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CONSOLIDATION OF THE VISCO-ELASTIC HALF-SPACE AT MIXED CONDITIONS OF PERMEABILITY OF THE BOUNDARY

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The exact form solution of axi-symmetric consolidation problem for a visco-elastic half-space at mixed conditions of the permeability of the boundary is presented. The process of consolidation is due to time-variable normal load acting on the boundary of the half-space. The load free boundary of the half-space is permeabile while its loaded part is not permeabile. On the basis of a coupled three-dimensional Biot's consolidation theory with taking into account the rheological properties of the skeleton the state of stress and deformation is evaluated and the elastic skeleton the diagrams for pressure in pores under loading for different moments of time and different rates of the load increments are given.

1. INTRODUCTION

In several papers devoted to the solution of the boundary problem of the consolidation theory, problems essential for applications were undertaken and effective solutions obtained. The interest here are the general solutions presented in [6, 12] formulated for the most general linear visco-elastic medium without ageing. From a theoretical point of view, the papers devoted to the structure of fundamental solutions have a special value. Among them, note papers [7] and [5]. The solution of problems with mixed discontinuous boundary conditions is of special interest. Attempts at solution of these problems have many times been initiated by several authors [2, 3, 15], mainly for contact problems, but a fully satisfactory solution for the axi-symmetric state of strain has been given only recently by J. GASZYŃSKI and SZEFER [8] and for the plane state of strain by GASZYŃSKI [9]. A shortcomming of all the papers indicated is cleare to take into account the mixed conditions of permeability of the boundary which, it seems, is necessary from a point of view of applications and is also attractive for the further development of a theory.

This paper deals with the process of consolidation of a porous viscoelastic saturated medium excited by a known normal load acting on the boundary of the half-space. The loaded part of the boundary is not permeable, while the unloaded boundary of the half-space is permeable. The problem under consideration belongs then to the group of mixed boundary-value problems for two-phase media. For description of the rheological properties of the skeleton, the most general linear visco-elastic model without ageing was considered, assuming different features of creep — namely shear creep, dilatational creep, and creep caused by liquid pressure in the pores. The problem was formulated and solved within the framework of Biot's coupled three-dimensional consolidation theory for an axi-symmetric state of strain.

2. Register of equations for a consolidating visco-elastic medium

We shall consider here a two-phase deformable medium composed of a porous isotropic homogeneous visco-elastic skeleton and liquid flowing slowly through the pores according to Darcy's law. We assume that the skeleton has rheological properties without ageing, with different creep features during a process of deformation — namely, shear creep, dilatational creep and creep caused by liquid pressure in the pores. We assume the most general model of a linear visco-elastic porous and saturated medium. It covers, according to the results of Zarecki's works [15] and Mestchian's investigations [13] a wide class of clays which reveal different features of shear and dilatational creep in the process of deformation.

The constitutive equations for the material characterized above have the form:

$$\sigma_{ij}=2N\varepsilon_{ij}+M\varepsilon_{kk}\,\delta_{ij}-Ap\delta_{ij}, \tag{2.1}$$

where N, M, A are the following Volterra type integral operators describing the rheological properties of the skeleton:

$$\begin{gathered} N=\mu\left[1-\int\limits_0^t R(t-\tau)\ldots d\tau\right], \\ M=\frac{1}{3}(A_v-2N), \\ A_v=\alpha_v\left[1-\int\limits_0^t R_v(t-\tau)\ldots d\tau\right], \\ A_p^{-1}=\frac{1}{\alpha_p}\left[1+\int\limits_0^t k_p(t-\tau)\ldots d\tau\right], \\ A=A_v\,A_p^{-1}, \end{gathered} \tag{2.2}$$

$R(t-\tau)$, $R_v(t-\tau)$ are the resolvents of the kernels of the shear and dilatational creep, respectively, $K_p(t-\tau)$ is the kernel of the creep caused by liquid pressure in the pores. The following notations have been introduced: σ_{ij} — the elements of the stress tensor for the two-phase medium, ε_{ij} — the elements of the strain tensor, p — the pressure of liquid in the pores, δ_{ij} — Kronecker's symbol, μ — shear modulus of the skeleton, α_p — bulk modulus caused by liquid pressure in the pores.

