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Discussion

Symmetry[1)]

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Abstract.[2)] The concept of symmetry was introduced already by the ancient Greeks in relation to spatial (geometric) systems. They understood it as *commensurability and proportionality* and linked it with the aesthetic categories of *harmony* and *beauty.* A spatial system (object) was considered symmetric if it consisted of *regular, repeatable parts* of comparable size, creating a *coordinated, ordered, larger whole.* Only two thousand years later, in the twentieth century, the *essence of the concept of symmetry* was identified. *Symmetry* is *invariance* (stability, durability, constancy) of a *feature* (geometric, physical, biological, informational, etc.) of *an object* (an object can here be a geometric system, a material thing, but also a natural phenomenon, physical law, social relation, etc.) after subjecting it to a *set of transformations* (transformations can be shifts, reflections, rotations, permutations, etc.), with respect to which symmetry is considered. The above observation led to the discovery of the *universal nature* of the concept of symmetry, which in a broader sense can be understood as a philosophical category, one of the fundamental regularities of mathematical character in the organization of the Universe. The contemporary understanding of symmetry has led to significant and non-obvious conclusions. For example, it turned out that the invariance (symmetry) of the laws of motion with respect to the shift in time is equivalent to the necessity of the existence of the principle of conservation of energy, the invariance (symmetry) of the laws of motion with respect to the shift in physical space proves to be equivalent to the existence of the principle of conservation of momentum. The Report provides an outline of the *general formal language of symmetry* applicable to the study of any situation in which this concept appears. The key elements of the mathematical apparatus of the algebraic theory of symmetry are defined and discussed, the notions of Γ-sets, orbits, orbital markers, invariants, and invariant functions. They provide versatile tools enabling the analysis of all types of symmetries. The Report concisely presents important results of the theory of symmetry, such as: the *ornament principle* – expressing the most straightforwardly the innermost property of complex symmetrical objects, the representation theorem for symmetric objects, the theorem on the *symmetry of causes and effects of physical laws*, the theorem on *invariant extension of any function.*

Keywords: definition of symmetry; concept of Γ-set; orbits of elements; orbit markers; ornament principle; motif; symmetry of causes and effects; invariant extension of function.

[1)] *Editorial note:* The present document is an English translation of Appendix B of Jan Rychlewski book titled *Dimensions and Similarity* (original Polish title *Wymiary i podobieństwo*), Wydawnictwo Naukowe PWN, Warszawa, 1991, pp. 171–184, ISBN 83-01-10557-7.

[2)] *Editorial note:* Abstract by Andrzej Ziółkowski.

Water, taken in moderation,
cannot hurt anybody.
Mark Twain

The concept of SYMMETRY – wherever it may appear – has a certain simple mathematical structure as its basis. We present here its outline.

1. Operation of a group in a set. Let $\mathcal{X}$ be a set and Γ a group with operation "$\circ$"[3)],

$$(\alpha, \beta) \to \alpha \circ \beta. \tag{B.1}$$

We will call the elements of the group operators. We will say that a group Γ operates in a set $\mathcal{X}$ if for each element x of the set $\mathcal{X}$ and each operator α of the group Γ, in the set $\mathcal{X}$, there is exactly one element $\alpha \cdot x$ such that

(i) for every $\alpha, \beta \in \Gamma$, $x \in \mathcal{X}$

$$(\alpha \circ \beta) \cdot x = \alpha \cdot (\beta \cdot x), \tag{B.2}$$

(ii) for the unity $\imath$ of the group Γ and every $x \in \mathcal{X}$

$$\imath \cdot x = x, \tag{B.3}$$

The elements of a set $\mathcal{X}$ we will sometimes call points.

Here we breathe the clear, thin air of pure Mathematics. The nature of the set $\mathcal{X}$ and the group Γ will therefore be irrelevant.

EXAMPLES

1° $\mathcal{X}$ is a set of all geometric figures in a point Euclidean space, Γ is a group of rigid movements of this space (translations, rotations, mirror images, and their combinations).

2° $\mathcal{X}$ is a set of all processes in which elementary particles take part, Γ is a special unitary group composed of all unitary and unimodular operators operating in the n-dimensional complex Euclidean space.

3° $\mathcal{X}$ is a set of first p natural numbers, Γ is a set of their permutations.

4° $\mathcal{X}$ is a set of all Euclidean tensors of n-th order, Γ is an orthogonal group of a Euclidean vector space.

5° $\mathcal{X}$ is a space-time of the theory of relativity, Γ is the Lorentz group.

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3) *Translator note:* In the original text, there is no *explicit* notation of the internal operation of the group of operators Γ. In the English translation of the text, such a symbol was introduced to facilitate understanding of the text and to prevent misconceptions. Hence, the symbol "$\circ$" denotes an *internal operation* of the group of operators Γ resulting in combining the operator's actions. The symbol "$\cdot$" denotes an *external operation* of an operator from the group Γ on an element of the *set* $\mathcal{X}$.

The concept of operation of a group in a set is one of the key concepts in Mathematics and Natural Science.

Triad $\{\mathcal{X},\ \varGamma,\ \cdot\}$ we will call a *$\varGamma$-set*. As usual, we will also apply this name to the base set, saying: $\varGamma$-set $\mathcal{X}$.

Mapping

$$x \to \alpha \cdot x, \qquad \alpha \in \varGamma, \qquad x, \alpha \cdot x \in \mathcal{X} \tag{B.4}$$

is a bijection of a set $\mathcal{X}$ on itself. This important fact occurs only due to the conditions (B.2), (B.3).

