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Large Deformation Constitutive Theory for a Two-Phase Shape Memory Alloy

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In this work we examine significant theoretical issues related to the constitutive modelling of a two-phase shape memory alloy which undergoes large deformations. For this purpose, we propose a new generalized plasticity based model. The model is based on a standard fractions approach and considers a local multiplicative decomposition of the deformation gradient into elastic and inelastic (transformation induced) parts, as its basic kinematic assumption. We also assess the ability of the model in simulating several patterns of the complex behavior of the material in question, by three representative numerical examples. These examples comprise a standard uniaxial tension problem, a torsion problem and an additional problem dealing with non-conventional pseudoelastic response.

Key words: two-phase shape memory alloy, martensitic transformations, pseudoelasticity, shape memory effect, generalized plasticity, finite deformations.

1. Introduction

Shape Memory Alloys (SMAs) are a unique class of metal alloys which can be deformed severely and afterwards recover their original shape after a thermomechanical cycle (shape memory effect) or a stress cycle within some appropriate temperature regimes (pseudoelasticity). The mechanisms of this recovery are either a diffusionless transformation between the austenite phase (which is a highly ordered phase and is also called the parent phase) and the martensite phase (which is a less ordered one) or the reorientation (detwinning) of martensite variants. These transformations in general are termed as martensitic and

may also be met in other metallic materials such as carbon steels and invar alloys.

The experimentally observed behavior of shape memory alloys in the course of martensitic transformations is extremely complex, a fact which in conjunction with the continuously increasing use of SMAs in innovating applications in many engineering fields results in a greater need for a better understanding of these materials. Due to recent rapid advances in computer technology complex constitutive representations can be considered, since their numerical implementation is no longer intractable, no matter how complex they may be. Nevertheless, the development of a material model for SMAs still is a non-trivial task; characteristic is the following comment stated by Abeyaratne *et al.* [1]: "*This subject requires an intimate mix of continuum and lattice theories, and in order to describe it satisfactorily one has to draw on tools from crystallography, lattice dynamics, thermodynamics, continuum mechanics and functional analysis. This provides for a remarkably rich subject which in turn has prompted analyses from various distinct points of view. The free energy function has multiple local minima, each minimum being identified with a distinct phase, and each phase being characterized by its own lattice. Crystallography plays a key role in characterizing the lattice structure and material symmetry, and restricts deformations through geometric compatibility. The thermodynamics of irreversible processes provides the framework for describing evolutionary processes. Lattice dynamics describes the mechanism by which the material transforms from one phase to the other. And eventually all this needs to be described in the continuum scale*".

As a result, for the past three decades there has been substantial activity to model martensitic transformations in shape memory alloys, and several constitutive models have been developed on the basis of non-isothermal elasticity (e.g., see [1, 2]), solid state physics (e.g., see [3–8]), non-equilibrium thermodynamics (e.g., see [9–18]), and plastic flow theories in both the microscopic (e.g., see [19–21]) and the macroscopic (e.g., see [22–27, 54–56]) regimes.

In [22] Lubliner and Auricchio presented an approach to the constitutive modelling of SMAs based on the employment of generalized plasticity [28, 29]. In particular, in [22] the authors reviewed the non-isothermal formulation of the theory [29] and proposed a rather simple three dimensional thermomechanical model which was able to simulate the basic features of the response of SMAs, such as pseudoelastic phenomena under both monotonic and cyclic loadings and the shape memory effect. Later on, the approach of Lubliner and Auricchio was revisited by Panoskaltsis and co-workers [23, 26, 27, 54–56] and was extended to a general thermomechanical framework that could accommodate almost all patterns of the complex behavior of SMAs at finite strains, such as the existence of multiple and possibly interacting loading mechanisms during phase transformations, rate of loading effects and transformation induced plasticity. In this paper,

being motivated by some developments given in [17, 30, 31], we will provide further insights on the application of the generalized plasticity within the *large deformation analysis* of SMAs by means of a new material model. The model is based on a straight forward extension to the finite regime of an infinitesimal model proposed by Panoskaltsis *et al.* [23] and considers a local *multiplicative decomposition* of the deformation gradient into elastic and inelastic (transformation induced) parts as its basic kinematic assumption. The ability of the model in simulating the basic patterns of the SMAs response is assessed by representative numerical examples. As a point of departure for the present analysis, we consider our previous work in Panoskaltsis *et al.* [26] in conjunction with some developments underlying an invariant structure of generalized plasticity within a stress-space formulation given in Panoskaltsis *et al.* [32]. In regard to many aspects of the phenomenology of plasticity and damage mechanics, the first of the authors has benefited a lot from a long friendship with George Voyiadjis.

2. A general material model for a two phase shape memory alloy

2.1. Large deformation generalized plasticity for a two-phase shape memory alloy material

Generalized plasticity is a local internal variable theory of rate-independent behavior which is motivated by loading-unloading irreversibility [28, 29] and is mathematically funded on set theory and topology [29, 32]. This general mathematical foundation provides to the theory the ability to deal with "non-standard" cases, such as non-connected elastic domains (e.g., see [22, 23, 26, 54–56]), which is exactly the challenge in modelling SMAs.

As in all internal variable theories, it is assumed that the local thermomechanical state in a body is determined uniquely by the couple $(\mathbf{\Lambda}, \mathbf{Q})$, where $\mathbf{\Lambda}$ stands for the controllable state variables and $\mathbf{Q}$ stands for the internal variables. Several identifications of $\mathbf{\Lambda}$ within the context of SMAs can be found in [22, 26, 27, 54, 55]. In this work, we start by considering a stress-space formulation of the model within a referential (material) setting; in this case $\mathbf{\Lambda}$ may be identified by the couple $(\mathbf{S}, T)$, where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor and T is the (absolute) temperature.

The identification of $\mathbf{Q}$ relies crucially on two basic assumptions. The first one is that the material in question is a two-phase alloy, so that we can confine our attention to phase transformations between the austenite and a single (favorably oriented) martensite variant, with volume fraction ξ, $(0 \leq \xi \leq 1)$. Accordingly, we only deal with two phase transformations, namely the forward austenite to martensite (A $\rightarrow$ M) transformation in which $\dfrac{d\xi}{dt} = \dot{\xi} > 0$, and the reverse

martensite to austenite (M $\rightarrow$ A) transformation where $\dot{\xi} < 0$. Models of this type have been proposed among others in [11–13, 15, 16, 18, 22, 23, 27, 54–56]. The second basic assumption consists of a local multiplicative decomposition of the deformation gradient into elastic and inelastic (transformation induced) parts

$$\mathbf{F} = \mathbf{F_e}\mathbf{F_{tr}},$$

where $\mathbf{F_e^{-1}}$ is considered as a tensor which releases elastically the stress on the current configuration. Multiplicative decompositions of this type within the context of SMAs have been considered in [18, 30, 31] and elsewhere. A similar decomposition has been also considered in the crystal models discussed in [20, 21]. By following the pioneering work of Simo [33] – see also [34, pp. 301–305] – we consider the inelastic right Cauchy-Green tensor

$$\mathbf{C_{tr}} = \mathbf{F_{tr}^T}\mathbf{F_{tr}},$$

which, since symmetric and positive definite, can be considered as a primary measure (metric) of inelastic deformation. As a result of these basic assumptions, the internal variables $\mathbf{Q}$ may be identified by the couple $(\mathbf{C_{tr}}, \xi)$.

Remark 1. By following [33] we can define the (spatial) elastic left Cauchy-Green tensor as

$$\mathbf{b_e} = \mathbf{F_e}\mathbf{F_e^T},$$

which is also symmetric and positive definite. Note that $\mathbf{b_e}$ is related to $\mathbf{C_{tr}}$ by

$$\mathbf{b_e} = \mathbf{F}\mathbf{C_{tr}^{-1}}\mathbf{F^T},$$

which means that $\mathbf{b_e}$ is the push-forward of $\mathbf{C_{tr}^{-1}}$ (e.g., see [35, pp. 82–84]) into the spatial configuration. Accordingly, $\mathbf{b_e}$ can be used as a primary measure of inelastic deformation as well. In this case, an equivalent description of the material model may be pursued in the spatial configuration in terms of the Kirchhoff stress tensor $\boldsymbol{\tau}$ defined as the push-forward of $\mathbf{S}$, that is $\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F^T}$, the tensor $\mathbf{b_e}$ and the (scalar invariant) quantities T and ξ.

