=1, ..., N\,. \tag{5.3}$$

Let us introduce the following positively definite matrix of order N

$$\Lambda_{\alpha\beta}\equiv\int_{\Omega}[\check{C}_{ijkl}\,\varepsilon_{kl}\{\boldsymbol{\psi}_{(\alpha)}\}]\,\varepsilon_{ij}(\boldsymbol{\psi}_{(\beta)})\,d\Omega\,. \tag{5.4}$$

To solve the non-linear set of algebraic equations (5.3), use can be made of the iteration methods described above. For this it is necessary that the finite dimensional analogue of the inequalities (4.19) should be satisfied. It is easy to observe that the first Fréchet differential of the left-hand part of the operator equation (5.3) is a Jacobian matrix $\partial\chi_m/\partial C_n$ $(m, n=1, ..., N)$.

We have

THEOREM 1. *The Jacobian matrix of the set of equations* (5.3) *is uniformly positively definite — that is, the following inequality holds for any N-dimensional vector* $\{a_1, ..., a_N\}$:

$$\frac{\partial\chi_i}{\partial C_j}a_j a_i\geqslant m\lambda_N a_j a_j\,, \qquad i,j=1, 1 ..., N\,, \tag{5.5}$$

where λ_N *is the least eigenvalue of the matrix* Λ (5.4). *In this case, the problem of solvability of* (5.3) *must additionally be studied.*

Let us consider now the problem of a finite-difference solution of the problem (4.4), (4.3). Le us replace the region Ω by a lattice region Ω^h, the boundary of which is Ξ^h and let us replace the sought for function vector $\mathbf{u}$ und the prescribed functions $\rho\mathbf{F}$, $\mathbf{S}^0$ by the lattice functions u^h, ρF^h, S^h. We assume that certain lattice analogues of the operators $\varepsilon_{ij}(\mathbf{u})$, $\check{\sigma}_{ij}(\mathbf{u})$ have been introduced in such a manner, that the inequalities (4.19) are satisfied. We shall not consider problems of approximation of these lattice operators and we shall consider only one of the possible methods for constructing a variational scheme for the problem (4.4), (4.3) and a method for solving the difference problem thus stated. The symbol $u_k^{(N)}$ means that three vector components are prescribed for each of the (N_1, N_2, N_3) nodes selected from the rectangular lattice, with the number (k_1, k_2, k_3). Let us write the expression for the difference analogue of the functional φ (4.47):

$$\Phi(u_k^{(N)}) = \sum_k A_k^{(N)} \check{\varphi}\{u_k^{(N)}\} + A^h\{u_k^{(N)}\}. \tag{5.6}$$

The condition of minimum of that functional leads to a non-linear set of algebraic equations:

$$\frac{\partial \Phi^h}{\partial u_k^{(N)}} \equiv \chi_k^{(N)} \equiv \sum_{\Omega_m^h} A_m^{(N)} \sum_{\Omega_k^h} \check{\sigma}_{ij}^h (u_k^{(N)}) B_{ij,k}^{(N)} - \sum_{\Omega_k^h} C_k^{(N)} \rho F_k^{(N)} - \sum_{\Xi_k^h} D_k^{(N)} S_k^{(N)},$$

where $A_k^{(N)}, C_k^{(N)}, D_k^{(N)}, B_{ij,k}^{(N)}$ are coefficients of the quadrature equations, where

$$B_{ij,k}^{(N)} = \frac{\partial \varepsilon_{ij}^h (u_m^{(N)})}{\partial u_k^{(N)}}. \tag{5.8}$$

THEOREM 2. *The Jacobian matrix of the set of equations* (5.7) *is uniformly positively definite — that is, for any N-dimensional vector we have*

$$\frac{\partial \chi_k^{(N)}}{\partial u_l^{(N)}} a_l a_k \geqslant \lambda a_l a_l. \tag{5.9}$$

It is essential to observe that if we require the existence of a second Fréchet derivative of the operator $\underset{\sim}{\check{\mathcal{F}}}$, use can be made of the rapidly convergent iteration method. If the difference analogue to the condition (4.20) is satisfied, it will be equivalent to the Newton-Kantorovich method for difference equations — that is, the number α in (4.31) will be equal to 1.

We have solved numerically a few quasi-static problems of the non-linear theory of thermo-viscoelasticity by means of the BESM-6 computer [8]. It should be observed that numerical solution of problems of the theory of viscoelasticity, in which a linearly elastic problem is solved at each iteration step, encounter considerable difficulties of storing data in the memory of the computer. Therefore in many cases a synthesis of theoretical and numerical methods is useful. In particular, by reducing the non-linear problem of viscoelasticity — see above — to a sequence of linear problems, use can, at each step of the method, be made of the numerical elastic solution [34], based on a known dependence of the elastic solution on the Poisson ratio.