2. Groups of symmetry. We will say that the element x is symmetric with respect to the operator α if

$$\alpha \cdot x = x, \tag{B.5}$$

(other names that sound good in different contexts are *invariant*, *stable*, *constant*, ...).

The symmetry group of an element x is the following subgroup in $\varGamma$:

$$\varGamma(x) \equiv \{\alpha \in \varGamma | \alpha \cdot x = x\}, \qquad \varGamma(x) \subset \varGamma, \tag{B.6}$$

(other good names: *isotropy group*, *stabilizer*, ...).

Sometimes it is convenient to operate with the complement

$$\overline{\varGamma}(x) \equiv \varGamma - \varGamma(x), \tag{B.7}$$

called the *set of dissymmetry* of an element x in a group $\varGamma$.

We will say that the element x is *symmetric with respect to the group* $\varGamma$ if $\varGamma(x) = \varGamma$.

3. Similarity. We will say that elements x, y are *mutually similar with respect to a group* $\varGamma$ if $y = \alpha \cdot x$ for some operator α (other good names: *comparable*, *equivalent*, *indistinguishable*, *mutually achievable*, ...).

4. Orbits. Due to (B.2) and (B.3), the similarity relation is *reflexive*, *symmetric* and *transitive*, and therefore it is *equivalence*. The equivalence classes are called here *orbits* of the group $\varGamma$ in a set $\mathcal{X}$ or, for short, $\varGamma$-orbits (other good names: *transitivity classes*, *group trajectories*, ...). An orbit passing through an element x is a set

$$\varGamma \cdot x \equiv \{\alpha \cdot x | \alpha \in \varGamma\}, \qquad \varGamma \cdot x \subset \mathcal{X}. \tag{B.8}$$

The set of all orbits in $\mathcal{X}$ we denote $\mathcal{X}/\varGamma$. It constitutes a *division* of $\mathcal{X}$, i.e.,

$$\mathcal{X} = \bigcup_{x \in \mathcal{X}} \varGamma \cdot x, \qquad \varGamma \cdot x_1 = \varGamma \cdot x_2 \quad \text{or} \quad \varGamma \cdot x_1 \cap \varGamma \cdot x_2 = \emptyset. \tag{B.9}$$

The subset $\mathcal{R}$ that intersects with each orbit at exactly one point is called the set of *orbits representatives.* The projection of the entire set $\mathcal{X}$ on the subset $\mathcal{R}$ is given by the formula

$$\vartheta(x) \equiv \mathcal{R} \cap (\varGamma \cdot x). \tag{B.10}$$

It is clear that in (B.9), the summation can be limited by taking only $x \in \mathcal{R}$ (Fig. B.1[4]).

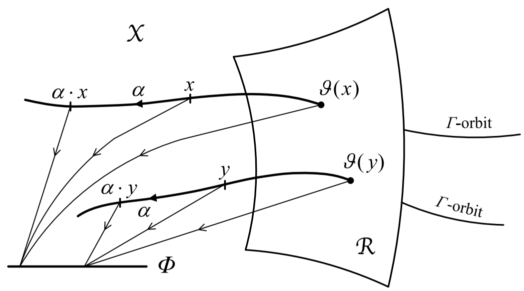

FIG. B.1.[5] Illustration of construction of a set of orbits representatives and orbits markers.

5. Markers of orbits. In applications, we need to conveniently distinguish one orbit from another. Let $\mathcal{M}$ be a set of features for the orbits. Every surjection

$$\varPhi : \mathcal{X} \to \mathcal{M}, \tag{B.11}$$

with property

$$\varPhi(x) = \varPhi(y) \Leftrightarrow \varGamma \cdot x = \varGamma \cdot y, \tag{B.12}$$

or, equivalently,

$$\varPhi(x) \equiv \varPhi(\upsilon(x)), \tag{B.13}$$

we will call the *orbits marker of the considered* $\varGamma$-set.

The orbits marker is, of course, the projection itself on $\mathcal{R}$: the feature of the orbit is its point of intersection with $\mathcal{R}$. However, explicit specification of the set of representatives $\mathcal{R}$ may be difficult or impossible.

Note. The very existence of a set $\mathcal{R}$ results from the axiom of choice, which so far is the "hot spot" of the Theory of Sets.

For $\varGamma$-sets encountered in applications, the cardinality of the set of orbits $\mathcal{X}/\varGamma$ is not "too high", let us say

$$\operatorname{card}(\mathcal{X}/\varGamma) \leq \mathfrak{c}, \tag{B.14}$$

[4] *Translator note*: Original Fig. B.1 has been supplemented with an illustration of operations of function $\varPhi$, cf. (B.13).

[5] *Translator note*: Captions under all figures were elaborated by translator.

where $\mathfrak{c}$ is the strength of the continuum. This gives the option of using numeric markers when

$$\mathcal{M} \subset R^m \quad \text{for some } m. \tag{B.15}$$

EXAMPLE: $\mathcal{X} \subset \otimes^2 \ni$ is a set of symmetric tensors, in terms of transposition, of the second order, where $\ni$ is a Euclidean vector space.

A well-known numeric marker for the orbits of a rotation group is

$$\Phi(x) \equiv (\mathrm{tr}\,(x),\ |x|,\ \det(x)), \tag{B.16}$$

where $\mathrm{tr}\,(\cdot)$ is a trace, $|\cdot|$ is a norm, $\det(\cdot)$ is a determinant of a tensor.