The central concept of generalized plasticity is that of the elastic range (e.g., see [29, 32]) which is defined at any material state as the region in the stress-temperature space comprising the values of $(\mathbf{S}, T)$ that can be attained elastically (i.e., with no change in the internal variables$(\mathbf{C_{tr}}, \xi)$) from the current material state. It is assumed that the elastic range is a regular set in the sense that it is the closure of an open set. The boundary of this set may be defined as a loading surface (e.g., see [29, 32]). In turn a material state may be defined as elastic if it is an interior point of its elastic range and inelastic if it is a boundary point of its elastic range; in the latter case the material state lies on

a loading surface and upon loading either the forward (A $\rightarrow$ M) or the reverse (M $\rightarrow$ A) can be activated. It should be added that the notion of process is introduced implicitly here. Lubliner and Auricchio [22] on the basis of the well known *critical stress-temperature phase diagram* for SMA's transformations (e.g., see [17, 22]), realized that for a two-phase alloy undergoing phase transformations the loading surfaces may be defined by a two parameter family of disjoint surfaces, which are given by expressions of the form

$$\begin{aligned} \Phi_{\mathbf{M}}(\mathbf{S},T) &= F_{\mathbf{M}}(\mathbf{S}) - C_{\mathbf{M}}T + R_{\mathbf{M}} = \text{const}, \\ \Phi_{\mathbf{A}}(\mathbf{S},T) &= F_{\mathbf{A}}(\mathbf{S}) - C_{\mathbf{A}}T + R_{\mathbf{A}} = \text{const}, \end{aligned} \tag{2.1}$$

where $C_{\mathbf{M}}$, $C_{\mathbf{A}}$ are two material constants, $R_{\mathbf{M}}$, $R_{\mathbf{A}}$ are the family parameters, while the $\Phi_{\mathbf{M}}$ surfaces are associated with the forward (A $\rightarrow$ M) transformation and the $\Phi_{\mathbf{A}}$ surfaces with the reverse (M $\rightarrow$ A), one. Note that in the present setting the loading surfaces are identified by the *isofractal surfaces*, that is the surfaces on the state space, on which ξ remains constant (see [13, 23]). Then, on the basis of the defining property of an inelastic state, and the irreversibility of an inelastic process from such a state Lubliner and Auricchio [22] further proved that the evolution of ξ in the course of martensitic transformations may be described in terms of the loading surfaces and their time derivatives by a general rate equation of the form

$$\dot{\xi} = H_{\mathbf{M}}(\Phi_{\mathbf{M}})\mathbf{L}_{\mathbf{M}}(\mathbf{S},T,\mathbf{C}_{\mathbf{Tr}},\xi)\left\langle \dot{\Phi}_{\mathbf{M}} \right\rangle + H_{\mathbf{A}}(\Phi_{\mathbf{A}})\mathbf{L}_{\mathbf{A}}(\mathbf{S},T,\mathbf{C}_{\mathbf{Tr}},\xi)\left\langle -\dot{\Phi}_{\mathbf{A}} \right\rangle, \tag{2.2}$$

where $< \cdot >$ stands for the Macauley bracket which is defined as

$$\langle x \rangle = \begin{cases} x & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

and $H_{\mathbf{M}}$, $H_{\mathbf{A}}$ stand for scalar functions which enforce the defining property of an inelastic state. Accordingly, their values must be positive at any inelastic state and zero at any elastic one. Finally, $L_{\mathbf{M}}$, $L_{\mathbf{A}}$ represent non-vanishing functions, which are associated with the properties of the forward (A $\rightarrow$ M) and the reverse (M $\rightarrow$ A) transformations, respectively.

Furthermore, on the basis of experimental evidence underlying the deformational properties of SMAs, (e.g., see [36]) it can be assumed that the evolution of the transformation induced deformation is directly proportional to the evolution of ξ. Accordingly, the corresponding expression for the evolution of $\mathbf{C_{tr}}$ may be stated in the form

$$\dot{\mathbf{C}}_{\mathbf{tr}} = \mathbf{M}(\mathbf{S},T,\mathbf{C}_{\mathbf{tr}},\xi)\dot{\xi}, \tag{2.3}$$

where $\mathbf{M}$ stands for a symmetric rank-2 tensorial function of the state variables.

The general material model described by Eqs. (2.2) and (2.3) constitutes the simplest constitutive representation of a two-phase shape memory alloy. In order to develop a particular model, it remains to specify the expressions for the loading surfaces (see Eq. (2.1)), which in turn will yield the functions $H_{\mathbf{M}}$ and $H_{\mathbf{A}}$, and the functions $\mathbf{L_M}$, $\mathbf{L_A}$, and $\mathbf{M}$.

As an application we first assume that the loading surfaces are given by a von-Mises type expression (e.g. see [32, 33]), that is:

$$\begin{aligned} \Phi_{\mathbf{M}}(\mathbf{S},T) &= \|DEV\mathbf{S}\| - C_{\mathbf{M}}T + R_{\mathbf{M}} = \text{const}, \\ \Phi_{\mathbf{A}}(\mathbf{S},T) &= \|DEV\mathbf{S}\| - C_{\mathbf{A}}T + R_{\mathbf{A}} = \text{const}, \end{aligned} \tag{2.4}$$

where $DEV(\cdot)$ stands for the deviatoric part in the material description and is defined as $DEV(\cdot) = (\cdot) - \frac{1}{3}[\mathbf{C} : (\cdot)]\mathbf{C}^{-1}$, in which $\mathbf{C}$ is the right Cauchy-Green deformation tensor ($\mathbf{C} = \mathbf{F^T F}$), which plays the role of a metric tensor for the reference configuration and $\|\cdot\|$ stands for the Euclidean norm.

In component form Eqs. (2.4) read

$$\Phi_{\mathbf{M}}(\mathbf{S},T) = \sqrt{S_{IJ}S_{KL}C_{IK}C_{JL} - \frac{1}{3}(S_{IJ}C_{IJ})^2} - C_{\mathbf{M}}T + R_{\mathbf{M}} = \text{const},$$

$$\Phi_{\mathbf{A}}(\mathbf{S},T) = \sqrt{S_{IJ}S_{KL}C_{IK}C_{JL} - \frac{1}{3}(S_{IJ}C_{IJ})^2} - C_{\mathbf{A}}T + R_{\mathbf{A}} = \text{const}.$$

As in [22], we introduce the following members of the family:

$$\begin{aligned} \Phi_{\mathbf{Ms}}(\mathbf{S},T) &= \|DEV\mathbf{S}\| - C_{\mathbf{Ms}}T + R_{\mathbf{Ms}} = 0, \\ \Phi_{\mathbf{Mf}}(\mathbf{S},T) &= \|DEV\mathbf{S}\| - C_{\mathbf{Mf}}T + R_{\mathbf{Mf}} = 0, \\ \Phi_{\mathbf{As}}(\mathbf{S},T) &= \|DEV\mathbf{S}\| - C_{\mathbf{As}}T + R_{\mathbf{As}} = 0, \\ \Phi_{\mathbf{Af}}(\mathbf{S},T) &= \|DEV\mathbf{S}\| - C_{\mathbf{Af}}T + R_{\mathbf{Af}} = 0, \end{aligned}$$

where

$$\begin{aligned} R_{\mathbf{Ms}} &= -C_{\mathbf{M}}(T - \mathrm{M_s}), & R_{\mathbf{Mf}} &= -C_{\mathbf{M}}(T - \mathrm{M_f}), \\ R_{\mathbf{As}} &= -C_{\mathbf{A}}(T - \mathrm{A_s}), & R_{\mathbf{Af}} &= -C_{\mathbf{A}}(T - \mathrm{A_f}), \end{aligned}$$

in which the parameters $\mathrm{M_s}$, $\mathrm{M_f}$, $\mathrm{A_s}$, and $\mathrm{A_f}$ correspond to the standard characteristic temperatures (martensite start, martensite finish, austenite start, austenite finish, respectively) for the alloy in question and can be determined by means of the critical stress-temperature phase diagram as well. Since $\Phi_{\mathbf{Mf}}$ is related to the finish values and $\Phi_{\mathbf{Ms}}$ to the starting values of the A $\rightarrow$ M transformation, the loading surfaces $\Phi_{\mathbf{Mf}} = 0$ and $\Phi_{\mathbf{Ms}} = 0$ may be considered as the boundaries

of the set of all states for which the A $\to$ M transformation can be active. Then the scalar $H_{\mathbf{M}}$ may be defined as

$$H_{\mathbf{M}} = \frac{\langle -\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}\rangle}{|\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}|}.$$

