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SAINT-VENANT'S PROBLEM FOR HETEROGENEOUS ELASTIC SOLIDS

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This paper gives a method for solving Saint-Venant's problem in the case of inhomogeneous and isotropic elastic cylinders when the elastic coefficients are independent of the axial coordinate. The cross-section of the cylinder is assumed to be occupied by different inhomogeneous and isotropic elastic materials. The problem is then generalized to the case when the cylinder is subjected to body forces and to surface tractions on the lateral surface.

1. Introduction

The importance of Saint-Venant's celebrated memoirs [1, 2] what has long since become known as Saint-Venant's problem, requires no emphasis. Most of the papers dealing with Saint-Venant's problem are restricted to homogeneous or piecewise homogeneous cylinders. However, some investigations (see, e.g., [3–6]) are devoted to Saint-Venant's problem for inhomogeneous cylinders where the elastic coefficients are independent of the axial coordinate, those being prescribed functions of the remaining coordinates. In this case the problem is entirely solved only when Poisson's ratio is constant.

This paper gives a solution to Saint-Venant's problem for inhomogeneous and isotropic elastic bodies, without considering the mentioned restriction. Moreover, we study the case of a composed cylinder where the generic cross-section is occupied by different inhomogeneous and isotropic elastic materials. We also consider a generalization of the preceding problem to the case when the cylinder is subjected to body forces and to surface tractions on the lateral surface and to appropriate stress resultants over its ends.

2. Statement of the problem

Throughout this paper R denotes the interior of a right cylinder of the length l with the generic cross-section Σ and the lateral boundary B. We call ∂R the boundary of R, and designate by n_i the components of the outward unit normal of ∂R. Moreover, a rectangular Cartesian coordinate system $0x_k$ ($k=1, 2, 3$) is used. The rectangular Cartesian coordinate frame is chosen such that the x_3-axis is parallel to the generators of R and the $x_1 0 x_2$-plane contains one of terminal cross-sections. We call $\Sigma^{(0)}$ the cross-section located at $x_3=0$ and $\Sigma^{(l)}$ the cross-section which lies in the plane $x_3=l$. We denote by L the boundary of the generic cross-section Σ.

We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2), whereas Latin subscripts—

over (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

In this paper we consider the linear theory of classical elastcity. Let u_i denote the components of the displacement vector field. The components of the strain tensor are given by

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}). \tag{2.1}$$

The stress-strain relations in the case of an isotropic elastic medium are

$$t_{ij}=\lambda e_{rr}\delta_{ij}+2\mu e_{ij}, \tag{2.2}$$

where t_{ij} are the components of the stress tensor, λ and μ are Lamé moduli and δ_{ij} is the Kronecker delta.

The equations of equilibrium, in absence of body forces, are

$$t_{ij,j}=0. \tag{2.3}$$

The surface tractions acting at a point x on the oriented surface S are given by

$$t_i=t_{ij}n_j, \tag{2.4}$$

where n_j are the direction cosines of the exterior normal to S at x.

We assume that Σ is a C^1-smooth domain [7]. Let L_1 and L_2 be two disjoint subsets of L such that $L=L_1\cup L_2$. Let Γ be a curve contained in Σ satisfying the condition that $L_\rho\cup\Gamma$ $(\rho=1, 2)$ is the boundary of a domain Σ_ρ contained in Σ such that $\Sigma_1\cap\Sigma_2=\varnothing$.

Suppose that Σ_1 and Σ_2 are occupied by two elastic materials. Let $\lambda^{(\rho)}$, $\mu^{(\rho)}$ be the Lamé moduli relative to Σ_ρ. We denote by R_ρ the domain occupied by the material with the Lamé moduli $\lambda^{(\rho)}$, $\mu^{(\rho)}$.

Throughout this paper we assume that

$$\lambda^{(\rho)}=\lambda^{(\rho)}(x_1, x_2),\ \mu^{(\rho)}=\mu^{(\rho)}(x_1, x_2) \text{ on } R_\rho. \tag{2.5}$$

The functions $\lambda^{(\rho)}$, $\mu^{(\rho)}$ are supposed to belong to C^∞, and the elastic potential corresponding to the body which occupies R_ρ is assumed to be a positive definite quadratic form.

We can consider Σ as being occupied by an elastic medium which, in general, has elastic coefficients discontinuous along Γ. We denote by Π the surface of separation of the two materials.

The displacement vector and the stress vector must be continuous in passing from one medium to another so that we have the conditions

$$[u_i]_1=[u_i]_2, \quad [t_{i\beta}]_1\nu_\beta=[t_{i\beta}]_2\nu_\beta \quad \text{on } \Pi, \tag{2.6}$$

where it has been indicated that the expressions in the parentheses are calculated for the domains R_1 and R_2, respectively, and ν_β are the direction cosines of the vector normal to Γ, outward to Σ_1.

The cylinder is supposed to be free from lateral loading so that we have the conditions

$$(2.7) \qquad t_{i\alpha} n_\alpha = 0 \quad \text{on } B.$$

The load of the cylinder is distributed over its ends in a way which fulfills the equilibrium conditions of a rigid body. Let the loading applied on $\Sigma^{(0)}$ be statically equivalent to a force $P(P_i)$ and a moment $M(M_i)$.

The aim of Saint-Venant's problem is to find out a solution for the equations (2.1)–(2.3) which would satisfy the conditions (2.6), (2.7) and the conditions on $\Sigma^{(0)}$. For the purpose of convenience the problem of extension and bending is treated separately in this paper.

3. Auxiliary plane strain problems

Let us consider the static problem of the plane strain in the cross-section Σ of the cylinder. For this problem we take into account the body force $f_\rho^{(\alpha)} \in C^\infty(\bar{\Sigma}_\rho)$. Here, the field equations are

equilibrium equations

$$(3.1) \qquad t_{\alpha\beta,\alpha} + f_\beta^{(\rho)} = 0,$$

stress-strain relations

$$(3.2) \qquad t_{\alpha\beta} = \lambda^{(\rho)} e_{\nu\nu} \delta_{\alpha\beta} + 2\mu^{(\rho)} e_{\alpha\beta},$$

geometrical relations

$$(3.3) \qquad 2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} \quad \text{in } \Sigma_\rho,$$

where $u_\alpha = u_\alpha(x_1, x_2)$.

If the displacement vector and the stress vector are assumed to be continuous in passing from one medium to another, then we obtain the conditions

$$(3.4) \qquad [u_\alpha]_1 = [u_\alpha]_2, \qquad [t_{\alpha\beta}]_1 \nu_\beta = [t_{\alpha\beta}]_2 \nu_\beta \quad \text{on } \Gamma.$$

We consider the following boundary conditions

$$(3.5) \qquad [t_{\alpha\beta} n_\beta]_\rho = h_\alpha^{(\rho)} \quad \text{on } L_\rho,$$

where $h_\alpha^{(\rho)}$ are C^∞-functions.