Note. If there is a marker $\Phi : \mathcal{X} \to R^m$, there is also a marker $\Phi : \mathcal{X} \to R$ characterizing orbits by single numbers because card R^m = card $R = \mathfrak{c}$. We discard such markers as useless. As a rule, continuous markers are needed in the appropriate topology.

Orbit equation we will write in the form

$$\Phi(x) = \text{idem} \tag{B.17}$$

(idem is a Latin equivalent of const, see notes for lecture 10[6]).

6. Numeric invariants. Function $\varphi : \mathcal{X} \to R$ we call a numeric invariant, or, for short, an invariant if

$$\varphi(\alpha \cdot x) = \varphi(x) \qquad \text{for every} \qquad x \in \mathcal{X},\ \ \alpha \in \Gamma. \tag{B.18}$$

Numeric marker for orbits

$$\Phi(x) = (\varphi_1(x), ..., \varphi_m(x)) \tag{B.19}$$

consists of invariants. It is often called a *functionally complete system of invariants* (numerical) because, for any invariant φ, there is a function $\widetilde{\varphi}$: $R^m \to R$ such that

$$\varphi(x) = \widetilde{\varphi}(\varphi_1(x), ..., \varphi_m(x)) \tag{B.20}$$

for every $x \in \mathrm{Dom}(\varphi)$.

7. Γ-set of objects. Each subset $\mathcal{D} \subset \mathcal{X}$ let us call an object. The set of all objects $2^{\mathcal{X}}$ is itself the Γ-set with respect to the natural operation

$$\mathcal{D} \to \alpha \bullet \mathcal{D} \equiv \{\alpha \cdot x | x \in \mathcal{D}\}. \tag{B.21}$$

[6] *Translator note*: This is reference to Lecture 10, "Presentations of self-similar phenomena: π-theorems", in book: Jan Rychlewski, *Dimensions and Similarity* [in Polish].

All the names we can therefore transfer from triad $\{\mathcal{X},\ \Gamma,\ \cdot\}$ to triad $\{2^{\mathcal{X}},\ \Gamma,\ \bullet\}$. Specifically, the *symmetry group* of an object $\mathcal{D}$ is

$$\Gamma(\mathcal{D}) = \{\alpha \in \Gamma | \alpha \bullet \mathcal{D} = \mathcal{D}\}. \tag{B.22}$$

It is clear that

$$\Gamma(\mathcal{D}) \supset \bigcap_{x\in\mathcal{D}} \Gamma(x). \tag{B.23}$$

We are interested in situations where this inclusion is not equality.

An object $\mathcal{D}$ is called *symmetric with respect to the group* Γ, when $\Gamma(\mathcal{D}) = \Gamma$ (other good names: *stable*, *invariant*, *isotropic*, *self-similar*, ...).

Example. Let $\mathcal{X}$ be a three-dimensional Euclidean point space $\mathcal{E}_{\mathsf{L}}$. Let us take $\mathcal{D}$ as the set of nodes of the infinite crystal lattice. Let Γ be a group of rigid movements in $\mathcal{E}_{\mathsf{L}}$. The group $\Gamma(\mathcal{D}) \subset \Gamma$ is called the *symmetry group* of the crystal under consideration.

8. Principle of ornament. Let $\Gamma(\mathcal{D})$ be a symmetry group of the object $\mathcal{D} \in \mathcal{X}$. Let $\mathcal{R}_{\mathcal{D}}$ be a set of representatives of $\Gamma(\mathcal{D})$-orbits. The intersection $\mathcal{D} \cap \mathcal{R}_{\mathcal{D}}$ we will call the *motif of an object* $\mathcal{D}$.

Principle of ornament. *Each object is generated by its motif, i.e.,*

$$\mathcal{D} = \Gamma(\mathcal{D}) \bullet \{\mathcal{D} \cap \mathcal{R}_{\mathcal{D}}\}, \tag{B.24}$$

for every $\mathcal{D}$.

This identity is self-evident, but because of its exceptional importance, it is useful to track the following sequence of equalities:

$$\begin{aligned}\mathcal{D} \equiv \Gamma(\mathcal{D}) \bullet \mathcal{D} = \Gamma(\mathcal{D}) \bullet \bigcup_{x\in\mathcal{D}} x = \bigcup_{x\in\mathcal{D}} \Gamma(\mathcal{D}) \cdot x = \bigcup_{x\in\mathcal{D}\cap\mathcal{R}_{\mathcal{D}}} \Gamma(\mathcal{D}) \cdot x \\ = \Gamma(\mathcal{D}) \bullet \bigcup_{x\in\mathcal{D}\cap\mathcal{R}_{\mathcal{D}}} x = \Gamma(\mathcal{D}) \bullet \{\mathcal{D} \cap \mathcal{R}_{\mathcal{D}}\}.\end{aligned} \tag{B.25}$$

In particular, for symmetric objects $\Gamma(\mathcal{D}) = \Gamma$, and therefore

$$\boxed{\mathcal{D} = \Gamma \bullet \{\mathcal{D} \cap \mathcal{R}\}.} \tag{B.26}$$

We can repeat the principle of ornament in the following form: an object is symmetric if and only if it consists of orbits.

Figures B.2 and B.3 show examples.

Principle of ornament contains in itself the most primal and the simplest, but also the most important information about the symmetry of objects. The artisans of antiquity used this principle perfectly (Figs. B.3, B.4).