By applying similar arguments for the reverse (M $\to$ A) transformation, $H_{\mathbf{A}}$ can be found to be

$$H_{\mathbf{A}} = \frac{\langle -\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}\rangle}{|\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}|}.$$

For the functions $\mathbf{L_A}$ and $\mathbf{L_M}$ several choices are possible. In this work, as in our previous ones (see [23, 27]), we use a linear type of expression which has been proposed by Likchachev and Koval [37] for the description of the hysteretic response of SMAs, that is

$$\mathbf{L_M} = -\frac{1-\xi}{\Phi_{\mathbf{Mf}}(\mathbf{S},T)}, \qquad \mathbf{L_A} = -\frac{\xi}{\Phi_{\mathbf{Af}}(\mathbf{S},T)},$$

so that the final form of Eq. (2.2) reads

$$\dot{\xi} = -\frac{\langle -\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}\rangle}{|\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}|}\frac{1-\xi}{\Phi_{\mathbf{Mf}}}\left\langle \dot{\Phi}_{\mathbf{M}}\right\rangle - \frac{\langle -\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}\rangle}{|\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}|}\frac{\xi}{\Phi_{\mathbf{Af}}}\left\langle -\dot{\Phi}_{\mathbf{A}}\right\rangle. \tag{2.5}$$

For the rate equation for the evolution of $\mathbf{C_{tr}}$, motivated by several models which are based on the infinitesimal theory (e.g., see [12, 22]; see also the relevant discussion given in Panoskaltsis [23]), we assume a normality flow rule in stress-space which within the present context is expressed in a somewhat surprising format in terms of the (reciprocal) metric $\mathbf{C}_{\mathbf{tr}}^{-1}$ as

$$\dot{\mathbf{C}}_{\mathbf{tr}}^{-1} = -\frac{1}{\beta}(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}) : \mathbf{N}\dot{\xi}, \tag{2.6}$$

where β is additional model parameter, $(\cdot)\otimes(\cdot)$ stands for the tensor product, and $\mathbf{N}$ is the projection of the outward normal vector to the loading surfaces in the stress space, that is $\mathbf{N} = \dfrac{\partial\Phi_{\mathbf{M}}}{\partial\mathbf{S}} = \dfrac{\partial\Phi_{\mathbf{A}}}{\partial\mathbf{S}}$. The component form of Eq. (2.6) reads

$$\dot{C}^{-1}_{\mathbf{tr}IJ} = -\frac{1}{\beta}C^{-1}_{IK}C^{-1}_{JL}\left(\frac{\partial\Phi}{\partial S_{KL}}\right)\dot{\xi}.$$

Remark 2. A further observation in the model governing equations reveals that the right Cauchy-Green tensor $\mathbf{C}$ is used as a basic state variable in addition to the second Piola-Kirchhoff stress tensor $\mathbf{S}$, since it is included among the arguments of the functions $H_{\mathbf{M}}$, $H_{\mathbf{A}}$, $\mathbf{L_M}$, $\mathbf{L_A}$, and $\mathbf{M}$. Such a formulation of a finite theory in terms of $\mathbf{S}$ and $\mathbf{C}$ is called the *convected representation* of

the theory (e.g. see [34, p. 261]). Convected representations have been also used within the context of finite plasticity in [33, 38] and more recently in [32]. Note that *the state variables* $\mathbf{S}$ *and* $\mathbf{C}$ *are not independent* since they are always related by means of the thermomechanical state equation.

The equivalent setting of the model in the spatial configuration can be determined by performing a push-forward operation to the basic Eqs. (2.4), (2.5), and (2.6) by the deformation gradient as

$$\phi_{\mathbf{M}}(\tau,T) = \|dev\boldsymbol{\tau}\| - C_{\mathbf{M}}T + R_{\mathbf{M}} = \sqrt{\tau_{ij}\tau_{kl}\delta_{ik}\delta_{jl} - \frac{1}{3}(\tau_{ij}\delta_{ij})^2} - C_{\mathbf{M}}T + R_{\mathbf{M}} = \text{const},$$

$$\phi_{\mathbf{A}}(\tau,T) = \|dev\boldsymbol{\tau}\| - C_{\mathbf{A}}T + R_{\mathbf{A}} = \sqrt{\tau_{ij}\tau_{kl}\delta_{ik}\delta_{jl} - \frac{1}{3}(\tau_{ij}\delta_{ij})^2} - C_{\mathbf{A}}T + R_{\mathbf{A}} = \text{const},$$

$$\dot{\xi} = -\frac{\langle -\phi_{\mathbf{Mf}}\phi_{\mathbf{Ms}}\rangle}{|\phi_{\mathbf{Mf}}\phi_{\mathbf{Ms}}|}\frac{1-\xi}{\phi_{\mathbf{Mf}}}\left\langle \dot{\phi}_{\mathbf{M}}\right\rangle - \frac{\langle -\phi_{\mathbf{Af}}\phi_{\mathbf{As}}\rangle}{|\phi_{\mathbf{Af}}\phi_{\mathbf{As}}|}\frac{\xi}{\varphi_{\mathbf{Af}}}\left\langle -\dot{\phi}_{\mathbf{A}}\right\rangle,$$

$$\mathrm{L}_{\mathbf{V}}\mathbf{b}_{\mathbf{e}} = -\frac{1}{\beta}\frac{\partial\phi}{\partial\tau}\dot{\xi},$$

where $\phi_{\mathbf{Mf}}$, $\phi_{\mathbf{Ms}}$, $\phi_{\mathbf{Af}}$, $\phi_{\mathbf{As}}$ are the spatial counterparts of $\Phi_{\mathbf{Mf}}$, $\Phi_{\mathbf{Ms}}$, $\Phi_{\mathbf{Af}}$, and $\Phi_{\mathbf{As}}$, respectively, and $\mathrm{L}_V(\cdot)$ stands for the Lie derivative which is defined as the convected derivative with respect to the spatial configuration (e.g. see [34, pp. 254–255]; [35, pp. 106–108]). Note that the push-forward of the right Cauchy-Green tensor $\mathbf{C}$ into the spatial configuration is the unit tensor $\mathbf{i}$ (with components δ_{ij}) and accordingly does not appear explicitly in the arguments of the state functions in the spatial description.

Remark 3. Experimental evidence (e.g., see [39, 40]) reveals an asymmetric tension-compression stress-strain curve, a response which cannot be simulated by the von-Mises loading surface family used herein. Nevertheless, such a response can be modelled within the proposed framework by using an alternative expression for the loading surfaces. A possible choice may be based on an expression discussed in [24], which within the present setting may be stated in the following (general) form:

$$\phi(\boldsymbol{\tau},T) = g(y_\tau)\,\|dev\boldsymbol{\tau}\| - CT + R = \text{const}, \tag{2.7}$$

where y_τ is the third invariant of the Kirchhoff stress tensor defined as $y_\tau = \det(\boldsymbol{\tau})$, and g is a function of y_τ, which may be stated in the form

$$g(y_\tau) = \cos\left\{\frac{\cos^{-1}[1-\alpha(1-y_\tau)]}{3}\right\},$$

in which α is an additional material parameter accounting for the tension-compression asymmetry. It is noted that for $\alpha = 0$ the loading surface family (2.7) is degenerated to the von-Mises family, while the case $\alpha = 1$ corresponds to a loading surface family which predicts the maximum asymmetry in the tension-compression stressstrain curves. The equivalent expression for the loading surfaces in the material description reads:

$$\Phi(\mathbf{S}, T) = g(Y_{\mathbf{S}})\, \|DEV\mathbf{S}\| - CT + R = \text{const},$$

where $Y_{\mathbf{S}}$ is the third invariant of the second Piola-Kirchhoff stress tensor $\mathbf{S}$ defined as $Y_{\mathbf{S}} = \dfrac{\det(\mathbf{S})}{\det(\mathbf{C})}$. More sophisticated expressions for the loading surfaces and related discussions may be found in [41].