It is known that if the domains Σ, Σ_ρ satisfy some conditions of regularity [7, p. 386], then the boundary value problem (3.1)–(3.5) has a solution $u_\alpha \in C^\infty(\Sigma_1 \cup L_1) \cap C^\infty(\Sigma_2 \cup L_2) \cap C^0(\Sigma)$ when, and only when,

$$(3.6) \qquad \sum_{\rho=1}^{2} \Big[\int_{\Sigma_\rho} f_\alpha^{(\rho)} d\sigma + \int_{L_\rho} h_\alpha^{(\rho)} ds \Big] = 0, \qquad \sum_{\rho=1}^{2} \Big[\int_{\Sigma_\rho} \varepsilon_{3\alpha\beta} x_\alpha f_\beta^{(\rho)} d\sigma + \int_{L_\rho} \varepsilon_{3\alpha\beta} x_\alpha h_\beta^{(\rho)} ds \Big] = 0,$$

where ε_{ijk} is the alternating symbol.

In what follows we assume that the domains Σ, Σ_ρ satisfy the conditions which insure this existence theorem.

It is easy to show that if the conditions (3.4) are replaced by

$$[u_\alpha]_1=[u_\alpha]_2, \quad [t_{\alpha\beta}]_1 \nu_\beta=[t_{\alpha\beta}]_2 \nu_\beta+g_\alpha \quad \text{on } \Gamma, \tag{3.7}$$

where g_α are C^∞-functions then the conditions (3.6) become

$$\begin{gathered}\sum_{\rho=1}^{2}\Big[\int_{\Sigma_\rho} f_\alpha^{(\rho)}\,d\sigma+\int_{L_\rho} h_\alpha^{(\rho)}\,ds\Big]+\int_\Gamma g_\alpha\,ds=0,\\ \sum_{\rho=1}^{2}\Big[\int_{\Sigma_\rho} \varepsilon_{3\alpha\beta}\,x_\alpha f_\beta^{(\rho)}\,d\sigma+\int_{L_\rho} \varepsilon_{3\alpha\beta}\,x_\alpha h_\beta^{(\rho)}\,ds\Big]+\int_\Gamma \varepsilon_{3\alpha\beta}\,x_\alpha g_\beta\,ds=0.\end{gathered} \tag{3.8}$$

We will have the occasion to use three special problems $P^{(s)}$ $(s=1,2,3)$ of plane strain. We denote by $v_i^{(s)}$, $\varepsilon_{ij}^{(s)}$ and $\sigma_{ij}^{(s)}$ the components of the displacement vector, the components of the strain tensor and the components of the stress tensor from the problem $P^{(s)}$.

The problem $P^{(s)}$ is characterized by the equations

$$\varepsilon_{\alpha\beta}^{(s)}=\frac{1}{2}\,(v_{\alpha,\beta}^{(s)}+v_{\beta,\alpha}^{(s)}), \tag{3.9}$$

$$\sigma_{\alpha\beta}^{(s)}=\lambda^{(\rho)}\,\varepsilon_{\eta\eta}^{(s)}\,\delta_{\alpha\beta}+2\mu^{(\rho)}\,\varepsilon_{\alpha\beta}^{(s)}, \tag{3.10}$$

$$\sigma_{\alpha\beta,\beta}^{(\eta)}+(\lambda^{(\rho)}x_\eta)_{,\alpha}=0, \quad \sigma_{\alpha\beta,\beta}^{(3)}+\lambda_{,\alpha}^{(\rho)}=0 \quad \text{in } \Sigma_\rho \quad (\eta,\rho=1,2; \quad s=1,2,3), \tag{3.11}$$

and the following conditions

$$[v_\alpha^{(s)}]_1=[v_\alpha^{(s)}]_2, \quad [\sigma_{\alpha\beta}^{(s)}]_1\,\nu_\beta=[\sigma_{\alpha\beta}^{(s)}]_2\,\nu_\beta+g_\alpha^{(s)} \quad \text{on } \Gamma, \tag{3.12}$$

$$[\sigma_{\alpha\beta}^{(\eta)}\,n_\beta]_\rho=-\lambda^{(\rho)}\,x_\eta n_\alpha, \quad [\sigma_{\alpha\beta}^{(3)}\,n_\beta]_\rho=-\lambda^{(\rho)}\,n_\alpha \quad \text{on } L_\rho, \tag{3.13}$$

where

$$g_\alpha^{(\eta)}=[\lambda^{(2)}-\lambda^{(1)}]\,x_\eta\nu_\alpha, \quad g_\alpha^{(3)}=[\lambda^{(2)}-\lambda^{(1)}]\,\nu_\alpha \quad (\eta=1,2). \tag{3.14}$$

It is easy to show that the necessary and sufficient conditions (3.8) for the existence of the solution are satisfied for each boundary value problem $P^{(s)}$. In what follows we assume that the functions $v_\alpha^{(s)}$, $\varepsilon_{\alpha\beta}^{(s)}$, $\sigma_{\alpha\beta}^{(s)}$ are known.

4. Extension and bending by terminal couples

Let the loading applied at the end $\Sigma^{(0)}$ be statically equivalent to the force $P(0,0,P_3)$ and the moment $M(M_1,M_2,0)$. Thus, for $x_3=0$ we obtain the following conditions

$$\int_\Sigma t_{\alpha 3}\,d\sigma=0, \tag{4.1}$$

$$\int_\Sigma t_{33}\,d\sigma=-P_3, \tag{4.2}$$

$$\int_{\Sigma} x_\alpha t_{33}\, d\sigma = \varepsilon_{\alpha\beta 3} M_\beta, \tag{4.3}$$

$$\int_{\Sigma} \varepsilon_{3\alpha\beta} x_\alpha t_{\beta 3}\, d\sigma = 0. \tag{4.4}$$

The problem consists in solving the equations (2.1)–(2.3) with the conditions (2.6), (2.7), (4.1)–(4.4).

We seek the solution in the form

$$u_\alpha = -\frac{1}{2} a_\alpha x_3^2 + \sum_{j=1}^{3} a_j v_\alpha^{(j)}, \qquad u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3, \tag{4.5}$$

where $v_\alpha^{(j)}$ are the components of the displacement vectors from the auxiliary plane strain problems $P^{(s)}$, and a_i are unknown constants.

From (2.1), (4.5) we get

$$e_{\alpha\beta} = \sum_{j=1}^{3} a_j \varepsilon_{\alpha\beta}^{(j)}, \qquad e_{\alpha 3} = 0, \qquad e_{33} = a_1 x_1 + a_2 x_2 + a_3. \tag{4.6}$$

Using (4.6), from (2.2) we obtain

$$\begin{gathered} t_{\alpha\beta} = \lambda^{(\rho)} (a_1 x_1 + a_2 x_2 + a_3)\, \delta_{\alpha\beta} + \sum_{j=1}^{3} a_j \sigma_{\alpha\beta}^{(j)}, \qquad t_{\alpha 3} = 0, \\ t_{33} = (\lambda^{(\rho)} + 2\mu^{(\rho)}) (a_1 x_1 + a_2 x_2 + a_3) + \lambda^{(\rho)} \sum_{j=1}^{3} a_j \varepsilon_{\alpha\alpha}^{(j)} \quad \text{in } \Sigma_\rho. \end{gathered} \tag{4.7}$$

The equilibrium equations and the boundary conditions (2.7) are satisfied on the basis of the relations (3.11), (3.13). The conditions (2.6) are satisfied in view of the relations (3.12), (3.14).