Let us denote by f any of the prescribed quantities ρF_i, S_i^0 and by φ any of the prescribed quantities $u_i^0, \alpha\vartheta_i$. It is assumed that

$$\begin{aligned} f(t, \mathbf{x}) &= \sum_i F_i(t) X_i(\mathbf{x}); \\ \varphi(t, \mathbf{x}) &= \sum_j \Phi_j(t) Y_j(\mathbf{x}) \end{aligned} \tag{5.10}$$

and that we have $f=0$ for all f and $\varphi_i(t)=0$ for all φ_i, except $\varphi_1(t)=1$. With such conditions we solve, by means of any method, the elastic problem (4.6), (4.7). Let us consider one of the components of the displacement vector $\mathbf{u}$.

We have [34]

$$u=(A_1+A_2 g_{1/2}+A_3 g_2+A_4 g_\beta+A_5/\omega+A_6\omega)\,\Phi_1\,, \tag{5.11}$$

where A_i $(i=1, ..., 6)$ are certain functions of the coordinates. Taking now definite values $\omega=\omega^{(1)}, ..., \omega^{(7)}$, we obtain, at every point of the body, a set of seven algebraic equations

$$u^{(j)}=A_1+\frac{A_2}{1+1/2\,\omega^{(j)}}+\frac{A_3}{1+2\omega^{(j)}}+ \\ +\frac{A_4}{1+\beta\omega^{(j)}}+\frac{A_5}{\omega^{(j)}}+A_6\omega^{(j)}, \quad j=1, ..., 7\,. \tag{5.12}$$

By solving the equations (5.12) we shall find the values of A_i, β at each point of the body $\mathbf{x}$. If the expression (5.11) has been written correctly, the quantity β will be the same at every point. Next let us set $\varphi_2(t)=1$ and the remaining $\varphi_j(t)$ and f equal to zero etc. Then, we write the solution of the problem of viscoelasticity. From the Eq. (5.11) we have

$$u(t,\mathbf{x})=A_1(\mathbf{x})\,\Phi_1(t)+A_2(\mathbf{x})\int_0^t g_{1/2}(t-\tau)\,d\Phi_1(\tau)+ \\ +A_3(\mathbf{x})\int_0^t g_2(t-\tau)\,d\Phi_1(\tau)+A_4(\mathbf{x})\int_0^t g_\beta(t-\tau)\,d\Phi_1(\tau)+ \\ +A_5(\mathbf{x})\int_0^t \Pi(t-\tau)\,d\Phi_1(\tau)+A_6(\mathbf{x})\int_0^t \omega(t-\tau)\,d\Phi_1(\tau)\,. \tag{5.13}$$

Several problems have been solved by this method [8, 35].

The solution of coupled problems is much more complicated. Only an insignificant number of quasi-static problems of the linear theory of thermo-viscoelasticity have as yet been solved [36]. In the domain of coupled dynamic problems of the linear theory of thermo-viscoelasticity the one-dimensional case [37, 38] is the only that has been solved.

It should be observed that the solution of problems of the theory of viscoelasticity has some particular features connected with multiple repetition of the computation according to a definite algorithm, the previous computation being disregarded. For the programming for such a problem the question of economy of computer memory and time is essential. There is much waste work in the preparation of the problem, for the programming and the programming itself. The application of general routine languages (Algol, Fortran etc.) simplifies the problem but very little. A program for the entire solution process of a problem of thermo-viscoelasticity, beginning with the derivation of the basic equations for prescribed constitutive equations and prescribed geometry of the body, and ending with the choice of a numerical method for solving the problem, requires many non-arithmetical operations.

The development of a specialized language for the automatization of the programming process for problems of the theory of thermo-viscoelasticity would essentially facilitate their solution. The elaboration of such a system for the solution of problems of the theory of elasticity and plasticity is in progress [39, 40].

The algorithmic system of thermo-viscoelasticity would include as a subset a system for solving problems of the theory of elasticity and certain theories of plasticity.

The most perfect would be a situation in which the research worker writes formulae describing a mechanical model of the continuum and the particular problem—that is, the region occupied by the body, the boundary and initial conditions etc. The computer performs, owing to a certain algorithmic system, the derivation of formulae convenient for numerical computation, selects the computation method by a certain optimum procedure, performs the computation, appraises the accuracy of the solution obtained, and plots graphs.