REMARK 4. Another interesting feature of the proposed model is that it can be combined with a viscoplastic model which is based on the *overstress concept* (e.g., see [42]) in order to account for the rate-effects which are met in the course of martensitic transformations in SMAs (e.g., see [43, 44]). If this is the case it may be assumed that the *same mechanisms within the material substructure which are responsible for the martensitic transformations are also responsible for rate effects*. Accordingly, it may be considered that both mechanisms may be described by the evolution of the (referential) metric $\mathbf{C_{tr}}$, which in turn may be given by a rate equation of the form:

$$\begin{aligned}(2.8)\quad \dot{\mathbf{C}}_{\mathbf{tr}}^{-1} = &-\frac{1}{\beta}(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}) : \mathbf{N}\dot{\xi} \\ &- \gamma(\xi, T, \dot{T})\left(\frac{\langle -\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}\rangle}{|\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}|}\Phi_{\mathbf{Ms}} - \frac{\langle -\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}\rangle}{|\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}|}\Phi_{\mathbf{As}}\right)(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}) : \mathbf{N},\end{aligned}$$

where $\gamma = \gamma(\xi, T, \dot{T})$ is the material fluidity. It is noted that the dependence of γ on $\dot{T}$ cannot be arbitrary, since it must be consistent with the experimentally observed fact – see further [44] – that a rate dependent behavior appears only in non-isothermal loadings. Accordingly, the dependence of γ on $\dot{T}$ must be such that at the limit $\dot{T} \to 0$, γ vanishes. It is further noted that within the present proposal, the loading surfaces $\Phi_{\mathbf{Ms}}$ and $\Phi_{\mathbf{As}}$, besides being the initial loading surfaces for the A $\to$ M and the M $\to$ A transformations, respectively, serve as yields surfaces and loading potentials for the viscoplastic part of the model. Finally, the equivalent expression of Eq. (2.8) in the spatial configuration may be written in the following (remarkably simple) form:

$$\mathrm{L}_{\mathbf{V}}\mathbf{b}_{\mathbf{e}} = -\frac{1}{\beta}\frac{\partial\varphi}{\partial\boldsymbol{\tau}}\dot{\xi} - \gamma(\xi, T, \dot{T})\left(\frac{\langle -\phi_{\mathbf{Mf}}\phi_{\mathbf{Ms}}\rangle}{|\phi_{\mathbf{Mf}}\phi_{\mathbf{Ms}}|}\phi_{\mathbf{Ms}} + \frac{\langle -\phi_{\mathbf{Af}}\phi_{\mathbf{As}}\rangle}{|\phi_{\mathbf{Af}}\phi_{\mathbf{As}}|}\phi_{\mathbf{As}}\right)\frac{\partial\varphi}{\partial\boldsymbol{\tau}}.$$

A general framework for combining mechanisms with different characteristic times within a large deformation formulation can be found in our recent work in PANOSKALTSIS *et al.* [45].

2.2. Further considerations on the structure of the evolution equation for the material martensite fraction ξ

To this end it is instructive to state some alternative forms for the rate equation for the evolution of the material martensite fraction ξ, which, as it has been noted in Lagoudas et al. [46], plays an essential role in the correct prediction of the material response. For this purpose we assume that Equation (2.2) may be restated in the following rather general form:

$$\dot{\xi} = \frac{\langle -\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}\rangle}{|\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}|}\frac{Q_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})}{P_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})}\left\langle \dot{\Phi}_{\mathbf{M}}\right\rangle + \frac{\langle -\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}\rangle}{|\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}|}\frac{Q_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})}{P_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})}\left\langle -\dot{\Phi}_{\mathbf{A}}\right\rangle, \tag{2.9}$$

where the state functions $P_{\mathbf{M}}$, $Q_{\mathbf{M}}$, $P_{\mathbf{A}}$, and $Q_{\mathbf{A}}$ must reflect the fact that the complete A $\to$ M transformation leads ξ from 0 (on the loading surface $\Phi_{\mathbf{Ms}} = 0$) to 1 (on the loading surface $\Phi_{\mathbf{Mf}} = 0$), while the reverse (M $\to$ A) transformation leads ξ from 1 (on $\Phi_{\mathbf{As}} = 0$) to 0 (on $\Phi_{\mathbf{Af}} = 0$). Such an equation can be always reduced to an exact differential equation and subsequently integrated in a closed form if there exist scalar functions (integrating factors) $\mu_{\mathbf{M}}$ and $\mu_{\mathbf{A}}$ such as the following integrability conditions:

$$\mu_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})\frac{\partial P_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})}{\partial \Phi_{\mathbf{M}}} + P_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})\frac{\partial \mu_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})}{\partial \Phi_{\mathbf{M}}}$$
$$= -\left[\mu_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})\frac{\partial Q_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})}{\partial \xi} + Q_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})\frac{\partial \mu_{\mathbf{M}}(\xi,\Phi_{\mathbf{M}})}{\partial \xi}\right]$$

and

$$\mu_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})\frac{\partial P_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})}{\partial \Phi_{\mathbf{A}}} + P_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})\frac{\partial \mu_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})}{\partial \Phi_{\mathbf{M}}}$$
$$= -\left[\mu_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})\frac{\partial Q_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})}{\partial \xi} + Q_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})\frac{\partial \mu_{\mathbf{A}}(\xi,\Phi_{\mathbf{A}})}{\partial \xi}\right],$$

hold.

A particular case of interest which encompasses Eq. (2.5) appears when Eq. (2.9) takes the following specific form:

$$\dot{\xi} = \frac{\langle -\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}\rangle}{|\Phi_{\mathbf{Mf}}\Phi_{\mathbf{Ms}}|}(1-\xi)A_{\mathbf{M}}(\Phi_{\mathbf{M}})\left\langle \dot{\Phi}_{\mathbf{M}}\right\rangle + \frac{\langle -\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}\rangle}{|\Phi_{\mathbf{Af}}\Phi_{\mathbf{As}}|}\xi A_{\mathbf{A}}(\Phi_{\mathbf{A}})\left\langle -\dot{\Phi}_{\mathbf{A}}\right\rangle,$$

where $(1-\xi)$ and ξ are termed as "first order reactions" (e.g., see [22, 47]). An equation of this form, which has been also discussed in [22], is the following:

$$\dot{\xi} = \beta_{\mathbf{M}} \frac{\langle -\Phi_{\mathbf{Mf}} \Phi_{\mathbf{Ms}} \rangle}{|\Phi_{\mathbf{Mf}} \Phi_{\mathbf{Ms}}|} \frac{(1-\xi)}{\Phi^2_{\mathbf{Mf}}} \left\langle \dot{\Phi}_{\mathbf{M}} \right\rangle - \beta_{\mathbf{A}} \frac{\langle -\Phi_{\mathbf{Af}} \Phi_{\mathbf{As}} \rangle}{|\Phi_{\mathbf{Af}} \Phi_{\mathbf{As}}|} \frac{\xi}{\Phi^2_{\mathbf{Af}}} \left\langle -\dot{\Phi}_{\mathbf{A}} \right\rangle,$$

where $\beta_{\mathbf{M}}$ and $\beta_{\mathbf{A}}$ are material constants. This equation can be integrated by separation of variables to yield for the complete A $\rightarrow$ M transformation an exponential type law for the evolution of ξ, that is

$$\xi = 1 - \exp\left[-\beta_{\mathbf{M}} \left(\frac{1}{C_{\mathbf{M}}(T - \mathrm{M_f}) - \|DEV\mathbf{S}\|} - \frac{1}{C_{\mathbf{M}}(\mathrm{M_s} - \mathrm{M_f})} \right) \right],$$

which is identical with an empirical expression proposed by Koistinen and Marburger [48], which has been extensively used for the description of phase transformations in austenitic steels (e.g., see [47, 49]).

Another particular case arises when the right hand side of Eq. (2.9) is independent of ξ. If this is the case, the rate of the material martensite fraction produced/dissolute during the phase transformations depends only on the external agents $\mathbf{S}$ and T (and their rates $\dot{\mathbf{S}}$ and $\dot{T}$). The simplest rate equation accounting for this case arises when the functions $A_{\mathbf{M}}(\Phi_{\mathbf{M}})$ and $A_{\mathbf{A}}(\Phi_{\mathbf{A}})$ are constants, that is,

$$\dot{\xi} = \gamma_{\mathbf{M}} \frac{\langle -\Phi_{\mathbf{Mf}} \Phi_{\mathbf{Ms}} \rangle}{|\Phi_{\mathbf{Mf}} \Phi_{\mathbf{Ms}}|} \left\langle \dot{\Phi}_{\mathbf{M}} \right\rangle + \gamma_{\mathbf{A}} \frac{\langle -\Phi_{\mathbf{Af}} \Phi_{\mathbf{As}} \rangle}{|\Phi_{\mathbf{Af}} \Phi_{\mathbf{As}}|} \left\langle -\dot{\Phi}_{\mathbf{A}} \right\rangle, \tag{2.10}$$

in which the constants $\gamma_{\mathbf{M}}$ and $\gamma_{\mathbf{A}}$ are given as

$$\gamma_{\mathbf{M}} = \frac{1}{C_{\mathbf{M}}(\mathrm{M_s} - \mathrm{M_f})} \qquad \text{and} \qquad \gamma_{\mathbf{A}} = \frac{1}{C_{\mathbf{A}}(\mathrm{A_s} - \mathrm{A_f})}.$$