From (4.2), (4.3) and (4.7) we obtain the following system for the unknown constants a_i

$$D_{\alpha j} a_j = \varepsilon_{3\alpha\beta} M_\beta, \qquad D_{3j} a_j = -P_3, \tag{4.8}$$

where

$$\begin{aligned} D_{\alpha\beta} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha [(\lambda^{(\rho)} + 2\mu^{(\rho)}) x_\beta + \lambda^{(\rho)} \varepsilon_{\eta\eta}^{(\beta)}]\, d\sigma, \\ D_{\alpha 3} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha [\lambda^{(\rho)} + 2\mu^{(\rho)} + \lambda^{(\rho)} \varepsilon_{\beta\beta}^{(3)}]\, d\sigma, \\ D_{3\alpha} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [(\lambda^{(\rho)} + 2\mu^{(\rho)}) x_\alpha + \lambda^{(\rho)} \varepsilon_{\beta\beta}^{(\alpha)}]\, d\sigma, \\ D_{33} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [\lambda^{(\rho)} + 2\mu^{(\rho)} + \lambda^{(\rho)} \varepsilon_{\beta\beta}^{(3)}]\, d\sigma. \end{aligned} \tag{4.9}$$

Let us prove that the system (4.8) uniquely determines the constants a_i. We use a similar procedure to that in [8, 9].

The elastic potentials corresponding to the considered materials

$$W^{(\rho)}(u)=\frac{1}{2}[t_{ij}e_{ij}]_\rho=\frac{1}{2}\lambda^{(\rho)}e_{rr}e_{ss}+\mu^{(\rho)}e_{ij}e_{ij}\quad(\rho=1,2), \tag{4.10}$$

are positive definite quadratic forms. Let us consider two elastic configurations $\{u'_i, e'_{ij}, t'_{ij}\}$ and $\{u''_i, e''_{ij}, t''_{ij}\}$. If we denote

$$W^{(\rho)}(u', u'')=\frac{1}{2}[t'_{ij}e''_{ij}]_\rho=\frac{1}{2}\lambda^{(\rho)}e'_{ss}e''_{rr}+\mu^{(\rho)}e'_{ij}e''_{ij}, \tag{4.11}$$

then

$$W^{(\rho)}(u', u'')=W^{(\rho)}(u'', u'),\qquad W^{(\rho)}(u, u)=W^{(\rho)}(u). \tag{4.12}$$

From (2.3), (2.4), (2.6), (4.11) and (4.12) we obtain

$$2\int_R W(u', u'')\,dv\equiv 2\sum_{\rho=1}^{2}\int_{R_\rho}W^{(\rho)}(u', u'')\,dv=\int_{\partial R}t'_i u''_i\,d\sigma=\int_{\partial R}t''_i u'_i\,d\sigma. \tag{4.13}$$

Obviously, the total elastic potential is

$$U=\int_R W(u)\,dv=\sum_{\rho=1}^{2}\int_{R_\rho}W^{(\rho)}(u)\,dv=\frac{1}{2}\int_{\partial R}t_i u_i\,d\sigma. \tag{4.14}$$

The relations (4.5)–(4.7) can be written in the form

$$u_i=\sum_{s=1}^{3}a_s u_i^{(s)},\qquad e_{ij}=\sum_{s=1}^{3}a_s e_{ij}^{(s)},\qquad t_{ij}=\sum_{s=1}^{3}a_s t_{ij}^{(s)}. \tag{4.15}$$

It is easy to show that the total elastic potential is given by

$$U=U_{ij}a_i a_j, \tag{4.16}$$

where

$$U_{ij}=\int_R W(u^{(i)}, u^{(j)})\,dv=U_{ji}. \tag{4.17}$$

From (4.5), (4.7) and (4.15) it follows that we have

$$\begin{gathered} u_\alpha^{(\beta)}=-\frac{1}{2}l^2\delta_{\alpha\beta}+v_\alpha^{(\beta)},\qquad u_3^{(\alpha)}=lx_\alpha,\qquad u_\alpha^{(3)}=v_\alpha^{(3)},\qquad u_3^{(3)}=l,\\ t_\alpha^{(s)}=0,\quad t_3^{(\alpha)}=(\lambda^{(\rho)}+2\mu^{(\rho)})x_\alpha+\lambda^{(\rho)}\varepsilon_{\beta\beta}^{(\alpha)},\quad t_3^{(3)}=\lambda^{(\rho)}+2\mu^{(\rho)}+\lambda^{(\rho)}\varepsilon_{\alpha\alpha}^{(3)}\ \text{ on } x_3=l, \end{gathered} \tag{4.18}$$

and

$$t_\alpha^{(s)}=0,\qquad u_3^{(s)}=0,\qquad u_\alpha^{(s)}=v_\alpha^{(s)}\quad\text{on } x_3=0. \tag{4.19}$$

Using the boundary conditions (2.7) and the relations (4.17), (4.13), (4.14), (4.18), (4.19), we can write

$$(4.20)\qquad \begin{aligned} 2U_{11} &= \int_{\partial R} t_i^{(1)} u_i^{(1)}\, d\sigma = \int_{\Sigma^{(0)}+\Sigma^{(l)}+B} t_i^{(1)} u_i^{(1)}\, d\sigma = \int_{\Sigma^{(l)}} t_3^{(1)} u_3^{(1)}\, d\sigma = lD_{11}\,, \\ 2U_{12} &= \int_{\partial R} t_i^{(1)} u_i^{(2)}\, d\sigma = lD_{12}\,. \end{aligned}$$

In a similar way we can prove the relations

$$(4.21)\qquad 2U_{ij} = lD_{ij}\,.$$

Taking into account the relations (4.21), (4.16) and (4.14), it follows that

$$(4.22)\qquad \det(D_{ij}) \neq 0\,,$$

so that the system (4.8) uniquely determines the constants a_i. The conditions (4.1) and (4.4) are satisfied on the basis of the relations (4.7). Thus, the problem is solved.

5. Torsion and flexure

Let the loading applied at the end $\Sigma^{(0)}$ be statically equivalent to the force $P(P_1, P_2, 0)$ and the moment $M(0, 0, M_3)$. In this case on the plane $x_3=0$ we have the following conditions:

$$(5.1)\qquad \int_\Sigma t_{\alpha 3}\, d\sigma = -P_\alpha\,,$$

$$(5.2)\qquad \int_\Sigma t_{33}\, d\sigma = 0\,,$$

$$(5.3)\qquad \int_\Sigma x_\alpha t_{33}\, d\sigma = 0\,,$$

$$(5.4)\qquad \int_\Sigma \varepsilon_{3\alpha\beta} x_\alpha t_{\beta 3}\, d\sigma = -M_3\,.$$

The problem consists in solving the equations (2.1)–(2.3) with the conditions (2.6), (2.7), (5.1)–(5.4).

We seek the solution in the form

$$(5.5)\qquad \begin{aligned} u_\alpha &= -\frac{1}{6} b_\alpha x_3^3 - \tau\varepsilon_{\alpha\beta 3} x_\beta x_3 + x_3 \sum_{j=1}^{3} b_j v_\alpha^{(j)}\,, \\ u_3 &= \frac{1}{2}(b_1 x_1 + b_2 x_2 + b_3) x_3^2 + F(x_1, x_2)\,, \end{aligned}$$

where $v_\alpha^{(j)}$ are the components of the displacement vectors from the auxiliary plane strain problems already considered in Sect. 3; b_i, τ are unknown constants and F is an unknown function.