References

1. А. А. Ильюшин, Б. Е. Победря, *Основы математической теории термо-вязко-упругости* [Foundations of the Mathematical Theory of Thermo-Viscoelasticity], М., Наука, 1970.
2. Б. Е. Победря, *О связанных задачах механики сплошной среды* [On coupled problems of mechanics of a continuum, Selected works: «Elasticity and non-elasticity»], в сб. Упругость и неупругость, вып. 2, Изд-во МГУ, 224–252, 1971.
3. А. А. Ильюшин, *Пластичность* (*Основы общей математической теории*), [Plasticity (Foundations of the general theory)], М., Изд-во АНСССР, 1963.
4. P. Lévy, *Problèmes concrets d'analyse fonctionnelle,* Deuxième edition, Paris 1951.
5. M. E. Gurtin, Eli Sternberg, *On the linear theory of viscoelasticity*, Arch. Ration Mech. and Analys, **11**, 4, 291 – 356, 1962.
6. C. Truesdell, W. Noll, *The non-linear field theories of mechanics*, Handbuch der Physik, Band III/3, 1965.
7. Б. Е. Победря, *Об уравнениях состояния в нелинейной теории вязко-упругости* [Equations of state in the non-linear theory of viscoelasticity], Механика полимеров, № 3, 1967.
8. Б. Е. Победря, *Математическая теория нелинейной вязко-упругости* [Mathematical theory of non-linear viscoelasticity. Selected works: «Elasticity and non-elasticity»], в сб. Упругость и неупругость, вып. 3, Изд-во МГУ, 95–173, 1973.
9. А. А. Ильюшин, *Пластичность* [Plasticity], М., Гостехиздат, 1948.
10. Д. Л. Быков, *Основные уравнения и теоремы для одной модели физически нелинейной среды* [Fundamental equations and theorems for a certain model of a physically non-linear medium], М. Т. Т., № 4, 1966.
11. А. А. Ильюшин, П. М. Огибалов, *Квазилинейная теория вязкоупругости и метод малого параметра* [Quasi-linear theory of viscoelasticity and the perturbation method], Механика поломеров, № 2, 1966.
12. А. А. Ильюшин, Б. Е. Победря, *К вопросу о нелинейной теории вязко-упругости* [A contribution to the problem of non-linear theory of viscoelasticity], в сб. Прочность и пластичность, М., Наука, 1971.
13. H. Leaderman, F. Mc Crakin, O. Nakada, *Large longitudinal retarted elastic deformation of rubberlike network polimers*, Trans. Soc. Rheology, N 7, 1963.
14. Б. Е. Победря, *Теорема единственности в связанной термо-вязко-упругости* [The uniqueness theorem of coupled thermo-viscoelasticity], Вестник Московского университета, Сер. матем., мех., № 6, 84–88, 1969.