Then, Eq. (2.10) can be integrated in a closed form to yield a linear law for the evolution of ξ, which for the complete A $\rightarrow$ M transformation reads

$$\xi = \gamma_{\mathbf{M}} \Phi_{\mathbf{M}} + \frac{\mathrm{M_s}}{\mathrm{M_s} - \mathrm{M_f}},$$

while for the complete reverse (M $\rightarrow$ A) transformation the solution is:

$$\xi = -\gamma_{\mathbf{A}} \Phi_{\mathbf{A}} - \frac{\mathrm{A_f}}{\mathrm{A_s} - \mathrm{A_f}}.$$

Finally, another rate equation may appear if we choose the functions $A_{\mathbf{M}}(\Phi_{\mathbf{M}})$ and $A_{\mathbf{A}}(\Phi_{\mathbf{A}})$ to be trigonometric ones. A possible choice is

$$\dot{\xi} = -\frac{1}{2}\delta_{\mathbf{M}} \frac{\langle -\Phi_{\mathbf{Mf}} \Phi_{\mathbf{Ms}} \rangle}{|\Phi_{\mathbf{Mf}} \Phi_{\mathbf{Ms}}|} \sin(\delta_{\mathbf{M}} \Phi_{\mathbf{Mf}}) \left\langle \dot{\Phi}_{\mathbf{M}} \right\rangle + \frac{1}{2}\delta_{\mathbf{A}} \frac{\langle -\Phi_{\mathbf{Af}} \Phi_{\mathbf{As}} \rangle}{|\Phi_{\mathbf{Af}} \Phi_{\mathbf{As}}|} \sin(\delta_{\mathbf{A}} \Phi_{\mathbf{As}}) \left\langle -\dot{\Phi}_{\mathbf{A}} \right\rangle, \tag{2.11}$$

where the constants $\delta_{\mathbf{M}}$ and $\delta_{\mathbf{A}}$ are defined as

$$\delta_{\mathbf{M}} = \frac{\pi}{\Phi_{\mathbf{Mf}} - \Phi_{\mathbf{Ms}}} = \frac{\pi}{C_{\mathbf{M}}(\mathrm{M_f} - \mathrm{M_s})} \quad \text{and} \quad \delta_{\mathbf{A}} = \frac{\pi}{\Phi_{\mathbf{As}} - \Phi_{\mathbf{Af}}} = \frac{\pi}{C_{\mathbf{A}}(\mathrm{A_s} - \mathrm{A_f})}.$$

The solution of Eq. (2.11) for the complete $\mathrm{A} \to \mathrm{M}$ transformation is

$$\xi = \frac{1}{2}\cos(\delta_{\mathbf{M}}\Phi_{\mathbf{Mf}}) + \frac{1}{2},$$

while for the complete reverse ($\mathrm{M} \to \mathrm{A}$) transformation the corresponding solution is

$$\xi = \frac{1}{2}\cos(\delta_{\mathbf{A}}\Phi_{\mathbf{As}}) + \frac{1}{2},$$

which resemble the expressions for the martensite function suggested in the so-called cosine model proposed by Liang and Rogers [10].

2.3. Thermomechanical state equations

As a final step we derive the model thermomechanical state equations. Following Simo [33] – see also [32] – we assume that the stress response is hyperelastic. For the present case of an SMA material, in accordance with the thermomechanical treatment of damage – e.g., see [50, 51] – the Helmholtz free energy is assumed to be decomposed into elastic ($\Psi_{\mathbf{e}}$) and inelastic ($\Psi_{\mathbf{tr}}$) parts as follows:

$$\rho_{\mathrm{ref}}\Psi(\mathbf{C}, T, \mathbf{C}_{\mathbf{tr}}^{-1}, \xi) = \rho_{\mathrm{ref}}\Psi_{\mathbf{e}}(\mathbf{C}, T, \mathbf{C}_{\mathbf{tr}}^{-1}, \xi) + \rho_{\mathrm{ref}}\Psi_{\mathbf{tr}}(T, \xi).$$

As it has been noted in Panoskaltsis *et al.* [23], this is not the conventional decomposition of the free energy function performed within the classical inelastic theories (viscoelasticity, plasticity, viscoplasticity) since the elastic part ($\Psi_{\mathbf{e}}$) depends on the internal variable ξ. In this work we shall deal only with isothermal cases, as we are interested only in the elastic part of the Helmholtz free energy. The latter is assumed to be given by an isotropic function in terms of the invariants of the tensor $\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1}$, (see [34, pp. 258–259]; [32]) as

$$\rho_{\mathrm{ref}}\Psi_{\mathbf{e}} = \lambda(\xi)\frac{\det(\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1}) - 1}{4} - \left[\frac{\lambda(\xi)}{2} + \mu(\xi)\right]\ln\left(\sqrt{\det(\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1})} + \frac{1}{2}\mu(\xi)[\mathrm{tr}(\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1}) - 3],$$

where ρ_{ref} is the material density and λ and μ are Lamé type of parameters ($\lambda > 0$, $\mu > 0$), which are defined in terms of the standard elastic constants E, ν as

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)},$$

and $\det(\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1})$ and $\mathrm{tr}(\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1})$ stand for the third and first invariants of $\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1}$. The dependence of the Lamé parameters on the martensite fraction ξ is assumed to be given by the following rule of mixtures (e.g., see [21, 27]):

$$\lambda(\xi) = \lambda_{\mathbf{A}} + \xi(\lambda_{\mathbf{M}} - \lambda_{\mathbf{A}}), \qquad \mu(\xi) = \mu_{\mathbf{A}} + \xi(\mu_{\mathbf{M}} - \mu_{\mathbf{A}}),$$

where $\lambda_{\mathbf{A}}$, $\mu_{\mathbf{A}}$ are the Lamé parameters when the material is fully austenite, and $\lambda_{\mathbf{M}}$, $\mu_{\mathbf{M}}$ are those when the material is fully martensite. Then, the second Piola-Kirchhoff stress tensor $\mathbf{S}$ can be found by the standard thermomechanical state equation (e.g., see [34, p. 256]; [35, p. 210]) $\mathbf{S} = 2\rho_{\mathrm{ref}}\dfrac{\partial \Psi}{\partial \mathbf{C}}$, as

$$\mathbf{S} = \lambda(\xi)\frac{\det(\mathbf{C}\mathbf{C}_{\mathbf{tr}}^{-1}) - 1}{2}\mathbf{C}^{-1} + \mu(\xi)(\mathbf{C}_{\mathbf{tr}}^{-1} - \mathbf{C}^{-1}). \tag{2.12}$$

By employing once more a standard push-forward operation to Eq. (2.12), the latter may be written equivalently in terms of the spatial quantities $\mathbf{b_e}$ and τ as

$$\boldsymbol{\tau} = \lambda(\xi)\frac{\det(\mathbf{b_e}) - 1}{2}\mathbf{i} + \mu(\xi)(\mathbf{b_e} - \mathbf{i}).$$

3. Numerical simulations

The basic objective of this section is to assess the capability of the proposed model to simulate several patterns of the behavior of SMAs under isothermal loading conditions. In particular, the model will be implemented numerically and will be used for the solution of three representative numerical examples which comprise a standard uniaxial tension problem, a torsion problem, and an additional problem dealing with non-conventional pseudoelastic response of an SMA material subjected to a strain cycle. A general integration scheme for the numerical implementation of a generalized plasticity model is discussed in [32], while particular computational implications related to the case of an SMA material, where the elastic domain is a non-connected set, can be found in [23].

3.1. Uniaxial tension

The first problem we study is that of uniaxial tension. This problem is a standard one within the context of large deformation analyses and it is defined as:

$$x_1 = (1 + \lambda_1)X_1, \qquad x_2 = (1 + \lambda_2)X_2, \qquad x_3 = (1 + \lambda_2)X_3,$$

where $1 + \lambda_1$ and $1 + \lambda_2$ are the straining parameters (principal stretches) along the axial and transverse directions, respectively. Note that in the infinitesimal case, λ_1 and λ_2 are equal to the principal normal strains ε_{11} and ε_{22}, respectively.