From (2.1), (2.2) and (5.5) we obtain

$$t_{\alpha\beta}=\lambda^{(\rho)}(b_1x_1+b_2x_2+b_3)x_3\delta_{\alpha\beta}+x_3\sum_{j=1}^{3}b_j\sigma_{\alpha\beta}^{(j)},$$

$$(5.6)\qquad t_{\alpha 3}=\mu^{(\rho)}\left(F_{,\alpha}-\varepsilon_{\alpha\beta 3}\,\tau x_\beta+\sum_{j=1}^{3}b_j v_\alpha^{(j)}\right),$$

$$t_{33}=(\lambda^{(\rho)}+2\mu^{(\rho)})(b_1x_1+b_2x_2+b_3)x_3+\lambda^{(\rho)}x_3\sum_{j=1}^{3}b_j\varepsilon_{\alpha\alpha}^{(j)}\quad\text{in }\Sigma_\rho.$$

Obviously, the conditions (5.2), (5.3) are identically satisfied.

Using the last of the equilibrium equations and the conditions (2.6), (2.7) we can write

$$(5.7)\qquad \int_\Sigma t_{\alpha 3}\,d\sigma=\int_\Sigma (t_{\alpha 3}+x_\alpha t_{3i,i})\,d\sigma=\int_\Sigma [(x_\alpha t_{3\beta})_{,\beta}+x_\alpha t_{33,3}]\,d\sigma=\int_\Sigma x_\alpha t_{33,3}\,d\sigma.$$

Taking into account (5.6) and (5.7) from (5.1), we obtain

$$(5.8)\qquad D_{\alpha j}b_j=-P_\alpha,$$

where $D_{\alpha j}$ are given by (4.9).

On the basis of the relations (3.11)–(3.14) it follows that the equilibrium equations and the conditions (2.6), (2.7) are satisfied if the function F satisfies the equation

$$(5.9)\qquad (\mu^{(\rho)}F_{,\alpha})_{,\alpha}=-p^{(\rho)}\quad\text{in }\Sigma_\rho,$$

and the following conditions

$$(5.10)\qquad [F]_1=[F]_2,\quad \mu^{(1)}\frac{\partial F}{\partial\nu}=\mu^{(2)}\frac{\partial F}{\partial\nu}+k\quad\text{on }\Gamma,$$

$$(5.11)\qquad \mu^{(\rho)}\frac{\partial F}{\partial n}=m^{(\rho)}\quad\text{on }L_\rho,$$

where

$$p^{(\rho)}=-\left[\mu^{(\rho)}\left(\varepsilon_{\alpha\beta 3}\,\tau x_\beta-\sum_{j=1}^{3}b_j v_\alpha^{(j)}\right)\right]_{,\alpha}+(\lambda^{(\rho)}+2\mu^{(\rho)})\times$$
$$\times(b_1x_1+b_2x_2+b_3)+\lambda^{(\rho)}\sum_{j=1}^{3}b_j\varepsilon_{\alpha\alpha}^{(j)},$$

$$(5.12)\qquad k=(\mu^{(1)}-\mu^{(2)})\left(\varepsilon_{\alpha\beta 3}\tau x_\beta-\sum_{j=1}^{3}b_j v_\alpha^{(j)}\right)\nu_\alpha,$$

$$m^{(\rho)}=\mu^{(\rho)}\left(\varepsilon_{\alpha\beta 3}\,\tau x_\beta-\sum_{j=1}^{3}b_j v_\alpha^{(j)}\right)n_\alpha.$$

The necessary and sufficient condition for the existence of the solution of the boundary value problem (5.9)–(5.11) is [7]

$$(5.13)\qquad \sum_{\rho=1}^{2}\left[\int_{\Sigma_\rho}p^{(\rho)}\,d\sigma+\int_{L_\rho}m^{(\rho)}\,ds\right]+\int_\Gamma k\,ds=0.$$

Taking into account (5.12) and the divergence theorem, we obtain

$$\sum_{\rho=1}^{2}\Big[\int_{\Sigma_\rho} p^{(\rho)}\,d\sigma+\int_{L_\rho} m^{(\rho)}\,ds\Big]+\int_{\Gamma} k\,ds=D_{3j}\,b_j, \tag{5.14}$$

where D_{3j} are given by (4.9).

The condition (5.13) is equivalent with the following:

$$D_{3j}\,b_j=0. \tag{5.15}$$

From the system (5.8), (5.15) we uniquely determine the constants b_i. In order to determine the constant τ we introduce the torsion function $\varphi(x_1, x_2)$ which satisfies the equation

$$(\mu^{(\rho)}\varphi_{,\beta})_{,\beta}=\varepsilon_{\alpha\beta3}(x_\beta\,\mu^{(\rho)})_{,\alpha} \quad \text{in } \Sigma_\rho, \tag{5.16}$$

and the conditions

$$[\varphi]_1=[\varphi]_2, \quad \mu^{(1)}\frac{\partial\varphi}{\partial\nu}=\mu^{(2)}\frac{\partial\varphi}{\partial\nu}+(\mu^{(1)}-\mu^{(2)})\,\varepsilon_{\alpha\beta3}\,x_\beta\,\nu_\alpha \quad \text{on } \Gamma, \tag{5.17}$$

$$\frac{\partial\varphi}{\partial n}=\varepsilon_{\alpha\beta3}\,x_\beta\,n_\alpha \quad \text{on } L_\rho. \tag{5.18}$$

It is easy to show that the necessary and sufficient condition for the existence of a solution to the boundary value problem (5.16)–(5.18) is satisfied.

Let us introduce the flexure function ψ by the relation

$$F=\tau\varphi+\psi. \tag{5.19}$$

From (5.9)–(5.12), (5.16)–(5.18) and (5.19) it follows that the function ψ satisfies the equation

(5.20)

$$(\mu^{(\rho)}\psi_{,\alpha})_{,\alpha}=-\Big[\mu^{(\rho)}\sum_{j=1}^{3}b_j v_\alpha^{(j)}\Big]_{,\alpha}-(\lambda^{(\rho)}+2\mu^{(\rho)})(b_1x_1+b_2x_2+b_3)-\lambda^{(\rho)}\sum_{j=1}^{3}b_j\,\varepsilon_{\alpha\alpha}^{(j)},$$

and the conditions

$$[\varphi]_1=[\psi]_2, \quad \mu^{(1)}\frac{\partial\psi}{\partial\nu}=\mu^{(2)}\frac{\partial\psi}{\partial\nu}-(\mu^{(1)}-\mu^{(2)})\sum_{j=1}^{3}b_j v_\alpha^{(j)}\nu_\alpha \quad \text{on } \Gamma, \tag{5.21}$$

$$\frac{\partial\psi}{\partial n}=-\sum_{j=1}^{3}b_j v_\alpha^{(j)}n_\alpha \quad \text{on } L_\rho. \tag{5.22}$$

