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Research Paper

Nonlinear Vibration of a Beam Resting on a Nonlinear Viscoelastic Foundation Traversed by a Moving Mass: A Homotopy Analysis

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In this study, the dynamic response of an Euler-Bernoulli beam resting on the nonlinear viscoelastic foundation under the action of a moving mass by considering the stretching effect of the beam's neutral axis is investigated. A Dirac-delta function is applied to model the location of the moving mass along the beam as well as its inertial effects. The Galerkin decomposition method is used to transform a partial dimensionless nonlinear differential equation of dynamic motion into an ordinary nonlinear differential equation. Subsequently, the well-known homotopy analysis method (HAM) is employed to obtain an approximate analytical solution of this equation. The validity and accuracy of the solution are examined numerically using the fourth-order Runge-Kutta method. Finally, several examples are provided to show the effects of parameters such as linear and nonlinear stiffness coefficients of a viscoelastic foundation, velocity of the moving mass as well as Coriolis force, centrifugal force and inertia force of the moving mass on the dynamic deflection of the beam.

Keywords: Euler-Bernoulli beam; nonlinear viscoelastic foundation; moving mass; homotopy analysis method.

1. Introduction

Engineers have investigated the dynamic behavior of the beam affected by moving load from the establishment of the railway in the 19th century and have studied different aspects of this behavior. A problem of structure that vibrates due to the movement of a load on it can be observed in many industries and engineering applications. Investigating the effect of moving load on bridge vibrations, pipes carrying fluid, construction cranes, wood saw, computer disks, vehicle brakes and cutting tools in machining are examples of applications of such modeling [1–5]. Today, considering economic constraints, it is common to study low-weight structures carrying a moving load, which increases the probability of forming large vibrations in these structures; thus, controlling such systems has become more important than ever [6–11]. The first study on bridge vibrations was performed by Willis [12]. This study formulated a movement equation for a railway bridge for the first time; the equation was proposed assuming a beam without mass and moving load with constant speed. Mackertich [13] introduced the effects of rotational inertia and shear deformation on the Timoshenko beam. In this study, the effect of Coriolis acceleration has not been considered. Green and Cebon [14] studied the bridge-vehicle interaction by considering a vehicle model as a lumped mass supported by a spring and damper. By using the iterative method, the authors analyzed a limited range of sprung mass properties. Auciello [15] analyzed the vibrations of an elastically restrained cantilever beam of varying cross-sections under concentrated axial force using the Rayleigh-Ritz method. Chen *et al.* [16] investigated the dynamic response of an infinite beam under harmonic moving force on a viscoelastic foundation by presenting similar and comprehensive studies. In their study, critical speed and resonance frequency are obtained. Azam *et al.* [17] presented equations regarding Timoshenko beam vibrations affected by a moving mass using Hamilton's developed principle. In their study, dynamic response affected by moving force, moving mass and suspending moving mass was examined. Nikkhoo *et al.* [18] investigated the dynamic response of the beam under moving mass with constant speed using the semi-analytic method. In their analysis, the effect of the critical speed of moving force on the displacement of the system was investigated. Eftekhari [19] applied the differential quadrature method to investigate the steady state of the linear and nonlinear vibration of the Euler-Bernoulli-von Kármán beam resting on an elastic Winkler foundation; the beam was subjected to a moving point load.

In recent years, semi-analytic or analytic solutions have attracted attention in investigating the effect of different parameters on engineering systems [20–22]. One of the most applicable analytic methods is the perturbation method. But this method has a great disadvantage: its dependency on nonlinear terms is weak. The iteration method, energy balancing method and the HAM are more recent analytic-approximation methods that have been widely used and tried to resolve the perturbation method's shortcoming. The HAM has attracted attention in recent years in the analysis of vibrational systems [23, 24]. Most studies on the homotopy method can be divided into two categories: studies that involve improving accuracy and studies that demonstrate the capability of the homotopy method to solve different equations [25–27]. Papers [28–31] can be mentioned among the studies that involved solving governing equations of dynamic systems using the homotopy method. The buckling properties of a single-layered graphene sheet by using of nonlocal integral first shear deformation theory on viscoelastic medium were analyzed by Rouabhia *et al.* [32]. In another paper, the buckling behavior of a single-layered graphene sheet in a visco-Pasternak elastic medium was studied by means of nonlocal theory for the four-unknown integral model by Moussa *et al.* [33]. Some of the papers discussed the Winkler, Pasternak, and Kerr elastic foundation. Interested readers are referred to [34–42]. Based on the authors' knowledge, the nonlinear vibration of the beam on a nonlinear viscoelastic foundation under the effect of moving mass using the homotopy method has not been conducted yet. Furthermore, the HAM is a nonperturbative analytical technique for obtaining series solutions to nonlinear equations. Its capacity to choose different base functions to approximate a nonlinear problem and its ability to control the convergence of the solution series have been very advantageous in solving highly nonlinear problems in science and engineering.