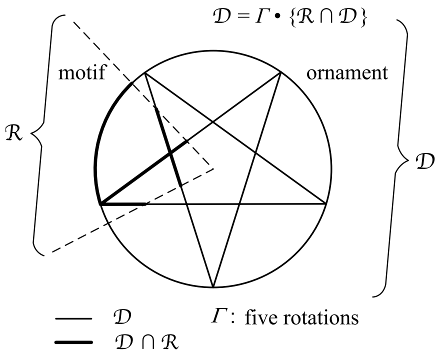

FIG. B.2. Illustration of the idea of a *motif* and subsequently generation of an *ornament* using specific motif and specific *repetition operation* – rotations.

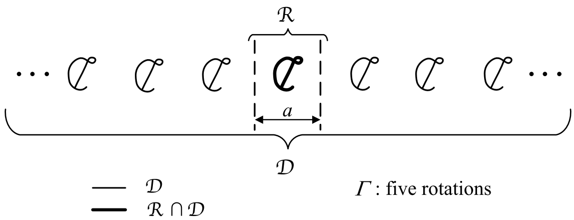

FIG. B.3. Illustration of the idea of a *motif* and subsequently generation of an *ornament* using specific motif and specific *repetition operation* – translations.

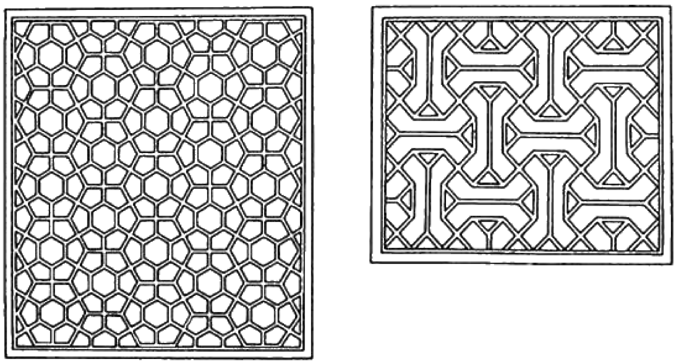

FIG. B.4. Examples of ornaments borrowed from Hermann Weyl's book, *Symmetry*, 1952, cf. [B.3].

All the so-called Representative theorems about symmetric objects (e.g., about isotropic tensor functions) are only consequences of the PRINCIPLE OF ORNAMENT, some direct consequences, others slightly more complex.

In Lecture 10, we show that the famous π-theorem of dimensional analysis is nothing more than an expression of the ornament principle for dimensional quantities.

9. Representation theorem. The immediate consequence of the principle of ornament is

A GENERAL REPRESENTATION THEOREM. *The object $\mathcal{D}$ is symmetric if and only if the condition*

$$x \in \mathcal{D}, \tag{B.27}$$

is equivalent to the condition

$$\Phi(x) \in \mathcal{M}_{\mathcal{D}}, \tag{B.28}$$

where $\mathcal{M}_{\mathcal{D}} = \Phi(\mathcal{D})$ is the set of attributes assigned by the marker Φ to all elements $x \in \mathcal{D}$.

It can be written differently. Let us introduce an indicator function

$$f(x) \equiv \begin{cases} 0 & \text{when} \quad x \in \mathcal{D}, \\ 1 & \text{when} \quad x \notin \mathcal{D}, \end{cases} \tag{B.29}$$

and

$$\widetilde{f}(p) \equiv \begin{cases} 0 & \text{when} \quad p \in \mathcal{M}_{\mathcal{D}}, \\ 1 & \text{when} \quad p \notin \mathcal{M}_{\mathcal{D}}. \end{cases} \tag{B.30}$$

Now the theorem can be formulated as follows: equation

$$f(x) = 0, \qquad x \in \mathcal{D}, \tag{B.31}$$

describes a symmetric object if and only if it is equivalent to the "groupless" equation

$$\widetilde{f}(\Phi(x)) = 0, \qquad x \in \mathcal{D}. \tag{B.32}$$

It is exactly the same but looks somehow more practical.

10. Cartesian products. In applications, we usually deal with several Γ-sets for the same group Γ but the nature of the sets and the way the groups operate in them may be completely different.

So let us take

$$\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_m, \tag{B.33}[7]$$

where each set $\mathcal{X}_1, \ldots, \mathcal{X}_m$ is a Γ-set for some fixed group Γ.

[7] *Editorial note*: There is an error in the numbering of the formulas in the original; formula (B.33) is omitted.

We will introduce for every

$$\mathbf{x} = (x_1, ..., x_m) \in \mathcal{X}, \tag{B.34}$$

its image under operation $\alpha \in \Gamma$:

$$\alpha \cdot \mathbf{x} \equiv (\alpha \cdot x_1, ..., \alpha \cdot x_m). \tag{B.35}$$

It is clear that the mapping

$$\mathbf{x} \to \alpha \cdot \mathbf{x} \tag{B.36}$$

meets the conditions (B.2), (B.3). In this way, the Cartesian product $\mathcal{X}$ also becomes Γ-set.

N o t e. For the orbits we have

$$\Gamma \cdot (x_1, ..., x_m) \subset (\Gamma \cdot x_1) \times ... \times (\Gamma \cdot x_m) \tag{B.37}$$

but equality occurs here only in special situations.