In what follows we assume that the function φ and ψ are known. From (5.19) and (5.6) we obtain

$$t_{\alpha3}=\tau\mu^{(\rho)}(\varphi_{,\alpha}-\varepsilon_{\alpha\beta3}\,x_\beta)+\mu^{(\rho)}\Big(\psi_{,\alpha}+\sum_{j=1}^{3}b_j v_\alpha^{(j)}\Big). \tag{5.23}$$

From (5.4) and (5.23) we get

$$\tau D = -M_3 - M^*, \tag{5.24}$$

where D is the torsional rigidity

$$D = \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \mu^{(\rho)} \varepsilon_{\alpha\beta3} x_\alpha (\varphi_{,\beta} - \varepsilon_{\beta\eta3} x_\eta)\, d\sigma, \tag{5.25}$$

and

$$M^* = \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \mu^{(\rho)} \varepsilon_{\alpha\beta3} x_\alpha \left(\psi_{,\beta} + \sum_{j=1}^{3} b_j v_\beta^{(j)}\right) d\sigma.$$

As in the case of homogeneous bodies [10] we can prove that $D>0$ so that the relation (5.24) determines the constant τ. Thus, Saint-Venant's problem is solved. The case of piecewise homogeneous cylinders was studied in [10].

6. Generalized Saint-Venant's problem

We shall now consider a generalization of the preceding problem to the case when the cylinder is subjected to body forces and to surface tractions on the lateral surface and to appropriate stress resultants over its ends. The problem of loaded homogeneous and isotropic cylinders was first undertaken by Almansi [11] and Michell [12] and was developed in various papers (see, e.g., [13, 14]).

In this case the equilibrium equations become

$$t_{ij,j} + f_i^{(\rho)} = 0 \quad \text{in } R_\rho, \tag{6.1}$$

where $f_i^{(\rho)}$ are the components of the body force vector.

On the lateral surface of the cylinder we have the conditions

$$[t_{i\alpha} n_\alpha]_\rho = \tilde{t}_i^{(\rho)} \quad \text{on } B_\rho, \tag{6.2}$$

where B_ρ are subsets of B and correspond to the two materials.

As in [9] we assume that the body forces and the tractions applied on the lateral surface are polynomials of the r degree in the axial coordinate x_3, namely

$$f_i^{(\rho)} = \sum_{k=0}^{r} F_{ik}^{(\rho)}(x_1, x_2)\, x_3^k, \tag{6.3}$$

$$\tilde{t}_i^{(\rho)} = \sum_{k=0}^{r} p_{ik}^{(\rho)}(x_1, x_2)\, x_3^k, \tag{6.4}$$

where $F_{ik}^{(\rho)}$ and $p_{ik}^{(\rho)}$ are prescribed functions which are supposed to belong to C^∞.

Let the loading applied on $\Sigma^{(0)}$ be statically equivalent to a force $P(P_i)$ and a moment $M(M_i)$. Thus, for $x_3=0$ we have the following conditions

$$\int_\Sigma t_{\alpha3}\, d\sigma = -P_\alpha, \tag{6.5}$$

$$\int_{\Sigma} t_{33}\, d\sigma = -P_3, \tag{6.6}$$

$$\int_{\Sigma} x_\alpha t_{33}\, d\sigma = \varepsilon_{\alpha\beta 3} M_\beta, \tag{6.7}$$

$$\int_{\Sigma} \varepsilon_{\alpha\beta 3} x_\alpha t_{\beta 3}\, d\sigma = -M_3. \tag{6.8}$$

The problem consists in finding a solution for the equations (2.1), (2.2), (6.1) which would satisfy the conditions (2.6), (6.2), (6.5)–(6.8).

Let us denote by (A) the problem of determining a solution for the equations (2.1), (2.2), (6.1) with body forces

$$f_i^{(\rho)} = F_{in}^{(\rho)}(x_1, x_2)\, x_3^n, \tag{6.9}$$

which can satisfy the conditions (2.6), (6.5)–(6.8) and

$$[t_{i\alpha} n_\alpha]_\rho = p_{in}^{(\rho)}(x_1, x_2)\, x_3^n \quad \text{on } B_\rho. \tag{6.10}$$

In (6.9) and (6.10) n is a positive integer or zero and the functions $F_{in}^{(\rho)}$, $p_{in}^{(\rho)}$ are prescribed.

Obviously, if we know the solution of the problem (A) for any n then, according to the linearity of the equations, we can determine the solution of the initial problem. We denote by $B^{(0)}$ the problem (A) for $n=0$ and by $B^{(s)}$ the problem (A) when $n=s$ $(s=1, 2, 3, \ldots, r)$ and $P_i = M_i = 0$. If the components of the displacement vector from the problem $B^{(k)}$ $(k=0, 1, \ldots, r)$ are u_{ik}, then, the components of the displacement vector of the initial problem are given by

$$u_i = \sum_{k=0}^{r} u_{ik}. \tag{6.11}$$

In order to solve the initial problem we make use of the induction method. In Sect. 7 we shall attempt to solve the problem $B^{(0)}$. In Section 8 we shall establish the solution of the problem $B^{(n+1)}$ once the solution of the problem $B^{(n)}$ (with $P_i = M_i = 0$) is known.

7. Theory of uniformly loaded cylinders

We assume that the body forces have the form

$$f_i^{(\rho)} = G_i^{(\rho)}(x_1, x_2) \quad \text{in } R_\rho, \tag{7.1}$$

and that the conditions on the lateral surface are

$$[t_{i\alpha} n_\alpha]_\rho = p_i^{(\rho)}(x_1, x_2) \quad \text{on } B_\rho. \tag{7.2}$$

In this section we establish a solution of the equations (2.1), (2.2), (6.1) which can satisfy the conditions (2.6), (6.5)–(6.8), (7.2), when the body forces are given by (7.1).

Taking into account the results from [9, 13], we seek the solution in the form

$$u_\alpha = -\frac{1}{2} a_\alpha x_3^2 - \frac{1}{6} b_\alpha x_3^3 - \frac{1}{24} A_\alpha x_3^4 - \tau \varepsilon_{3\alpha\beta} x_\beta x_3 - \frac{1}{2} C \varepsilon_{3\alpha\beta} x_\beta x_3^2 + \sum_{k=1}^{3} \left(a_k + b_k x_3 + \frac{1}{2} A_k x_3^2\right) v_\alpha^{(k)} + v_\alpha(x_1, x_2),$$

(7.3)

$$u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3 + \frac{1}{2}(b_1 x_1 + b_2 x_2 + b_3) x_3^2 + \frac{1}{6}(A_1 x_1 + A_2 x_2 + A_3) x_3^3 + F(x_1, x_2) + x_3 \Phi(x_1, x_2),$$

where $v_\alpha^{(k)}$ are the components of the displacement vectors from the auxiliary plane strain problems considered in Sect. 3; v_α, F, Φ are unknown functions and a_i, b_i, τ, A_i, C are unknown constants.