The main objective of this study is to obtain the analytical expressions for nonlinear dynamic response of the Euler-Bernoulli beam resting on the nonlinear viscoelastic foundation subject to the action of a moving mass. In the derived dynamic equation, a geometric type of nonlinearity is considered. which is due to the stretching effect of the mid-plane of the beam. The first partial differential equation of beam is reduced to a typical nonlinear differential equation via the Galerkin decomposition technique. The latter equation is solved analytically in the time domain using the HAM. Afterward, in order to accurately assess and examine the precision of calculation, an analytical solution is compared with the numerical solution by the fourth-order Runge-Kutta method. Finally, effect parameters such as linear and nonlinear stiffness coefficients of a viscoelastic foundation, velocity of the moving mass as well as inertial effects of moving mass on the dynamic deflection of the beam are studied.

2. Formulation of problem

The mathematical model of a simply supported Euler-Bernoulli beam resting on a non-linear viscoelastic foundation and subjected to a moving mass is considered, as shown in Fig. 1. The moving mass travels in a straight line in

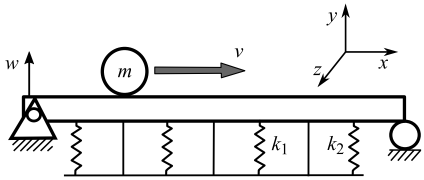

Fig. 1. A beam under the influence of moving concentrated mass on a linear and nonlinear viscoelastic foundation.

the horizontal direction, this movement is known, and the beam only vibrates in the y-direction. Let $w(x,t)$ denote the transverse displacement of the beam and x and t represent the axial and the time coordinates, respectively. The governing equation for the transverse vibration of the Euler-Bernoulli beam model under a moving mass of weight mg and velocity v by considering the stretching effect of the beam's neutral axis and the nonlinear viscoelastic foundation is as follows [43, 44]:

$$EI_z \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} \frac{EA}{2l} \int_0^l \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x + k_1 w(x,t) + k_2 w^3(x,t) = m\left(g - \frac{\partial^2 w(x,t)}{\partial t^2} - 2v \frac{\partial^2 w(x,t)}{\partial t\, \partial x} - v^2 \frac{\partial^2 w(x,t)}{\partial x^2}\right) \delta(x - vt), \tag{2.1}$$

where ρ is the beam's density, A is the cross-sectional area, E is Young's modulus of elasticity, I_z is the moment of inertia, and $\delta(.)$ is the Dirac delta function. Also, the parameters k_1 and k_2 are the linear and nonlinear parts of the foundation stiffness, respectively.

There are several methods for converting partial differential equations to ordinary differential equations, and one of the best and most practical of these methods is the Galerkin method, whose precision and accuracy have been tested in a variety of problems. Using the Galerkin method, Eq. (2.1) is reduced to the ordinary differential equation. The transverse displacement is assumed in the following form [45]:

$$w(x,t) = \sum_{i=1}^{n} \varphi_i(x) q_i(t), \tag{2.2}$$

where $q_i(t)$ is the generalized coordinate in accordance with the i-th modal shape function $\varphi_i(x)$. The modal shape functions that satisfy the basic boundary conditions of a simply supported beam can be considered as follows [45]:

$$\varphi_i(x) = \sin\left(\frac{i\pi x}{l}\right), \qquad i = 1, 2, 3, ... \tag{2.3}$$

In accordance with the Galerkin decomposition technique, Eq. (2.2) is placed in Eq. (2.1), and as a result a typical differential equation can be obtained

$$M(t)\ddot{q}(t) + C(t)\dot{q}(t) + K(t)q(t) + F\left(\ddot{q}(t), \dot{q}(t), q(t)\right) = f(t). \tag{2.4}$$