The important for symmetry groups is a simple formula

$$\boxed{\Gamma(x_1, ..., x_m) = \Gamma(x_1) \cap ... \cap \Gamma(x_m).} \tag{B.38}$$

Indeed, $\alpha \cdot \mathbf{x} = \mathbf{x}$ means that $\alpha \cdot x_i = x_i$ for every $i = 1, ..., m$.

An object $\mathcal{F} \subset \mathcal{X}_1 \times ... \times \mathcal{X}_m$ is symmetric if and only if

$$\boxed{\alpha \bullet \mathcal{F} = \mathcal{F} \quad \text{for every} \quad \alpha \in \Gamma.} \tag{B.39}$$

The operation of groups in products (B.33) and their division into orbits is a fundamental topic for describing physical phenomena.

11. Functions as graphs. In Physics, we constantly encounter causal laws (also in Quantum Mechanics!). Their ideal images are functions. In specific situations, instead of the word function, other words are used, depending on the context: *mapping*, *operator*, *functional*, *form*, *law*, *field*, *motion*, ...

All these objects are functions of a special kind.

The simplest way to describe the symmetry (invariance) of functions is through their most primal definition. This is the definition of Peano: a *function* is any object $f \subset \mathcal{X} \times \mathcal{Y}$ with the following property: for any $x \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$ if

$$(x, y_1) \in f, \qquad (x, y_2) \in f, \tag{B.40}$$

then

$$y_1 = y_2. \tag{B.41}$$

A quantity y unequivocally defined for a given $x \in \mathcal{X}$ by condition $(x, y) \in f$ is usually written in the form $f(x)$ (much less frequently in the form) $(x)f$). Thus,

$$y = f(x) \quad \Leftrightarrow \quad (x, y) \in f. \tag{B.42}$$

In ordinary working language, an object $f \subset \mathcal{X} \times \mathcal{Y}$, here called a function, is called a graph of a function f (Fig. B.5).

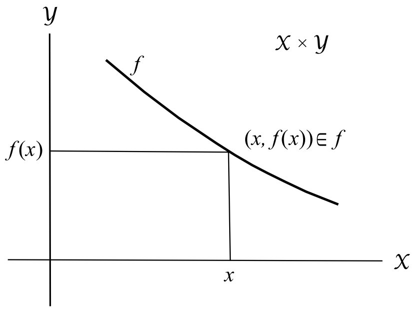

FIG. B.5. Illustration of Peano's definition of a *function*.

12. Invariance of functions. Let us take a situation where the same group (e.g., one of the basic groups of Physics – Lorentz group, group $SO(n)$, ...) acts on both the arguments x and the values of the function $f(x)$:

$$x \to \alpha \cdot x, \qquad f(x) \to \alpha \times f(x). \tag{B.43}$$

The main question here is (see Fig. B.6)

$$\alpha \times f(x) = ?? = f(\alpha \cdot x). \tag{B.44}$$

If

$$\boxed{\alpha \bullet \mathrm{Dom}\, f = \mathrm{Dom}\, f} \tag{B.45}$$

and

$$\boxed{f(\alpha \cdot x) = \alpha \times f(x) \quad \text{for every} \quad \alpha \in \mathit{\Gamma}, \;\; x \in \mathrm{Dom}\, f,} \tag{B.46}$$

then the function is called *invariant* with respect to α. So the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \alpha \cdot \downarrow & & \downarrow \alpha \times \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{B.47}$$

is commutative.

At first glance, the condition (B.46) looks quite artificial, or at least it looks not being directly associated with the previous definitions of objects symmetry. We will show that it is, in fact, *a special form of the condition* (B.39).

When a group Γ operates in $\mathcal{X}$ and $\mathcal{Y}$, then according to (B.35) it also operates in the Cartesian product $\mathcal{X} \times \mathcal{Y}$. But therefore, it works on subsets as well, according to (B.21). As a result, we get Γ-set $\{2^{\mathcal{X}\times\mathcal{Y}},\ \Gamma,\ \bullet\}$ and for each function $f \in 2^{\mathcal{X}\times\mathcal{Y}}$

(B.48) $$f \to \alpha \bullet f.$$

It is not difficult to show that if f is a function, then $\alpha \bullet f$ also is a function, with

(B.49) $$(\alpha \bullet f)(\alpha \cdot x) = \alpha \times f(x).$$

Please see Fig. B.6.

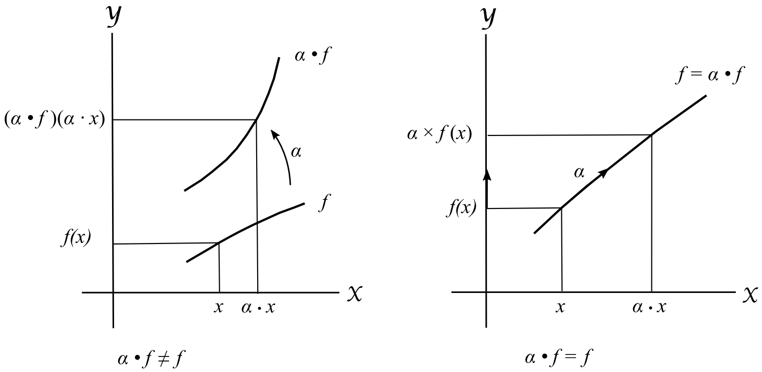

FIG. B.6. Illustration of the concept of invariance of a function subject to a group of transformations.