From (2.1) and (7.3) we obtain

$$e_{\alpha\beta} = \sum_{j=1}^{3} \left(a_j + b_j x_3 + \frac{1}{2} A_j x_3^2\right) \varepsilon_{\alpha\beta}^{(j)} + \gamma_{\alpha\beta},$$

(7.4)

$$2e_{\alpha 3} = F_{,\alpha} - \tau \varepsilon_{3\alpha\beta} x_\beta + [\Phi_{,\alpha} - C \varepsilon_{3\alpha\beta} x_\beta] x_3 + \sum_{j=1}^{3} (b_j + A_j x_3) v_\alpha^{(j)},$$

$$e_{33} = a_1 x_1 + a_2 x_2 + a_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3 + \frac{1}{2}(A_1 x_1 + A_2 x_2 + A_3) x_3^2 + \Phi,$$

where

(7.5) $$2\gamma_{\alpha\beta} = v_{\alpha,\beta} + v_{\beta,\alpha},$$

and $\varepsilon_{\alpha\beta}^{(j)}$ are given by (3.9).

The components of the stress tensor have the expressions

$$t_{\alpha\beta} = \lambda^{(\rho)} \left[a_1 x_1 + a_2 x_2 + a_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3 + \frac{1}{2}(A_1 x_1 + A_2 x_2 + A_3) \times x_3^2\right] \delta_{\alpha\beta} + \lambda^{(\rho)} \Phi \delta_{\alpha\beta} + \sum_{j=1}^{3} \left(a_j + b_j x_3 + \frac{1}{2} A_j x_3^2\right) \sigma_{\alpha\beta}^{(j)} + \sigma_{\alpha\beta},$$

(7.6)

$$t_{\alpha 3} = \mu^{(\rho)} \left[F_{,\alpha} - \tau \varepsilon_{3\alpha\beta} x_\beta + (\Phi_{,\alpha} - C \varepsilon_{3\alpha\beta} x_\beta) x_3 + \sum_{i=1}^{3} (b_i + A_i x_3) v_\alpha^{(i)}\right],$$

$$t_{33} = (\lambda^{(\rho)} + 2\mu^{(\rho)}) \left[a_1 x_1 + a_2 x_2 + a_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3 + \frac{1}{2}(A_1 x_1 + A_2 x_2 + A_3) x_3^2\right] + (\lambda^{(\rho)} + 2\mu^{(\rho)}) \Phi + \lambda^{(\rho)} \sum_{j=1}^{3} \left(a_j + b_j x_3 + \frac{1}{2} A_j x_3^2\right) \varepsilon_{\alpha\alpha}^{(j)} + \lambda^{(\rho)} \gamma_{\alpha\alpha},$$

where

$$\sigma_{\alpha\beta}=\lambda^{(\rho)}\gamma_{\nu\nu}\delta_{\alpha\beta}+2\mu^{(\rho)}\gamma_{\alpha\beta}. \tag{7.7}$$

Taking into account (7.6), the equilibrium equations lead to the following equations:

$$\sigma_{\alpha\beta,\beta}+H_\alpha^{(\rho)}=0, \tag{7.8}$$

$$(\mu^{(\rho)}F_{,\alpha})_{,\alpha}=g^{(\rho)}, \tag{7.9}$$

$$(\mu^{(\rho)}\Phi_{,\alpha})_{,\alpha}=h^{(\rho)} \quad \text{in } \Sigma_\rho, \tag{7.10}$$

where

$$\begin{aligned} H_\alpha^{(\rho)}&=G_\alpha^{(\rho)}+(\lambda^{(\rho)}\Phi)_{,\alpha}+\mu^{(\rho)}\Phi_{,\alpha}-\mu^{(\rho)}C\varepsilon_{3\alpha\beta}x_\beta+\mu^{(\rho)}\sum_{j=1}^{3}A_j v_\alpha^{(j)},\\ g^{(\rho)}&=\tau\varepsilon_{3\alpha\beta}(\mu^{(\rho)}x_\beta)_{,\alpha}-\sum_{j=1}^{3}b_j[(\mu^{(\rho)}v_\alpha^{(j)})_{,\alpha}+\lambda^{(\rho)}\varepsilon_{\alpha\alpha}^{(j)}]-\\ &\qquad -(\lambda^{(\rho)}+2\mu^{(\rho)})(b_1x_1+b_2x_2+b_3)-G_3^{(\rho)},\\ h^{(\rho)}&=C\varepsilon_{3\alpha\beta}(\mu^{(\rho)}x_\beta)_{,\alpha}-\sum_{j=1}^{3}A_j[(\mu^{(\rho)}v_\alpha^{(j)})_{,\alpha}+\lambda^{(\rho)}\varepsilon_{\alpha\alpha}^{(j)}]-\\ &\qquad -(\lambda^{(\rho)}+2\mu^{(\rho)})(A_1x_1+A_2x_2+A_3). \end{aligned} \tag{7.11}$$

The conditions (2.6) are satisfied if

$$[v_\alpha]_1=[v_\alpha]_2, \qquad [\sigma_{\alpha\beta}]_1\nu_\beta=[\sigma_{\alpha\beta}]_2\nu_\beta+(\lambda^{(2)}-\lambda^{(1)})\Phi\nu_\alpha, \tag{7.12}$$

$$[F]_1=[F]_2, \qquad \mu^{(1)}\frac{\partial F}{\partial\nu}=\mu^{(2)}\frac{\partial F}{\partial\nu}+(\mu^{(1)}-\mu^{(2)})\Big(\varepsilon_{\alpha\beta3}\tau x_\beta-\sum_{j=1}^{3}b_j v_\alpha^{(j)}\Big)\nu_\alpha, \tag{7.13}$$

$$[\Phi]_1=]\Phi]_2, \qquad \mu^{(1)}\frac{\partial\Phi}{\partial\nu}=\mu^{(2)}\frac{\partial\Phi}{\partial\nu}+(\mu^{(1)}-\mu^{(2)})\Big(\varepsilon_{\alpha\beta3}Cx_\beta-\sum_{j=1}^{3}A_j v_\alpha^{(j)}\Big)\nu_\alpha \quad \text{on } \Gamma. \tag{7.14}$$

From the conditions on the lateral surface (7.2) we obtain the following boundary conditions:

$$[\sigma_{\alpha\beta}n_\beta]_\rho=P_\alpha^{(\rho)}, \tag{7.15}$$

$$\mu^{(\rho)}\frac{\partial F}{\partial n}=Q^{(\rho)}, \tag{7.16}$$

$$\mu^{(\rho)}\frac{\partial F}{\partial n}=K^{(\rho)} \quad \text{on } L_\rho, \tag{7.17}$$

where

$$P_\alpha^{(\rho)}=p_\alpha^{(\rho)}-\lambda^{(\rho)}\Phi n_\alpha, \tag{7.18}$$

$$Q^{(\rho)}=p_3^{(\rho)}+\mu^{(\rho)}\Big(\tau\varepsilon_{3\alpha\beta}x_\beta-\sum_{j=1}^{3}b_j v_\alpha^{(j)}\Big)n_\alpha, \tag{7.19}$$

$$K^{(\rho)}=\mu^{(\rho)}\Big(C\varepsilon_{3\alpha\beta}x_\beta-\sum_{j=1}^{3}A_j v_\alpha^{(j)}\Big)n_\alpha. \tag{7.20}$$