For the first vibrational mode of the Euler-Bernoulli beam resting on the nonlinear viscoelastic foundation of Eq. (2.4) is derived from the following form [45]:

$$\begin{aligned}(2.5)\quad &\left(1+b_1\sin^2(\tau)\right)\ddot{q}(\tau)+b_1\sin(2\tau)\dot{q}(\tau)\\ &\qquad+\left(b_2-b_1\sin^2(\tau)\right)q(\tau)+b_3q^3(\tau)=b_4\sin(\tau).\end{aligned}$$

The coefficients used in Eq. (2.5), which are expressed as dimensionless, are respectively defined as follows:

$$(2.6)\quad \begin{gathered} b_1=\frac{2m}{\rho Al},\qquad b_2=\frac{\pi^2EI_z}{\rho Av^2l^2}+\frac{l^2k_1}{\rho Av^2\pi^2},\qquad b_3=\frac{3l^2k_2}{4\rho Av^2\pi^2}+\frac{E\pi^2}{4\rho l^2v^2},\\ b_4=\frac{2mgl}{\rho Av^2\pi^2},\qquad \tau=\frac{\pi v}{l}t.\end{gathered}$$

Equation (2.5) is the dimensionless differential equation of motion governing the nonlinear vibration of the Euler-Bernoulli beam under a moving mass. The center of the beam is subjected to the following initial conditions:

$$(2.7)\qquad q(0)=0,\quad \dot{q}(0)=0.$$

Note that nonlinear terms did not affect the governing equation of the beam when the moving mass passed the end of the beam.

3. Homotopy analysis method

3.1. An overview

Among the methods used to find the analytic solution of nonlinear differential equations, the HAM is one of the best methods. This method transforms a nonlinear differential equation into an indefinite number of linear differential equations with auxiliary parameter p varying between 0 and 1. As the value of p increases from 0 to 1, the solution of the problem moves from initial guess to the exact solution. To show the basic ideas of the HAM, consider the following non-linear differential equation:

$$(3.1)\qquad N[q(\tau)]=0$$

in which N is a nonlinear differential operator and $q(\tau)$ is an unknown function of a variable τ. The homotopy equation is generally defined as follows:

$$(3.2)\qquad \overline{H}[\phi;p,h,H(\tau)]=(1-p)L[\phi(\tau;p)-q_0(\tau)]-p\,h\,H(\tau)N[\phi(\tau;p)],$$

where h is a non-zero auxiliary parameter, $H(\tau)$ is a non-zero auxiliary function and L denotes an auxiliary linear operator. As p increases from 0 to 1, $\phi(\tau;p)$

varies from the initial approximation to the exact solution. In other words, $\phi(\tau;0) = q_0(\tau)$ is the solution of the $\overline{H}[\phi;p,h,H(\tau)]\big|_{p=0} = 0$ and $\phi(\tau;1) = q(\tau)$ is the solution of the $\overline{H}[\phi;p,h,H(\tau)]\big|_{p=1} = 0$. Similarly, $\omega(p)$, the nonlinear frequency of the beam, varies from the initial guess frequency ω_0 to the physical frequency ω. Then, $\phi(\tau;p)$ and $\omega(p)$ can be expanded in a power series of p using Taylor's theorem as follows:

$$\begin{aligned} \phi(\tau;p) &= \phi(\tau;0) + \sum_{k=1}^{\infty} \frac{1}{k!}\frac{\partial^k \phi(\tau;p)}{\partial p^k}\bigg|_{p=0} p^k = q_0(\tau) + \sum_{k=1}^{\infty} q_k(\tau)p^k, \\ \omega(p) &= \omega_0 + \sum_{k=1}^{\infty} \frac{1}{k!}\frac{\partial^k \omega(p)}{\partial p^k}\bigg|_{q=0} p^k = \omega_0 + \sum_{k=1}^{\infty} \omega_k p^k, \end{aligned} \tag{3.3}$$

where $q_k(\tau)$ and ω_k are called the k-order deformation derivative. Setting $\overline{H}[\phi;p,h,H(\tau)] = 0$, the zero-order deformation equation is constructed:

$$(1-p)L[\phi(\tau;p) - q_0(\tau)] = p\,h\,H(\tau)\,N[\phi(\tau;p)] \tag{3.4}$$

with the following initial conditions:

$$\phi(0;p) = 0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\bigg|_{0,p} = 0. \tag{3.5}$$