The model parameters are set equal to those reported in the work of BOYD and LAGOUDAS [12] for a generic SMA, that is:

$$E_{\mathrm{M}}=13,000 \text{ MPa}, \quad E_{\mathrm{A}}=30,000 \text{ MPa}, \quad \nu=0.30, \quad \mathrm{M_f}=5^{\circ}\mathrm{C}, \quad \mathrm{M_s}=23^{\circ}\mathrm{C},$$

$$\mathrm{A_f}=51^{\circ}\mathrm{C}, \quad \mathrm{A_s}=29^{\circ}\mathrm{C}, \quad \mathrm{C_M}=11.3 \text{ MPa}/^{\circ}\mathrm{C}, \quad \mathrm{C_A}=4.5 \text{ MPa}/^{\circ}\mathrm{C}.$$

All numerical tests start with the specimen in the parent (austenite) phase ($\xi = 0$). As a first simulation, we will show the ability of the model in predicting pseudoelastic response. For this purpose we perform a complete loading-unloading cycle by holding the temperature constant at a value above $\mathrm{A_f}$ ($T = 55^{\circ}\mathrm{C}$). The results are shown for three different values of the model parameter β in Fig. 1, where the Kirchhoff stress τ_{11} (in MPa) is plotted versus the axial strain λ_1. By referring to these results, the ability of the model in simulating pseudoelastic phenomena for a wide range of values of the model parameter β is easily demonstrated. More specifically, we note that the value of β controls the direction and speed of the transformation process: the higher the value of β, the higher the speed of transformation, which in turn leads to phase transformations which occur in a narrower displacement range under higher values of the applied stress. It is interesting to note that for small values of β the model can predict phase transformation processes which are accompanied by the ap-

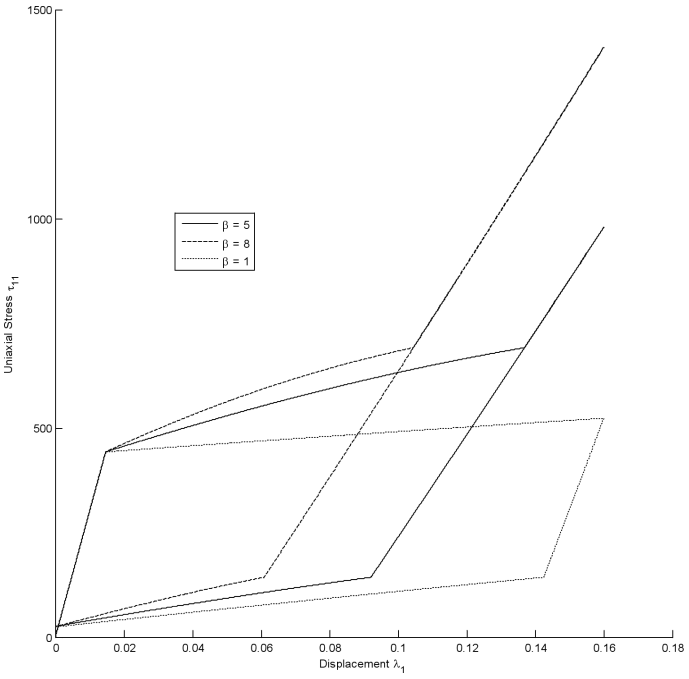

FIG. 1. Uniaxial tension. Uniaxial stress τ_{11} vs. axial strain λ_1 for different values of β.

pearance of very large deformations; for instance for a value of β equal to 1 the specimen attained an elongation of the order of 15% (see Fig. 1) by being only 30% martensite. Such a case may find application in magnetically actuated SMAs (e.g., see ARNDT *et al.* [8]), which exhibit very narrow (lens type) hysterisis loops at slightly higher temperatures that pseudoelastic SMAs.

As a further step we consider three additional simulations starting at three different values of the (reference) temperature T, namely $T = 51°\text{C}$, i.e. $T = \mathrm{A_f}$; $T = 40°\text{C}$, i.e. $\mathrm{A_s} < T < \mathrm{A_f}$, and $T = 25°\text{C}$, i.e. $T < \mathrm{A_s}$. The value of β for these simulations is set equal to 5. In the first test (at $T = \mathrm{A_f}$) the specimen, as in the previous tests, is subjected to a complete loading-unloading cycle exhibiting pseudoelastic response. In this case, since the temperature *has been set exactly equal to the austenite finish temperature*, the *inverse* ($\mathrm{M} \to \mathrm{A}$) *transformation ends at zero stress*. In the next two tests the specimen is first subjected to a stress cycle, keeping the temperature constant, and then to a temperature cycle at zero stress, consisting of heating to a temperature greater than $\mathrm{A_f}$, where martensite is unstable, and by cooling it back to the initial temperature. After the complete stress and temperature cycles, no permanent deformation exists and the material is fully austenite. This is the phenomenon of shape memory effect. The results of this stress cycle are shown in Fig. 2.

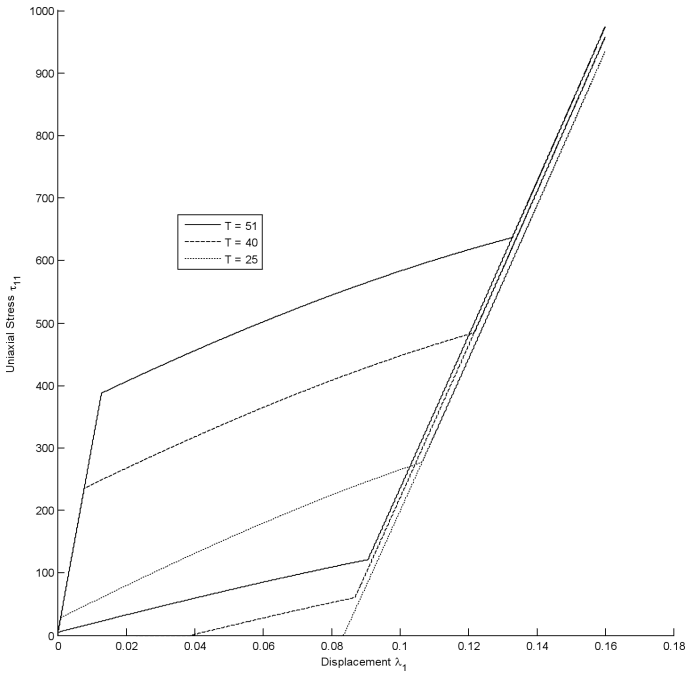

FIG. 2. Uniaxial tension. Uniaxial stress τ_{11} vs. axial strain λ_1 for different values of the (reference) temperature T.

3.2. Torsion of a SMA tube

As a second problem we consider the response in *finite torsion of an* SMA *tube*. A similar problem has been also considered within the context of the infinitesimal theory by LAGOUDAS and ENTCHEV [15]. This problem is defined conveniently in cylindrical polar coordinates as

$$r = R, \qquad \theta = \Theta + \omega Z, \qquad z = Z,$$

where R, Θ, Z, and r, θ, z stand for material coordinates and spatial coordinates respectively, and ω stands for the angle of twist per unit undeformed length. The later is related to the relative twist angle ϕ of the two end sections of the tube by $\omega = \dfrac{\phi}{L}$, where L is the length of the tube. The dimensions of the tube are considered equal to those reported in LAGOUDAS and ENTCHEV [15], namely: inner radius: 2.5 mm, outer radius: 3.17 mm, length: 0.67 mm, while the tube is twisted by controlling implicitly – via the angle ω – the relative twist angle ϕ. The temperature is set to a value higher than $\mathrm{A_f}$ ($T = 55°\mathrm{C}$), while the model parameters are set equal to those used in the uniaxial tension problem with the value of β being equal to 5. The SMA tube is subjected to *two-sided cyclic loading* exhibiting *two-sided* pseudoelastic response. The results are shown in Fig. 3, where the shear stress ($\tau_{\theta z}$) is plotted versus the twist angle ϕ.