Let us consider the boundary value problem (7.10), (7.14), (7.17). The necessary and sufficient conditions to solve this problem reduce to

$$D_{3i} A_i = 0, \tag{7.21}$$

where D_{3i} are given by (4.9). Let us now turn to the plane strain problem (7.5), (7.7), (7.8), (7.12), (7.15). The necessary and sufficient conditions to solve this problem reduce to

$$\begin{aligned} &\sum_{\rho=1}^{2} \Big\{ \int_{\Sigma_\rho} G_\rho \, d\sigma + \int_{L_\rho} p_\alpha^{(\rho)} \, ds \Big\} + \int_{\Sigma} t_{\alpha 3,3} \, d\sigma = 0, \\ &\sum_{\rho=1}^{2} \Big\{ \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta} x_\alpha G_\beta^{(\rho)} \, d\sigma + \int_{L_\rho} \varepsilon_{3\alpha\beta} x_\alpha p_\beta^{(\rho)} \, ds \Big\} + \int_{\Sigma} \varepsilon_{3\alpha\beta} x_\alpha t_{\beta 3,3} \, d\sigma = 0. \end{aligned} \tag{7.22}$$

On the basis of the equilibrium equations we can write

$$\begin{aligned} t_{\alpha 3,3} &= t_{\alpha 3,3} + x_\alpha (t_{i3,i} + G_3^{(\rho)})_{,3} = (t_{\alpha 3} + x_\alpha t_{i3,i})_{,3} = \\ &= [(x_\alpha t_{\beta 3})_{,\beta} + x_\alpha t_{33,3}]_{,3} = (x_\alpha t_{\beta 3,3})_{,\beta} + x_\alpha t_{33,33}. \end{aligned} \tag{7.23}$$

Using (7.23) and (7.17) we have

$$\int_{\Sigma} t_{\alpha 3,3} \, d\sigma = \int_{\Sigma} x_\alpha t_{33,33} \, d\sigma, \tag{7.24}$$

so that the first two relations (7.22) become independent of Φ. Taking note of (7.6) and (7.24) from the first two conditions (7.22), we obtain

$$D_{\alpha i} A_i = - \sum_{\rho=1}^{2} \Big\{ \int_{\Sigma_\rho} G_\alpha^{(\rho)} \, d\sigma + \int_{L_\rho} p_\alpha^{(\rho)} \, ds \Big\}, \tag{7.25}$$

where $D_{\alpha i}$ are given by (4.9). The system (7.21), (7.25) uniquely determines the constants A_i. In what follows we assume that the constants A_i are known. Let us determine the constant C. We introduce the function Ψ by the relation $\Phi = \Psi + C\varphi$ where φ is the torsion function. From (5.16)–(5.18), (7.10), (7.14), (7.17) it follows that the function Ψ is independent of C. It depends on the constants A_i but these are determined. We consider the function Ψ to be known. From the last of the conditions (7.22) we obtain

$$\begin{aligned} CD = - \sum_{\rho=1}^{2} \Big\{ &\int_{\Sigma_\rho} \varepsilon_{3\alpha\beta} x_\alpha G_\beta^{(\rho)} \, d\sigma + \int_{L_\rho} \varepsilon_{3\alpha\beta} x_\alpha p_\beta^{(\rho)} \, ds + \\ &+ \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta} x_\alpha \mu^{(\rho)} \Big[\Psi_{,\beta} + \sum_{j=1}^{3} A_j v_\beta^{(j)} \Big] d\sigma \Big\}, \end{aligned} \tag{7.26}$$

where D is the torsional rigidity. The constant C is determined by (7.26). In what follows we assume that the functions v_α, $\sigma_{\alpha\beta}$, Φ and the constants A_i, C are known.

Let us consider the boundary value problem (7.9), (7.13), (7.16). The necessary and sufficient conditions to solve this problem reduce to

$$D_{3i} b_i = - \sum_{\rho=1}^{2} \Big\{ \int_{\Sigma_\rho} G_3^{(\rho)} \, d\sigma + \int_{L_\rho} p_3^{(\rho)} \, ds \Big\}. \tag{7.27}$$

Let us now study the conditions (6.5). We can write

$$\int_{\Sigma} t_{\alpha 3}\, d\sigma = \int_{\Sigma} [t_{\alpha 3} + x_\alpha (t_{i3,i} + G_3^{(\rho)})]\, d\sigma = \int_{\Sigma} x_\alpha t_{33,3}\, d\sigma + \sum_{\rho=1}^{2} \Big\{ \int_{\Sigma_\rho} x_\alpha G_3^{(\rho)}\, d\sigma + \int_{L_\rho} x_\alpha p_3^{(\rho)}\, ds \Big\}. \tag{7.28}$$

From (6.5), (7.6), (7.28) we obtain

$$D_{\alpha i} b_i = -P_\alpha - \sum_{\rho=1}^{2} \Big\{ \int_{\Sigma_\rho} x_\alpha G_3^{(\rho)}\, d\sigma + \int_{L_\rho} x_\alpha p_3^{(\rho)}\, ds \Big\}. \tag{7.29}$$

The system (7.27), (7.29) determines the constants b_i. From (6.6), (6.7) and (7.6) we get

$$D_{ij} a_j = s_i , \tag{7.30}$$

where

$$\begin{aligned} s_\alpha &= \varepsilon_{3\alpha\beta} M_\beta - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha [(\lambda^{(\rho)} + 2\mu^{(\rho)}) \Phi + \lambda^{(\rho)} \gamma_{\alpha\alpha}]\, d\sigma, \\ s_3 &= -P_3 - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [(\lambda^{(\rho)} + 2\mu^{(\rho)}) \Phi + \lambda^{(\rho)} \gamma_{\alpha\alpha}]\, d\sigma . \end{aligned} \tag{7.31}$$

The constants a_i are determined by (7.30). Let us introduce the function Λ by the relation $F = \Lambda + \tau\varphi$, where φ is the torsion function. The function Λ is independent of τ. We can consider the function Λ to be known. From (6.8), (7.6) we obtain

$$\tau D = -M_3 - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta} x_\alpha \mu^{(\rho)} \Big[\Lambda_{,\beta} + \sum_{j=1}^{3} b_j v_\beta^{(j)} \Big]\, d\sigma , \tag{7.32}$$

where D is the torsional rigidity. The constant τ is determined by (7.32).