By differentiating the zero-order deformation equation with respect to p and putting $p = 0$, yields the first-order deformation equation which gives the first-order approximation of $q(\tau)$:

$$L\left[q_1(\tau)\right] = h\,H(\tau)\,N\left[\phi(\tau;p)\right]\big|_{p=0}, \tag{3.6}$$

$$q_1(0) = 0, \qquad \frac{\mathrm{d}q_1}{\mathrm{d}\tau}\bigg|_{0} = 0. \tag{3.7}$$

The higher-order approximations can be achieved by calculating the k-order $(k > 1)$ deformation equation which can be calculated by differentiating Eqs. (3.4) and (3.5) k times with respect to p that is expressed as follows:

$$L\left(q_k(\tau) - q_{k-1}(\tau)\right) = h\,H(\tau)\,R_k\left(q_{k-1},\omega_{k-1}\right), \tag{3.8}$$

where q_{k-1}, ω_{k-1}, and $R_k\left(q_{k-1},\omega_{k-1}\right)$ are defined as follows:

$$\begin{aligned} R_k\left(q_{k-1},\omega_{k-1}\right) &= \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}p^{k-1}} N[\phi(\tau;k),\omega(k)]\bigg|_{k=0}, \\ q_{k-1} &= \{q_0, q_1, q_2, ..., q_{k-1}\}, \\ \omega_{k-1} &= \{\omega_0, \omega_1, \omega_2, ..., \omega_{k-1}\}. \end{aligned} \tag{3.9}$$

Initial conditions are considered in the following form:

$$q_k(0) = 0, \qquad \left.\frac{\mathrm{d}q_k}{\mathrm{d}\tau}\right|_0 = 0. \tag{3.10}$$

3.2. Application of the HAM

In this section, we apply the HAM to solve the nonlinear differential Eq. (2.4). In order to obtain the answer to Eq. (2.4), the first guess of $q(\tau)$ is chosen as follows:

$$\begin{gathered} q_0(\tau) = a_1 \sin(\tau), \\ q_0(0) = 0, \qquad \left.\frac{\mathrm{d}q_0}{\mathrm{d}\tau}\right|_0 = 0. \end{gathered} \tag{3.11}$$

To construct the homotopy function, the auxiliary linear operator is selected as:

$$L\left[\phi(\tau;p)\right] = \omega_0^2 \left[\frac{\partial^2 \phi(\tau;p)}{\partial \tau^2} + \phi(\tau;p)\right]. \tag{3.12}$$

The auxiliary linear operator L is chosen in such a way that the solution of the equation exists and can be expressed by the general form of the base function [46]. From Eq. (2.4), the nonlinear operator is defined as:

$$\begin{aligned} N\left[\phi(\tau;p), \omega(p)\right] = \left(1 + b_1 \sin^2(\tau)\right) \frac{\partial^2 \phi(\tau;p)}{\partial \tau^2} + b_1 \sin(2\tau) \frac{\partial \phi(\tau;p)}{\partial \tau} \\ + \left(b_2 - b_1 \sin^2(\tau)\right) \phi(\tau;p) + b_3 \phi^3(\tau;p) - b_4 \sin(\tau). \end{aligned} \tag{3.13}$$

Having assumed auxiliary function $H(\tau) = 1$ and the auxiliary parameter $h = 1$, the first-order transformation equation with regard to Eqs. (3.6) and (3.7) can be written as follows:

$$\begin{aligned} \omega_0^2 \left[\frac{\mathrm{d}^2 q_1(\tau)}{\mathrm{d}\tau^2} + q_1(\tau)\right] = \left(1 + b_1 \sin^2(\tau)\right) \frac{\mathrm{d}^2 q_0(\tau)}{\mathrm{d}\tau^2} + b_1 \sin(2\tau) \frac{\mathrm{d}q_0(\tau)}{\mathrm{d}\tau} \\ + \left(b_2 - b_1 \sin^2(\tau)\right) q_0(\tau) + b_3\, q_0^3(\tau) - b_4 \sin(\tau). \end{aligned} \tag{3.14}$$

$$q_1(0) = 0, \qquad \left.\frac{\mathrm{d}q_1}{\mathrm{d}\tau}\right|_0 = 0. \tag{3.15}$$