Now we see that the conditions (B.45), (B.46) can be written in the following compact equivalent form:

(B.50) $$\boxed{\alpha \bullet f = f.}$$

So the *invariance* of a function with respect to the operator α is nothing but the symmetry of its graph with respect to that operator.

According to the general definition (B.22)

(B.51) $$\Gamma(f) \equiv \{\alpha \in \Gamma |\ \alpha \bullet f = f\}$$

is a *symmetry group* of function f (other good names: *stabilizer*, *isotropy group*, *invariance group*, ...).

13. Symmetry of causes and effects. While this appendix is about pure Mathematics, we will not refrain for now from using the more pictorial language of Natural Science:

$$\begin{array}{ccc} x & \to & f(x) \\ \text{cause} & \text{law} & \text{effect} \end{array} \tag{B.52}$$

The following simple observation is of great importance.

Theorem on the symmetry of Causes and Effects. *The effect is at least as symmetric as the pair* (*cause*, *law*), *i.e.*,

$$\boxed{\Gamma(f(x)) \supset \Gamma(f) \cap \Gamma(x)} \tag{B.53}$$

for every $x \in \operatorname{Dom} f$.

Proof. For any $x \in \operatorname{Dom} f$ and any operator $\alpha \in \Gamma$, if $\alpha \bullet f = f$ and $\alpha \cdot x = x$, then

$$\alpha \times f(x) \equiv (\alpha \bullet f)(\alpha \cdot x) = f(x). \quad \blacksquare \tag{B.54}$$

This theorem is used by every intelligent researcher, usually without being openly aware of it.

If $\Gamma(f) = \Gamma$, then

$$\Gamma(f(x)) \supset \Gamma(x), \tag{B.55}$$

so *if a law is symmetric* (*invariant*, *isotropic*, *self-similar*, ...), *then the effect is at least as symmetric as its cause.*

Using the dissymmetry set (B.7), we have

$$\overline{\Gamma(f(x))} \subset \overline{\Gamma(x)}, \tag{B.56}$$

so *if the law is symmetric*, *then the dissymmetry of the result is a consequence of the dissymmetry of the causes.*

Formulas (B.55) and (B.56) are the exact expression of the so-called *Curie symmetry principle.*

Theorem (B.55) can be strengthened as follows:

Theorem. *The invariant function is an injection, i.e.,*

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2 \tag{B.57}$$

if and only if

(i) *for every* $x \in \operatorname{Dom} f$

$$\Gamma(f(x)) = \Gamma(x), \tag{B.58}$$

(ii) *mapping the orbits of the set $\mathcal{X}$ into the orbits of the set $\mathcal{Y}$*

$$\Gamma \cdot x = \Gamma \times f(x) \tag{B.59}$$

is a bijection.

14. General form of invariant functions. Let $\mathcal{R}$ be a set of representatives of orbits in $\mathcal{X}$.

PRINCIPLE OF ORNAMENT FOR FUNCTION. *A function f is invariant if and only if*

$$f = \Gamma \bullet g, \tag{B.60}$$

where g is its motif,

$$g \equiv f|_{\mathcal{R} \cap \mathrm{Dom}\, f}. \tag{B.61}$$

The working form of the formula (B.60) is, according to (B.49),

$$f(x) = \alpha \times f(\vartheta(x)), \tag{B.62}$$

where α is any operator satisfying the equation

$$\alpha \cdot \vartheta(x) = x \tag{B.63}$$

(see Fig. B.7).

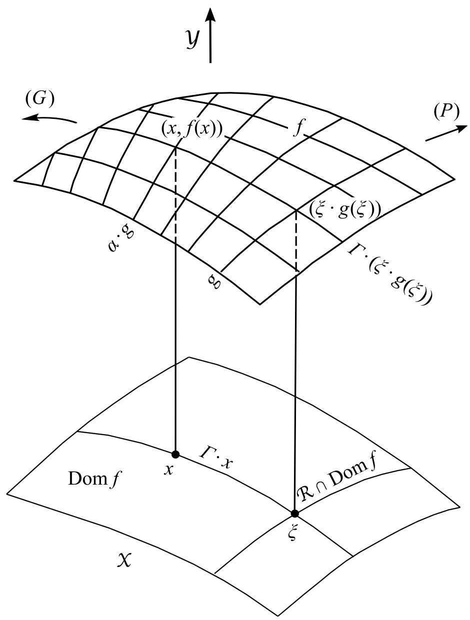

FIG. B.7. Illustration of the idea of "ornamental" function generated from some motif.

A key supplement to this theorem is the description of the admissible motifs.

THEOREM. *The function g defined on the set of representatives of the orbits,* $\operatorname{Dom} g \subset \mathcal{R}$*, is a motif of a certain invariant law:*

$$f = \Gamma \bullet g \tag{B.64}$$

if and only if it preserves symmetry, i.e., when

$$\Gamma(g(\xi)) \supset \Gamma(\xi) \quad \textit{for every} \quad \xi \in \operatorname{Dom} g. \tag{B.65}$$

Thus, each invariant function can be constructed as follows: we define it on $\mathcal{R}$ with the condition (B.65) and extend it along the orbits using the formula (B.62).

15. Invariant extension of any function. Let $\mathcal{P}$ be Γ-set such that for any subgroup $\Lambda \subset \Gamma$ in $\mathcal{P}$ there is an element p such that $\Gamma(p) = \Lambda$. The set defined in this way, the elements of which we will call the *asymmetry parameters*, allows us to reduce any function to a certain invariant function.