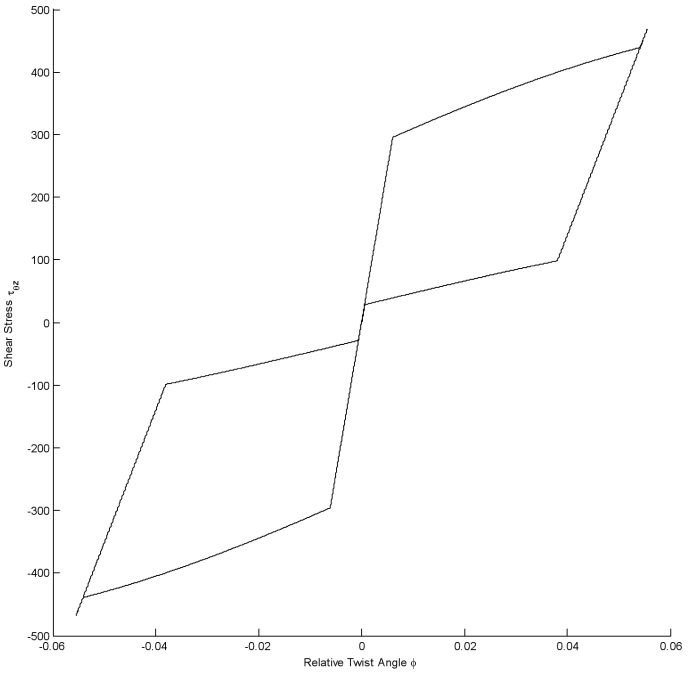

FIG. 3. Torsion of an SMA tube: Shear stress $\tau_{\theta z}$ vs. relative angle of twist ϕ.

3.3. Response under a strain cycle

As a final example, we examine the predictions of the model in the case where an SMA material is subjected to a *strain cycle.* For this purpose we discuss two problems proposed within the context of a hypoelastic formulation (e.g., see [34, pp. 256–258]) in a paper by MEYERS *et al.* [52] and subsequently used as reference problems for the case of a SMA material by PANOSKALTSIS *et al.* [27].

The first problem (Cycle 1) deals with a square element of size $H \times H$, which is imposed into a strain cycle by rotating both upper corners along a cycle of radius r. In this problem, the element submitted to both combined extension along the X_2 axis and 1–2 shear preserves its original (parallelogram) shape (see Fig. 1 in MEYERS *et al.* [52]). This problem is defined as follows:

$$x_1 = X_1 + \frac{\sin\phi(r/H)}{1 + (1 - \cos\phi)(r/H)} X_2,$$

$$x_2 = [1 + (1 - \cos\phi)(r/H)]X_2, \qquad x_3 = X_3.$$

To show the computational versatility of the model, we work here with another set of parameters, namely those used in TRAN *et al.* [53].

$$E_{\mathrm{M}} = E_{\mathrm{A}} = 62,300 \text{ MPa}, \quad \nu = 0.30, \quad \mathrm{M_f} = 263 \text{ K}, \quad \mathrm{M_s} = 248 \text{ K},$$

$$\mathrm{A_f} = 301 \text{ K}, \quad \mathrm{A_s} = 296 \text{ K},$$

while the remaining parameters are those of the previous problems i.e.:

$$\mathrm{C_M} = 11.3 \text{ MPa/K}, \qquad \mathrm{C_A} = 4.5 \text{ MPa/K},$$

and the value of β is set equal to 10.

The corresponding stress-angle of rotation curves are given for three different values of the ratio r/H in Figs. 4, 5, and 6, while the evolution of the material martensite fraction is plotted in Fig. 7. By referring to Figs. 4, 5, and 6, we realize that at the end of the *strain cycle, the stresses go back to zero and the material by obtaining its original stress free state is giving the false impression of being elastic.* However, this recovery has its origins in the martensitic transformations, since, as it is clear from Fig. 7, both the A→M and the M→A transformations have been activated during this strain cycle, resulting in zero final stresses. This is a *non-conventional manifestation of the pseudoelastic phenomenon*, since, unlike the previous (conventional) simulations where the material was subjected to stress cycles, in this case the material is subjected to a strain cycle; however, the exhibited response is identical.

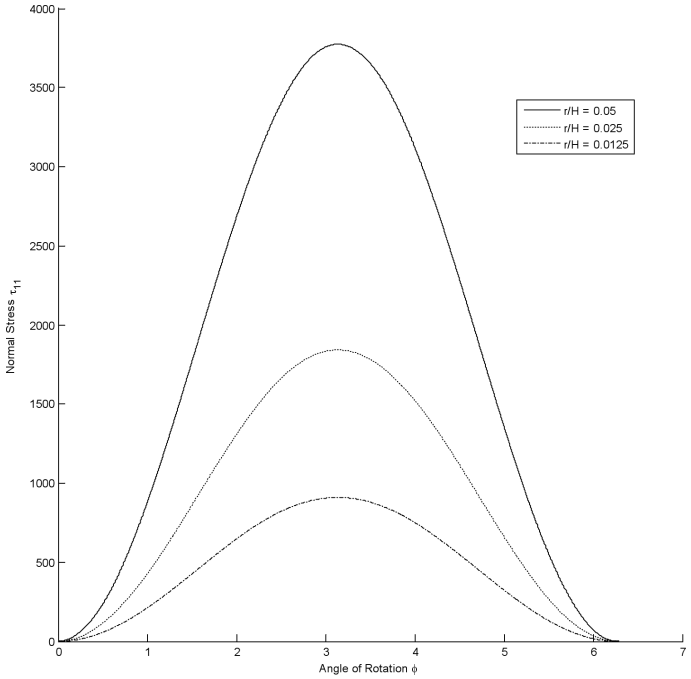

FIG. 4. Cycle 1: Normal stress τ_{11} vs. angle of rotation ϕ.

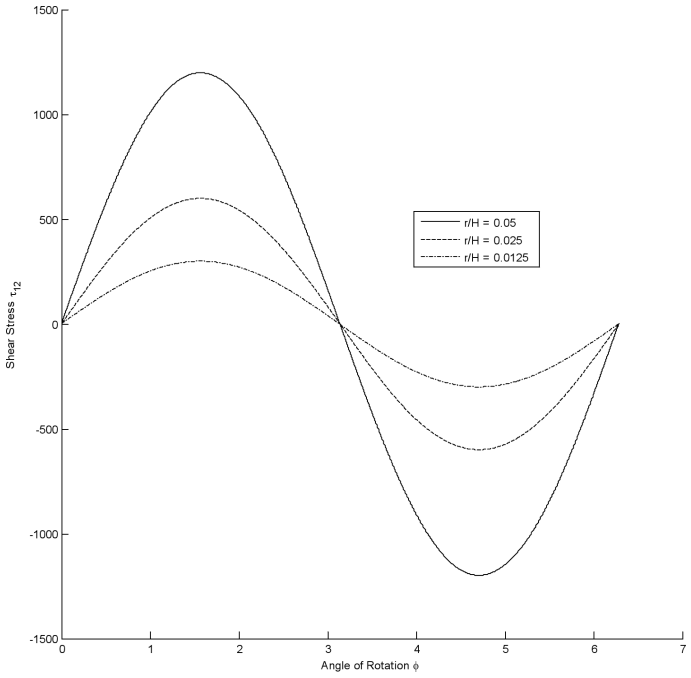

FIG. 5. Cycle 1: Shear stress τ_{12} vs. angle of rotation ϕ.

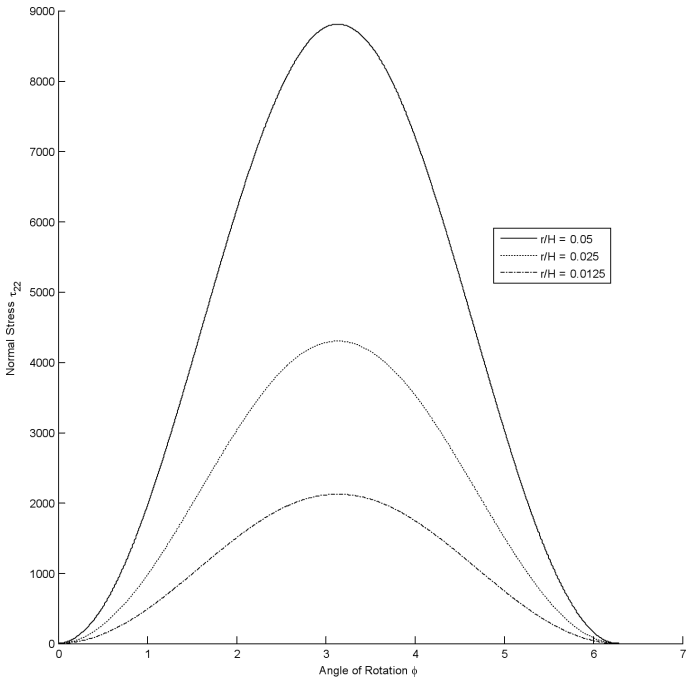

FIG. 6. Cycle 1: Normal stress τ_{22} vs. angle of rotation ϕ.