The solution of Eq. (3.14) should obey the general form of the base function. Therefore, the coefficient of the secular term must be zero. After eliminating secular terms and solving the differential Eq. (3.14), $q_1(\tau)$, and ω_0 are obtained as follows:

$$\omega_0^2 = \frac{3a_1^2 b_3 - 2b_1}{2b_1}, \tag{3.16}$$

$$q_1(\tau) = a_2 \sin(3\tau) + a_3 \sin(\tau) + a_4 \sin(2\tau) - a_5 \sin(2\tau) + a_6 \sin(3\tau) - a_7 \sin(\tau) + a_8 \sin(\tau). \tag{3.17}$$

The coefficients a_i $(i = 1, 2, ..., 8)$ are given in Appendix. The higher-order approximations are obtained similarly. Assuming $k = 2$ in Eq. (3.9) leads to the following result for the first-order approximation of non-linear frequency (ω_1):

$$\omega_1 = \frac{a_1^2(9a_8b_3 + 3a_8b_3) + b_3a_2a_1(-12a_2 - 6a_6 + 6a_7) + 6a_8a_2^2b_3}{8\omega_0 a_1} - \frac{(2a_8 + a_3 + a_4)b_1(1 + \omega_0^2) + 3(a_3 + a_4)a_2^2b_3 - 2(a_3 - a_4)b_1\omega_0}{8\omega_0 a_1}. \tag{3.18}$$

In accordance with Eq. (3.3), the first-order approximation of $q(\tau)$ and ω become as follows:

$$q(\tau) = q_0(\tau) + q_1(\tau), \tag{3.19}$$

$$\omega = \omega_0 + \omega_1. \tag{3.20}$$

4. Numerical results and discussion

In this section, in order to show the accuracy and effectiveness of HAM, some results are obtained for nonlinear vibration of the simply supported beam under the effect of moving mass on a nonlinear viscoelastic foundation. The geometric and mechanical properties of the Bernoulli-Euler beam resting on the nonlinear viscoelastic foundation are listed in Table 1.

Table 1. Geometrical and mechanical properties of the beam and non-linear foundation.

Item	Property	Notation	Value
Beam	Length	l	1 m
	Young's modulus	E	$207 \cdot 10^9$ Pa
	Mass density	ρ	7700 kg/m^3
	Cross-sectional area	A	0.01 m^3
	Second moment of area	I_z	$2.08 \cdot 10^{-6}$ m^4
Foundation	Linear stiffness	k_1	10^6 N/m^2
	Non-linear stiffness	k_2	10^{17} N/m^4
Moving mass	Mass	m	150 kg

In all plots, the horizontal and vertical axes represent the dimensionless position of the moving mass and the dimensionless dynamic deflection of the beam, respectively. The normalized dimensionless dynamic deflection can be expressed as $w\,/w_s$, where $w_s = \frac{mgl^2}{48EI_z}$ represents the static deflection of the beam under a mass load at midspan. In the first example, to demonstrate the accuracy of the obtained analytical results, the authors also calculate the variation of the dimensionless dynamic response of the beam using the fourth-order Runge-Kutta method. Figure 2 illustrates the comparison between these results. As can be seen in this figure, data from the analytical method are entirely consistent with numerical results and represent a very high accuracy and a very high convergence rate of this method.

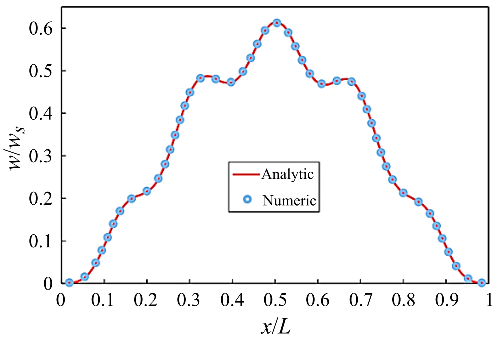

FIG. 2. Comparison of analytical solution and numerical results for the dimensionless dynamic response of the beam.