THEOREM ON THE INVARIANT EXTENSION OF ANY FUNCTION. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a function and p a parameter of asymmetry from $\mathcal{P}$ such that*

$$\Gamma(f) = \Gamma(p). \tag{B.66}$$

Let $g\colon \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$ be a function such that

$$\operatorname{Dom} g = \{\mathcal{R} \cap \operatorname{Dom} f\} \times \{p\}, \tag{B.67}$$

where $\mathcal{R}$ is the set of orbits representatives in $\mathcal{X}$ and

$$g(\xi, p) \equiv f(\xi) \quad \text{for} \quad \xi \in \mathcal{R} \cap \operatorname{Dom} f. \tag{B.68}$$

Then,

$$\boxed{f^I \equiv \Gamma \bullet g} \tag{B.69}$$

is an invariant function, wherein

$$f(x) = f^I(x, p) \tag{B.70}$$

for every $x \in \operatorname{Dom} f$.

In other words: *any non-invariant function can be obtained by presetting parameters in some invariant function f^I* (Fig. B.8).

This means that in applications, the lack of symmetry (invariance) should be described with the appropriate parameters of the asymmetry. The stronger

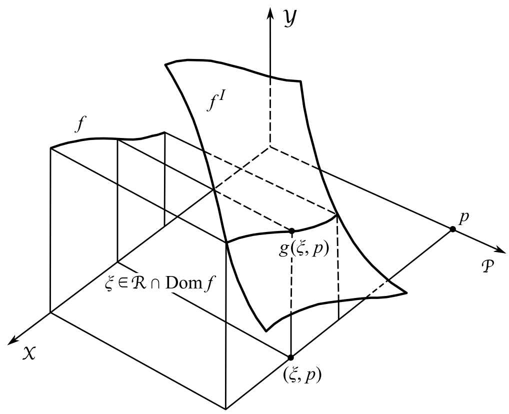

FIG. B.8. Illustration of the idea of extension of any non-invariant function to invariant function.

the theorem, the richer the additional structure of the sets $\mathcal{X}$ and $\mathcal{Y}$ under consideration. Let us consider an example of its effectiveness.

EXAMPLE (for professionals in mechanics and continuum physics). Let us take the constitutive relations of the material continuum theory describing its reactions to mechanical, electromagnetic, thermodynamic and chemical processes. These relations are in the form of tensor functions and functionals. The operating group is the orthogonal group Orth $\mathfrak{Э}$. The symmetry group of a functional $\mathcal{F}$ is usually one of the crystal symmetry groups. Sequences of tensors of order ≤ 4 can be used as parameters of anisotropy

(B.71) $$p = (P_1, ..., P_s), \quad s \leq 3.$$

Such sequences are sufficient to describe all the necessary subgroups of an orthogonal group. According to the theorem, each anisotropic function will be a special case of an isotropic function, i.e.,

(B.72) $$\mathcal{F}(X_1, ..., X_m) = \mathcal{F}^I(X_1, ..., X_m;\ P_1, ..., P_s),$$

where
(B.73)
$$\mathrm{Orth}_{Э}\ \mathcal{F} = \mathrm{Orth}_{Э}\ (P_1, ..., P_s) = \bigcap_{i=1}^{s} \mathrm{Orth}_{Э}\ P_i, \quad \mathrm{Orth}_{Э}\ \mathcal{F}^I = \mathrm{Orth}\,Э\ .$$

The situation is completely different in the following situation.

EXAMPLE. One of Pasteur's great discoveries was the observation that there are living organisms that distinguish the left and right forms of objects. From the point of view of the theorem about formal extensions, we should find a parameter that distinguishes proper and improper rotations. There are plenty of such parameters. Which of them may claim to be the true parameter remains a great mystery of Biophysics so far.

16. Final remarks. We have given a layout of the general language that talks about symmetry, with some extra considerations given to the needs of the Theory of Dimensions and Similarity. SYMMETRY deserves it.

Bibliographical notes and literature

1. The concept of Γ-set can be found, e.g., in [B.1]. The charming book [B.2] presents Geometry as a theory of special Γ-sets. H. Weyl's book [B.3] was my first and decisive contact with symmetry. From it, I also took Fig. B.4.
2. There are many extensive monographs on symmetry in Physics (see, e.g., [B.3]–[B.14]). Our presentation of the subject describes only the most important and primal concepts.
3. It took me a surprisingly long time to understand with full clarity the role of the most primal and simple facts connected with symmetry, which I have called the Ornament Principle and Theorem on the Symmetry of Cause and Effect. I described my understanding in the book [B.15]. This appendix is a brief summary of it.
4. The definition of a function given by G. Peano is commonly adopted today in teaching [B.16], [B.17].
5. P. Curie's profound comments on symmetry were given in [B.18]. Their concise presentation was delivered by M. Skłodowska-Curie in [B.19]. Shubnikov and his school [B.20], [B.21] promoted the Curie principle with due enthusiasm and proper understanding. Interesting remarks about misinterpretations of Curie's statements can be found in [B.22].
6. The non-equivalence of the right and left forms for living organisms, seen through the eyes of a physicist, is described in [B.23].
7. In work [B.24] and then in work [B.27], the systems (B.72) are presented for all classes of crystal symmetry.
8. The theorem on formal expansion was presented in its entirety in [B.15]. Earlier, its specific forms were announced in [B.25], [B.26].