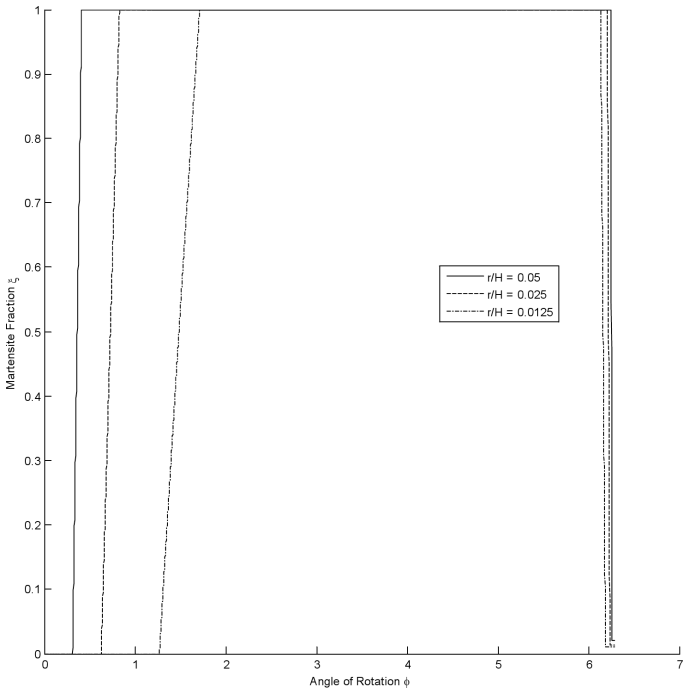

FIG. 7. Cycle 1: Material martensite fraction vs. angle of rotation ϕ.

As a further example, the response of an SMA specimen which is subjected to 10 straining cycles ($r/H = 0.025$), is illustrated in Fig. 8. As it was expected the aforementioned non-conventional pseudoelastic response appears once more, and the corresponding stress angle of rotation curves have the same qualitative characteristics with those of an elastic material; see for instance Fig. 5 in MEYERS *et al.* [52].

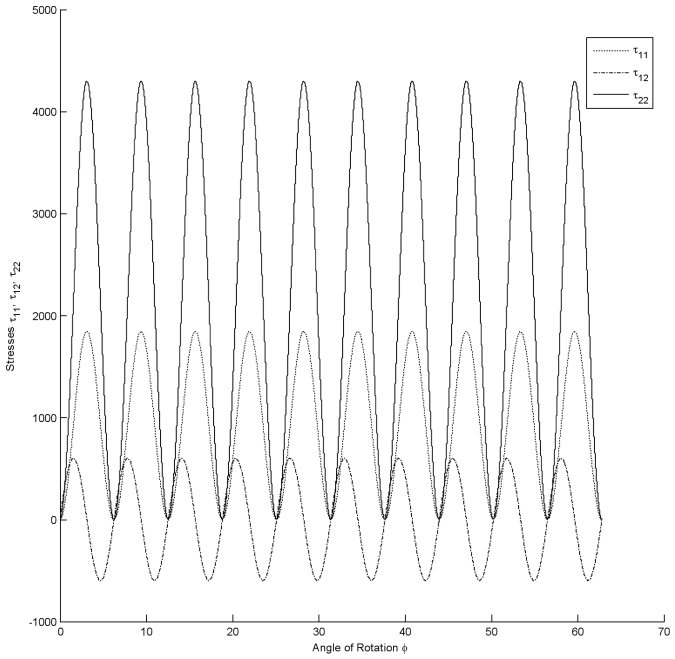

FIG. 8. Cycle 1 stresses vs. angle of rotation.

In order to provide a further insight to this kind of non-conventional pseudoelastic response, a second strain cycle (Cycle 2) which is accompanied by large rotations is considered as well. In this cycle the square element is subjected to a deformation according to which the two upper corners are rotated by a cycle of radius r to their right size; see Fig. 2 in MEYERS *et al.* [52]. As in the previous problem and in this one, the material element preserves its initial (parallelogram) shape. This problem is defined as:

$$x_1 = X_1 + \frac{(1-\cos\phi)(r/H)}{1+\sin\phi(r/H)}X_2,$$

$$x_2 = (1+\sin\phi(r/H))X_2, \qquad x_3 = X_3.$$

The corresponding results are illustrated for both one-cycle and 10-cycle cyclic loadings in Figs. 9 and 10. By referring to these figures, the aforementioned pattern of non-conventional pseudoelastic response, can be easily verified.

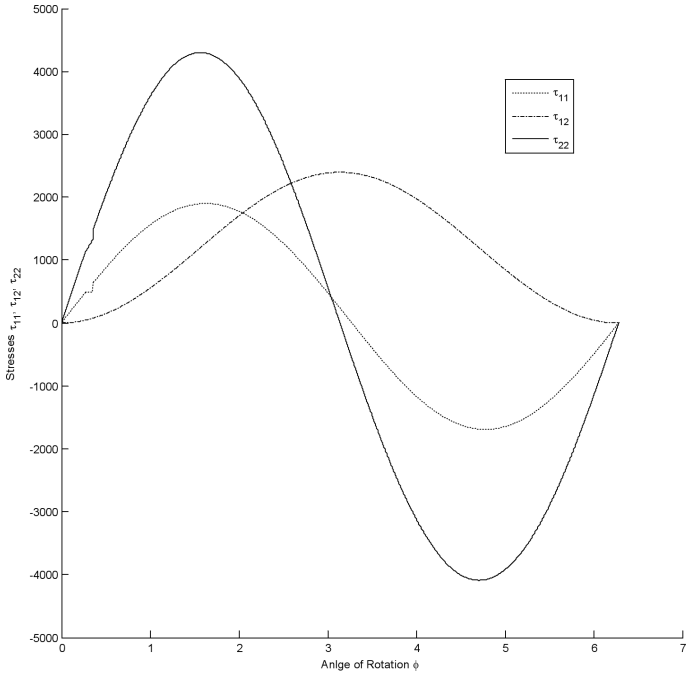

FIG. 9. Cycle 2 ($r/H = 0.05$): Stresses vs. angle of rotation ϕ.

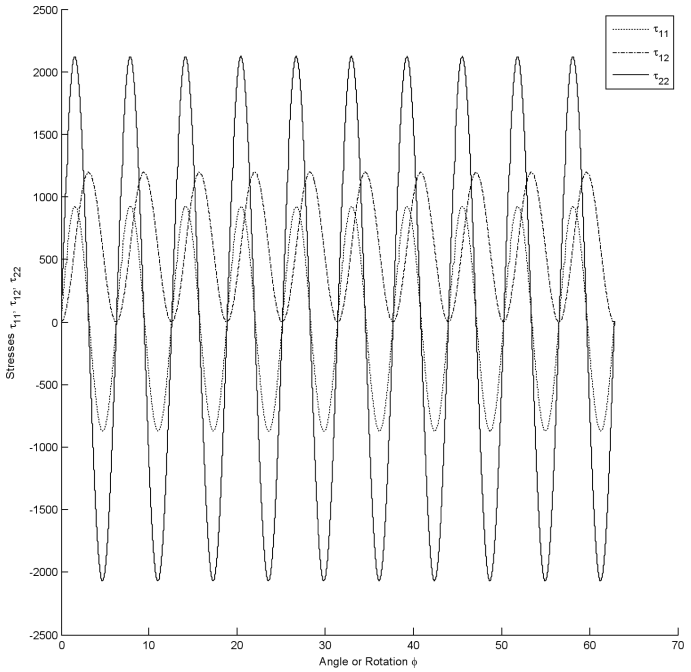

FIG. 10. Cycle 2 ($r/H = 0.025$): Stresses vs. angle of rotation ϕ.

4. Concluding remarks

The basic contribution of this paper is the provision of several insights into the constitutive modelling of shape memory alloys within the context of large deformation generalized plasticity. In particular, in this paper by assuming the multiplicative decomposition of the deformation gradient into elastic and inelastic (transformation induced) parts as a basic kinematic assumption and a standard fractions approach:

1. We have introduced a rather general model accounting for diffusionless phase transformations in a two-phase shape memory alloy. The model has been developed in an invariant setting, that is in a setting where the basic equations have an identical format in both the reference and the spatial configurations.
2. We have shown possible extensions of the model in order to deal with more complicated phenomena appearing in SMAs such as the tension-compression asymmetry and the rate of loading effects.
3. We have implemented the model numerically and we have shown its ability in predicting several patterns of the extremely complex response of this material under both monotonic and cyclic loadings.

Moreover, we have also exploited mathematically the formulation of the rate equation underlying the evolution of the material martensite fraction, which plays an essential role in the correct prediction of the material response.

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