Plots of the dimensionless dynamic deflection of the beam versus the normalized position of the moving mass are provided in Figs. 3a and 3b to further investigate the effect of the nonlinear foundation stiffness and velocity of the

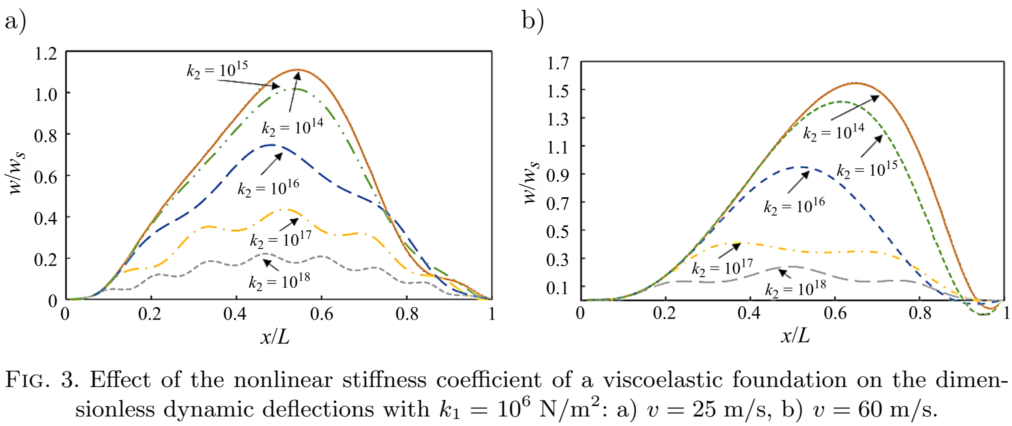

FIG. 3. Effect of the nonlinear stiffness coefficient of a viscoelastic foundation on the dimensionless dynamic deflections with $k_1 = 10^6$ N/m^2: a) $v = 25$ m/s, b) $v = 60$ m/s.

moving mass. The results are presented for $k_2 = 10^{14}$, 10^{14}, 10^{15}, 10^{16}, 10^{17}, 10^{18} N/m^4, and two levels of the moving mass velocity $v = 25$ m/s and 60 m/s. It can be seen that the nonlinear foundation parameter has a considerable influence on the normalized dynamic response of the beam. So, the dimensionless dynamic deflection of the beam decreases with the increasing values of the nonlinear foundation stiffness. Additionally, for definite values of the nonlinear foundation stiffness, it can be concluded that fluctuations of dynamic deflection are reduced with the increasing velocity of moving mass.

Figures 4a and b illustrate the effect of changing the stiffness of a linear foundation on the dimensionless dynamic response of the beam versus the normalized position of the moving mass. By ignoring the nonlinear foundation parameter in the movement equation ($k_2 = 0$), the results are presented for various values of the linear stiffness coefficient $k_1 = 0$, 10^7, 10^9 N/m^2 and constant velocities of the moving mass $v = 25$ m/s and 60 m/s. It is clear from Figs. 4a and 4b that the dimensionless dynamic deflection of the beam decreases with the increasing values of the linear stiffness coefficient. Also, as it might be observed, for the definite value of linear stiffness coefficient, the magnitude of dimensionless dynamic deflection increases with the increasing velocity of the moving mass.

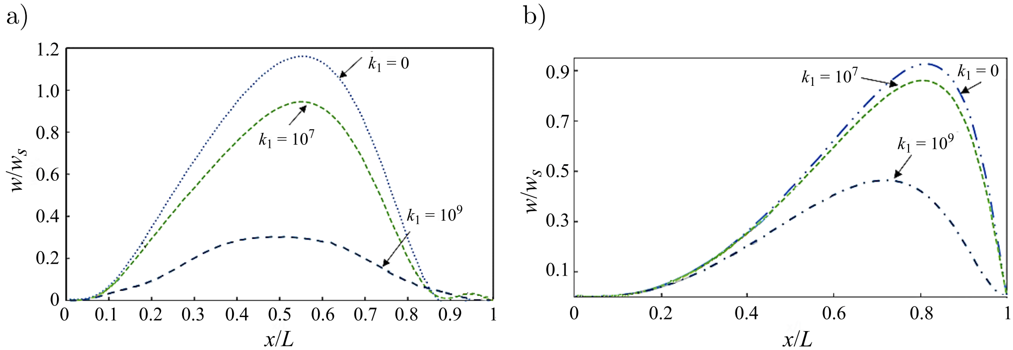

FIG. 4. Effect of the linear stiffness coefficient of a viscoelastic foundation on the dimensionless dynamic deflections with $k_2 = 0$: a) $v = 25$ m/s, b) $v = 60$ m/s.