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SUPPLEMENT[8)]

As proposed in the presented Report on algebraization of the concept of symmetry, a fundamental role is played by the concept of a *group*, thanks to which it is possible to introduce the concept of symmetry in a strict mathematical sense. The concept of a group is one of the most important concepts widely used in elaborating theories (models) of real physical phenomena. The second basic concept is the *equivalence relation* which enables to decompose the elements of the set under consideration into symmetry (equivalence) classes.

In order to make the Report as self-contained as possible, the definitions of group and equivalence relation and related concepts are recalled below.

DEFINITION. An algebraic structure $\Gamma \equiv (\{A\}, \circ)$ consisting of a non-empty *set of elements* $\{A\}$, equipped with an *operation* "$\circ$" assigning an element from $\{A\}$ to any pair of elements from $\{A\}$ ($\circ\colon \{A\}\times\{A\} \ni (\alpha_1,\ \alpha_2)\to\alpha_1 \circ \alpha_2 \in \{A\}$) is called a *group* when the operation "$\circ$" satisfies the following axioms:

$$\text{(i)} \quad \bigwedge_{\alpha_1,\alpha_2,\alpha_3\in G} \qquad \alpha_1 \circ (\alpha_2 \circ \alpha_3) = (\alpha_1 \circ \alpha_2) \circ \alpha_3,$$

$$\text{(ii)} \quad \bigvee_{e\in G} \quad \bigwedge_{\alpha\in G} \qquad e \circ \alpha = \alpha \circ e = \alpha\ ,$$

$$\text{(iii)} \quad \bigwedge_{\alpha\in G} \quad \bigvee_{\alpha^{-1}\in G} \qquad \alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = e,$$

i.e., the operation "$\circ$" is *associative* (i), there exists an *identity element* (unity) of the group (ii), and for each element of the group, there *exists a uniquely defined inverse element* (iii).

A *monoid* is an algebraic structure for which condition (i) is satisfied, i.e., in the set $\{A\}$, it is defined associative operation.

A *semi-group* is an algebraic structure for which conditions (i) and (ii) are satisfied, i.e., an associative operation is defined in the set $\{A\}$ and there exists identity element, but not for every element of the set $\{A\}$ there exists unambiguously defined inverse element.

When the group operation "$\circ$" is commutative

$$\text{(iv)} \quad \bigwedge_{\alpha_1,\alpha_2\in G} \qquad \alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$$

then we call such a Group commutative group (*Abelian* Group).

A *subgroup* Λ in a group Γ it is called a set $\Lambda \subset \Gamma$ that is closed with respect to the group operation and reciprocality, i.e.,

$$\alpha_1 \in \Lambda \ \wedge\ \alpha_2 \in \Lambda \ \Rightarrow\ \alpha_1 \circ \alpha_2 \in \Lambda \quad \text{and} \quad \alpha \in \Lambda \Rightarrow \alpha^{-1} \in \Lambda.$$

[8)] *Editorial note:* This supplement has been elaborated by Translator based on information in the supplement to the book by Jan Rychlewski titled *Symmetry of Causes and Effects* (in Polish), Polish Scientific Publishers PWN, Warsaw, 1991.

The common part of any subgroups is a subgroup.

A *binary relation* is any subset $R \subset \{S \times T\}$ of the Cartesian product of sets $\{S\}$, $\{T\}$. Often, instead of the designation $(s,t) \in R$, the designation $s\,R\,t$ is used.

Definition. A *binary relation* is called an *equivalence relation* $E \subseteq \{X \times X\}$ operating in a certain set $\{X\}$ if and only if it is

(i) $\bigwedge_{x \in \{X\}} \quad x\,E\,x,$

(ii) $\bigwedge_{x,\, y \in \{X\}} \quad x\,E\,y \Rightarrow y\,E\,x,$

(iii) $\bigwedge_{x,\, y,\, z \in \{X\}} \quad (x\,E\,y) \wedge (y\,E\,z) \Rightarrow y\,E\,z,$

i.e., the relation E is *reflexive* (i), *symmetric* (ii), *transitive* (iii). Two elements $x, y \in \{X\}$ such that $(x,y) \in E$ are called *equivalent* and often symbolically denoted $x \sim y$.

Class of the equivalence of an element $x \in \{X\}$ with respect to a relation E is the set of all $y \in \{X\}$ such that $y\,E\,x$. According to the axiom of choice, it is always possible to indicate a *set of representatives* $\mathcal{R}$ containing exactly one element from each equivalence class.

The *distribution* of a set $\{X\}$ is called a family of its subsets $\mathcal{D}_\alpha$, $\alpha \in I$ that meets the conditions

$$1^\circ \quad \{X\} = \bigcup_{\alpha \in I} \mathcal{D}_\alpha, \qquad\qquad 2^\circ \quad \bigwedge_{\alpha,\beta \in I} \quad \mathcal{D}_\alpha \cap \mathcal{D}_\alpha = \emptyset \ \text{ or } \ \mathcal{D}_\alpha = \mathcal{D}_\beta.$$

The set of all *equivalence classes* $\{X\}/E$ is the *decomposition* of the set $\{X\}$.

Reference

Polish language translation of Pierre Curie paper [B.18] is available provided with A. Ziółkowski extended commentary:

Curie P., O symetrii zjawisk fizycznych, symetrii pola elektrycznego i pola magnetycznego (A. Ziółkowski, tłumacz), *Studia Historiae Scientiarum*, **22**, 2023 doi: 10.4467/2543702XSHS.23.002.17693.

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