15. Н. Н. Ворович, Ю. П. Красовский, *О методе упругих решений*, [On the method of elastic solutions], ДАН СССР **126**, № 4, 1959.
16. Б. Е. Победря, *О сходимости метода «упругих решений» в нелинейной вязко-упругости* [The convergence of the method of elastic approximations in non-linear viscoelastiьcity], ДАН СССР, **195**, № 2, 1970.
17. А. И. Кошелев, *О некоторых вопросах существования и приближенного решения для квазилунейных эллиптических уравнений и систем в пространствах С. Л. Соболева* [Some problems of existence and approximate solution for quasi-linear elliptic equations and sets of equations in S. L. Sobolev's spaces], Сиб. матем. журнал, IX, № 5. 1968.
18. I. Babuška, I. Hlaváček, *On the existencense and uniqueness of solution in the theory of viscoelasticity*, Arch. Mech. Stos., 1, N **18**, 1966.
19. Б. Е. Победря, *О новом методе решения некоторых квазистатических задач нелинейной механики сплошной среды* [On a new method for solution of some quasi-static problems in non-linear continuum mechanics], ДАН СССР, **197**, № 2, 1971.
20. А. Н. Тихонов, А. А. Самарский, *Уравнения математической физики* [Equations of mathematic physics], Наука, 1966.
21. С. К. Годунов, *Уравнения математической физики* [Equations of mathematic physics], Наука, 1971.
22. А. А. Самарский, *Введение в теорию разностных схем* [Introduction into the theory of difference schemes], Наука, 1971.
23. С. К. Годунов, В. С. Рябенький, *Введение в теорию разностных схем* [Introduction into the theory of difference schemes], Физматгиз, 1962.
24. Н. Н. Яненко, *Метод дробных шагов решения многомерныS задач математической физики* [Method of small steps for solving multi-dimensional problems of theoretical physics], Новосибирск, Наука, 1967.
25. Е. Г. Дьяконов, *Разностные методы решения краевых задач* [Difference methods for solving boundary-value problems], Вып. 1, Москва, ВЦ МГУ (ротапринт), 1971.
26. В. Ф. Дьяченко, *Основные понятия вычислительной математики* [Fundamental notions of numerical analysis], Москва, Наука, 1971.
27. А. Н. Коновалов, *Численное решение задач теории упругости* [Numerical solutions of problems of the theory of elasticity], Новосибирск, Наука (ротапринт), 1968.
28. А. Г. Угодчиков, М. И. Длучач, А. Е. Степанов, *Решение краевых задач плоской теории упругости на цифровых и аналоговых машинах* [Solution of boundary value problems of the plane theory of elasticity by means of digital and analogue computers], М., Зысшая школа, 1970.
29. Б. Е. Победря, *Численные методы в теории вязко-упругости* [Numerical methods in the theory of viscoelasticity], Механика полимеров, № 3, 1973.
30. Е. Г. Дьяконов, *О разностных методах решения некоторых нестационарных систем* [Difference methods for solving certain nonstationary systems. Selected works: «Applied mathematics and programming»], в сб. Прикладная математика и программирование, вып. 6, Кишинев, РНО АН Молд. ССР. 1971.
31. С. Г. Михлин, *Численная реализация вариационных методов* [Numerical application of variational methods], М., Наука, 1966.
32. O. C. Zienkiewicz, Y. K. Cheung, *The finite element method in structural and continuum mechanics*, Mc Graw-Hill Publ. Comp. Lim., N.-Y, 1967.
33. J. T. Oden. *The finite element method in non-lineare continuum mechanics*, Mc Graw-Hill Company, N.Y., 1972.
34. Б. Е. Победря, *О зависимости решения задачи теории упругости от упругих постоянных* [Dependence of a solution in the theory of elasticity on the elastic constants], Проблемы прочности, № 3, 1972.
35. Б. Е. Победря, *Расчет вязко-упругих систем по численной упругой реализации* [Computation of viscoelastic systems according to the numerical elastic method], Проблемы прочности, № 4, 1973.

36. В. Н. Кузнецов, П. М. Огибалов, Б. Е. Победря, *Проблема связанности в механике полимеров* [Problem of coupling in the mechanics of polymers], Механика полимеров, № 1, 1971.
37. R. C. Petrof, S. Gratch, *Wawe propagation in a viscoelastic material with temperature-dependent properties and thermomechanical coupling*, J. Appl. Mech., Series E, **31**, 3, 1964.
38. Б. Е. Победря, *Численный метод решения связанных задач термо-вязко-упругости* [Numerical method for solving coupled problems of thermo-viscoelasticity], M.T.T., № 6, 1973.
39. В. К. Кабулов, *Алгоритмизация в теории упругости и деформационной теории пластичности* [Algorithms in the theory of elasticity and the strain theory of plasticity], Ташкент-Изд-во ФАН, 1969.
40. В. К. Кабулов, В. А. Толок, *Алгоритмическая система для решения на ЭВМ задач теории упругости и пластичности* [Algorithm system for solving, by means of an electronic computer, problems of the theory of elasticity and plasticity. Selected works: «Problems of numerical and applied mathematics»,]В сб. Вопросы вычислительной и прикладной математики, вып. 4. Ташкент 1971.

Streszczenie

PEWNE METODY ROZWIĄZANIA ZAGADNIEŃ NIELINIOWEJ TERMO-LEPKOSPRĘŻYSTOŚCI

Podano sformułowanie zagadnień nieliniowej teorii termo-lepkosprężystości z problemami sprzężonymi włącznie. W tym celu zanalizowano zastosowanie termodynamiki procesów nieodwracalnych do ośrodków lepkosprężystych. Do rozwiązania zagadnień nieliniowej teorii termolepkosprężystości zaproponowano niektóre metody kolejnych przybliżeń, w tym metodę "szybkozbieżną", i zbadano szybkość ich zbieżności. Rozważono również metody numeryczne, głównie metodę siatek i elementów skończonych.

Резюме

НЕКОТОРЫЕ МЕТОДЫ РЕШЕНИЯ ЗАДАЧ НЕЛИНЕЙНОЙ ТЕРМО-ВЯЗКО-УПРУГОСТИ

Даётся постановка задач нелинейной теории термо-вязко-упругости, в том числе и связанных задач. Для этого анализируется применение термодинамики необратимых процессов к вязко-упругим средам. Предложены некоторые методы последовательных приближений для решения задач нелинейной теории термо-вязко-упругости, в том числе „быстросходящийся" метод и анализируется скорость их сходимости. Рассмотрены также численные методы, в основном метод сеток и метод конечных элементов.