The influence of the velocity of the moving mass on the dimensionless dynamic deflection is shown in Fig. 5. This figure depicts the results for $v = 60$, 120, 176, 256 m/s and two values of the nonlinear stiffness coefficient $k_2 = 0$ and $k_2 = 10^{17}$ N/m^4. According to Fig. 5, the dimensionless dynamic deflection of the beam in the nonlinear model at each speed has smaller values compared to the linear model. Also, it is worth mentioning that at the low velocities of the moving mass, the difference in the dimensionless dynamic response of the beam between the linear and nonlinear models is noteworthy, while at high speed of moving mass this difference is insignificant.

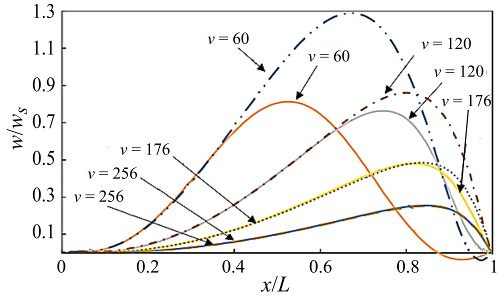

Fig. 5. Effect of the velocity of the moving mass on the dimensionless dynamic deflections: linear model ($k_1 = 10^6$ N/m^2, $k_2 = 0$) – non-continuous lines, nonlinear model ($k_1 = 10^6$ N/m^2, $k_2 = 10^{16}$ N/m^4) – continuous lines.

The graphs shown in Fig. 6 display the variations of the non-dimensional mass parameter ($\alpha = m/\rho Al$) on the dimensionless dynamic deflection of the beam with linear and nonlinear models. In this case, the velocity of the moving mass $v = 25$ m/s and other parameters were chosen according to Table 1. As can be seen in Fig. 6, the dimensionless dynamic deflection for the linear model of the beam increases with the increasing non-dimensional mass parameter, while for nonlinear model of the beam, the dimensionless dynamic deflection is reduced with the increasing non-dimensional mass parameter. Additionally, in this figure, we observe that for various values of the non-dimensional mass parameter, the difference of the dimensionless dynamic response of the nonlinear model of the beam is considerable, while for the linear beam, this difference is minor.

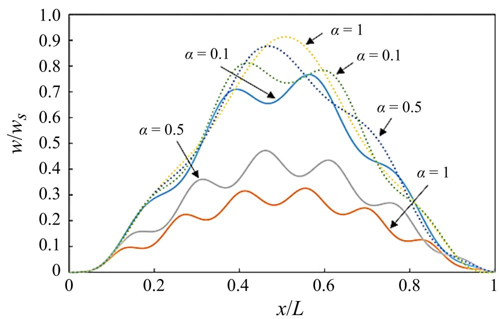

Fig. 6. Effect of non-dimensional mass parameter on the dimensionless dynamic deflections with $v = 25$ m/s: linear model ($k_1 = 10^6$ N/m^2, $k_2 = 0$) – non-continuous lines, nonlinear model ($k_1 = 10^6$ N/m^2, $k_2 = 10^{17}$ N/m^4) – continuous lines.

The effects of Coriolis force, centrifugal force and inertia force terms on the dimensionless dynamic deflection of the beam with linear and nonlinear foundation are plotted in Fig. 7. As can be seen in Fig. 7, for both models of foundation, while ignoring inertia force of the moving mass in movement equation,