8. Recurrence process

Let us establish the solution of the problem $B^{(n+1)}$ assuming that the solution of the problem $B^{(n)}$ (in which $P_i = M_i = 0$) is known. We denote by $u_i^*, e_{ij}^*, t_{ij}^*$, respectively, the components of the displacement vector, the components of the strain tensor and the components of the stress tensor of the problem $B^{(n)}$ and by u_i, e_{ij}, t_{ij} the analogous functions of the problem $B^{(n+1)}$. As the solution of the problem $B^{(n)}$ is known for any $F_{in}^{(\rho)}, p_{in}^{(\rho)}$, we can find the solution of the problem $B^{(n)}$ for $f_i^{(\rho)} = F_{i(n+1)}^{(\rho)}(x_1, x_2)\, x_3^n$ and $\tilde{t}_i^{(\rho)} = p_{i(n+1)}^{(\rho)}(x_1, x_2)\, x_3^n$. Thus, the problem can be presented as follows: to find the functions u_i which satisfy the equations

$$\begin{aligned} & t_{ij,j} + F_i^{(\rho)}(x_1, x_2)\, x_3^{n+1} = 0, \\ & t_{ij} = \lambda^{(\rho)} e_{rr} \delta_{ij} + 2\mu^{(\rho)} e_{ij}, \quad 2e_{ij} = u_{i,j} + u_{j,i} \quad \text{in } R_\rho \end{aligned} \tag{8.1}$$

and the conditions

$$[u_i]_1=[u_i]_2, \quad [t_{i\alpha}]_1\nu_\alpha=[t_{i\alpha}]_2\nu_\alpha \quad \text{on } \Pi, \tag{8.2}$$

$$[t_{i\alpha}n_\alpha]_\rho=p_i^{(\rho)}(x_1,x_2)\,x_3^{n+1} \quad \text{on } B_\rho, \tag{8.3}$$

$$\int_\Sigma t_{i3}\,d\sigma=0, \quad \int_\Sigma \varepsilon_{ijk}x_j t_{k3}\,d\sigma=0 \quad \text{on } \Sigma^{(0)}, \tag{8.4}$$

when the solution of the equations

$$\begin{gathered} t^*_{ij,j}+F_i^{(\rho)}(x_1,x_2)\,x_3^n=0, \\ t^*_{ij}=\lambda^{(\rho)}e^*_{rr}\delta_{ij}+2\mu^{(\rho)}e^*_{ij}, \quad 2e^*_{ij}=u^*_{i,j}+u^*_{j,i} \quad \text{in } R_\rho, \end{gathered} \tag{8.5}$$

with the conditions

$$[u^*_i]_1=[u^*_i]_2, \quad [t^*_{i\alpha}]_1\nu_\alpha=[t^*_{i\alpha}]_2\nu_\alpha \quad \text{on } \Pi, \tag{8.6}$$

$$[t^*_{i\alpha}n_\alpha]_\rho=p_i^{(\rho)}(x_1,x_2)\,x_3^n \quad \text{on } B_\rho, \tag{8.7}$$

$$\int_\Sigma t^*_{i3}\,d\sigma=0, \quad \int_\Sigma \varepsilon_{ijk}x_j t^*_{k3}\,d\sigma=0 \quad \text{on } \Sigma^{(0)}, \tag{8.8}$$

is known. In the above relations $F_i^{(\rho)}$ and $p_i^{(\rho)}$ are prescribed functions which belong to C^∞. We seek the solution of the problem (8.1)–(8.4) in the form

$$u_i=(n+1)\Big[\int_0^{x_3} u^*_i\,dx_3+v_i\Big], \tag{8.9}$$

where $v_i(x_1,x_2,x_3)$ are unknown functions. The components of the stress tensor are given by

$$t_{ij}=(n+1)\Big[\int_0^{x_3} t^*_{ij}\,dx_3+\pi_{ij}+k_{ij}^{(\rho)}\Big], \tag{8.10}$$

where

$$\pi_{ij}=\lambda^{(\rho)}\gamma_{rr}\delta_{ij}+2\mu^{(\rho)}\gamma_{ij}, \quad 2\gamma_{ij}=v_{i,j}+v_{j,i}, \tag{8.11}$$

$$\begin{gathered} k_{\alpha\beta}^{(\rho)}=\lambda^{(\rho)}u^*_3(x_1,x_2,0)\,\delta_{\alpha\beta}, \quad k_{\alpha3}^{(\rho)}=\mu^{(\rho)}u^*_\alpha(x_1,x_2,0), \\ k_{33}^{(\rho)}=(\lambda^{(\rho)}+2\mu^{(\rho)})\,u^*_3(x_1,x_2,0). \end{gathered} \tag{8.12}$$

By using the relations (8.5), the equilibrium equations reduce to

$$\pi_{ij,j}+\chi_i^{(\rho)}=0 \quad \text{on } R_\rho, \tag{8.13}$$

where

$$\chi_i^{(\rho)}(x_1,x_2)=k_{i\alpha,\alpha}^{(\rho)}+t^*_{i3}(x_1,x_2,0). \tag{8.14}$$

The conditions (8.2) lead to the following conditions for the functions v_i and $\pi_{i\alpha}$

$$(8.15)\qquad [v_i]_1=[v_i]_2, \quad [\pi_{i\alpha}]_1 \nu_\alpha=[\pi_{i\alpha}]_2 \nu_\alpha+\kappa_i \quad \text{on } \Pi$$

$$(8.16)\qquad [\pi_{i\alpha} n_\alpha]_\rho=\tau_i^{(\rho)} \quad \text{on } B_\rho,$$

where

$$(8.17)\qquad \kappa_i=\kappa_i(x_1, x_2)=(k_{i\alpha}^{(2)}-k_{i\alpha}^{(1)})\nu_\alpha, \quad \tau_i^{(\rho)}=\tau_i^{(\rho)}(x_1, x_2)=-k_{i\alpha}^{(\rho)} n_\alpha.$$

From the conditions (8.4) we obtain

$$(8.18)\qquad \int_\Sigma \pi_{i3}\, d\sigma=-T_i, \quad \int_\Sigma \varepsilon_{ijk} x_j \pi_{k3}\, d\sigma=-N_i,$$

where

$$(8.19)\qquad T_i=\sum_{\rho=1}^{2} \int_{\Sigma_\rho} k_{i3}^{(\rho)}\, d\sigma, \quad N_i=\sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{ijs} x_j k_{s3}^{(\rho)}\, d\sigma.$$

Thus, the functions v_i are the components of the displacement vector in the problem characterized by the equations (8.11), (8.13), (8.15), (8.16), (8.18). In this problem the load is independent of x_3. If κ_i were to vanish, then this problem would reduce to the one solved in Section 7. However, it is easy to see that, for $\kappa_i \neq 0$ as well the solution is (7.3). Moreover, in this case the solution has the form (7.3) in which $A_i=C=b_i=0$, $\Phi=0$. Thus, the problem is solved.

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STRESZCZENIE

PROBLEM SAINT-VENANTA DLA NIEJEDNORODNYCH I SPRĘŻYSTYCH CIAŁ STAŁYCH

W niniejszej pracy przedstawiono metodę rozwiązania problemu Saint-Venanta dla niejednorodnych i izotropowych walców sprężystych, gdy współczynniki sprężyste są niezależne od współrzędnej osiowej. Założono, że przekrój cylindra wypełniony jest przez różne niejednorodne i izotropowe materiały sprężyste. Zbadano również możliwość uogólnienia problemu na przypadek, gdy powierzchnia boczna walca poddana jest siłom masowym i napięciom powierzchniowym.

Резюме

ЗАДАЧА СЭН-ВЕНАНА ДЛЯ НЕОДНОРОДНЫХ И УПРУГИХ ТВЕРДЫХ ТЕЛ

В настоящей работе представлен метод решения задачи Сэн-Венана для неоднородных и изотропных упругих цилиндров, когда упругие коэффициенты не зависят от осевой координаты. Предположено, что сечение цилиндра заполнено разными неоднородными и изотропными упругими материалами. Исследована тоже возможность обобщения задачи на случай, когда боковая поверхность цилиндра подвергнута массовым силам и поверхностным напряжениям.

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