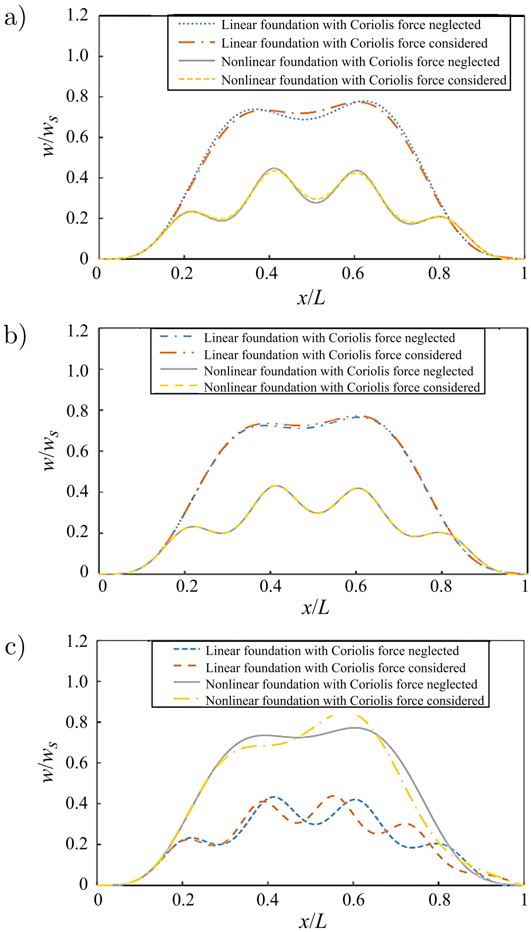

FIG. 7. Effect of: a) Coriolis force, b) centrifugal force, c) inertia force terms on the dimensionless dynamic deflections with $v = 25$ m/s; linear model ($k_1 = 10^6$ N/m^2, $k_2 = 0$) – non-continuous lines, nonlinear model ($k_1 = 10^6$ N/m^2, $k_2 = 10^{16}$ N/m^4) – continuous lines.

the fluctuations of dynamic response increase partially, while the magnitude of the dynamic response does not change considerably. In comparison with the previous case, it is interesting to note that the Coriolis force and centrifugal force of the moving mass do not have a considerable influence on the dynamic deflection of the beam.

5. Conclusions

In this paper, an Euler-Bernoulli beam resting on a nonlinear viscoelastic foundation subjected to a moving mass by considering the stretching effect of the beam's neutral axis was analyzed. The Galerkin method was used to transform the nonlinear partial differential equation of motion into an ordinary nonlinear differential equation, and consequently, an approximate analytical solution was obtained for the nonlinear dynamic deflection of the beam using the HAM. The efficiency and accuracy of the method were demonstrated by a comparison between solutions obtained by the HAM and the fourth-order numerical Runge-Kutta method. Numerical simulation was carried out to investigate the effects of linear and nonlinear stiffness coefficients of a viscoelastic foundation, velocity of the moving mass as well as Coriolis force, centrifugal force and inertia force of moving mass on the dynamic deflection of the beam. From the numerical results, the following key points were observed:

1) The dynamic deflection of the beam decreased with the increasing values of the linear and nonlinear foundation stiffness.
2) When comparing linear and nonlinear models of the beam, the dynamic deflection of the nonlinear beam was smaller than the linear model.
3) For the definite value of linear and nonlinear stiffness coefficient, the magnitude of dynamic deflection increases with the increasing velocity of the moving mass.
4) For both models of the beam, the dynamic deflection increased and decreased with the increasing non-dimensional mass parameter, respectively.
5) The inertia force parameter has a considerable influence on the fluctuations of the dynamic response of the beam, while the Coriolis force and centrifugal force do not have a remarkable effect on the dynamic deflection of the beam.

Appendix

The coefficients in Eqs. (3.15) and (3.16) are defined as:

$$a_1 = \frac{-b_4}{\omega_0(\omega_0^2 - 1)}, \qquad a_2 = \frac{hb_3 a_1^3}{32\omega_0^2},$$

$$a_3 = -\frac{h\left(3b_3(f_1+4)a_2^3 + a_2(-4f_1^2 + 6b_3 f_1 a_1^2 - 8f_1 + b_3 a_1^2) - 4b_4 f_1\right)}{4f_1\omega_0(\omega_0^2 - 1)},$$

$$a_4 = -\frac{ha_1(3a_1^2 b_3 + 4\omega_0 f_1)}{32(\omega_0 + 1)}, \qquad a_5 = \frac{ha_1(3a_1^2 b_3 - 4\omega_0 f_1)}{32(\omega_0 - 1)},$$

$$a_6 = -\frac{ha_1(b_3 a_2^2 - 4f_1)}{4(\omega_0^2 - 9)}, \qquad a_7 = -\frac{ha_1^2 a_2 b_3(9\omega_0^2 + 12\omega_0 + 3)}{36\omega_0^2 - 40\omega_0 + 4},$$

$$a_8 = \frac{ha_1^2 a_2 b_3(9\omega_0^2 - 12\omega_0 + 3)}{36\omega_0^2 - 40\omega_0 + 4}.$$

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