To the constitutive equations we join, further,
the equations of equilibrium:

$$\sigma_{ij,j}=0, \tag{2.3}$$

the geometrical relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.4}$$

and the filtration equation:

$$\frac{k}{\gamma} \Delta p = \frac{3n}{\alpha_w} \dot{p} + \dot{\varepsilon}, \tag{2.5}$$

in which u_i are the coordinates of the displacement vector, n denotes the porosity, k — filtration coefficient, γ — the voluminal weight of the liquid, α_w is a compressibility modulus of fluid, Δ denotes the Laplace operator and a dot $(\cdot)$ denotes the partial differentiation with respect to time.

The Eqs. (2.1)–(2.5) constitute the complete set of equations of the consolidation theory for a medium having a visco-elastic skeleton.

In a further work it will be more convenient to use the well known system of displacement equations:

$$N\Delta u_i + (N+M)\varepsilon_{,i} - Ap_{,i} = 0, \qquad i = 1, 2, 3 \tag{2.6}$$

obtained by substitution of the Eq. (2.4) in to the Eq. (2.1), and the result subsequently in to the Eq. (2.3). The Eqs. (2.6) together with the filtration equation (2.5) form the basic system of equations which will be convenient for use in further considerations.

3. Formulation of the problem

In a cylindrical system of coordinates $Or\varphi z$, consider the half-space $z \geqslant 0$, the smooth boundary of which is the known loading function $Q(r, t)$ (Fig. 1). The Eqs. (2.6) and (2.5) are the basis for further considerations and in the case of axial symmetry have the following form:

$$\begin{gathered} N\left(\Delta u - \frac{u}{r^2}\right) + (N+M)\varepsilon_{,r} = Ap_{,r}, \\ N\Delta w + (N+M)\varepsilon_{,z} = Ap_{,z}, \\ \frac{k}{\gamma}\Delta p = \frac{3n}{\alpha_w}\dot{p} + \dot{\varepsilon}, \end{gathered} \tag{3.1}$$

where

$$\varepsilon = u_{,r} + \frac{u}{r} + w_{,z}, \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

and u and w denote the radial and vertical displacement, respectively. The Eqs. (3.1) constitute a complete system of three linear integro-partial differential equations of mixed eliptico-parabolic type for the three unknown functions u, w, p. The solution of this system of equations will enable evaluation of the state of stress and strain in the consolidating medium.

The boundary conditions for the problem considered are as follows:

$$\sigma_{rz}(r,0,t)=0, \quad \sigma_z(r,0,t)=Q(r,t), \tag{3.2}$$

for the pressure in pores

$$\left.\frac{\partial p(r,z,t)}{\partial z}\right|_{z=0}=0, \quad r<R,$$

$$p(r,0,t)=0, \quad r>R.$$

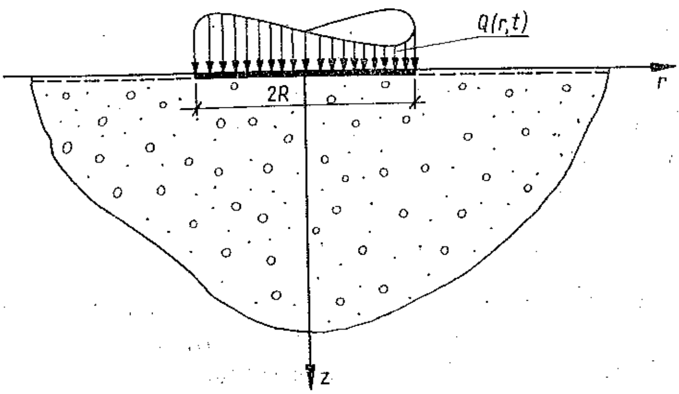


FIG. 1.

Thus, the problem formulated, which will be the subject of further considerations, belongs to the group of problems with mixed boundary conditions. Further we require that at infinity the following conditions be satisfied:

$$\lim_{R_z\to\infty}(u,w,p)=0, \quad \text{where} \quad R_z=\sqrt{r^2+z^2},$$

assuming that the loading $Q(r,t)$ increases with time in a continuous manner from zero to a finite value and that the following homogeneous initial conditions are fulfilled

$$u(r,z,0)=w(r,z,0)=p(r,z,0)=0. \tag{3.3}$$

There conditions satisfy the compatibility equations [7], required for the system of equations of the consolidation theory.

4. General solution of the problem formulated

In constructing the general solution of the system (3.1), we shall apply the Nankel integral transforms with respect to the variable r, and the Laplace transforms with respect to the variable t. Using the well known relations for the Bessel functions, the form of the operators N, M, A, a convolution theorem for the Laplace transforms and the initial conditions (3.3) will be obtained from the Eq. (3.1):

$$\bar{N}\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\tilde{u}}+(\bar{N}+\bar{M})(-\rho)\bar{\tilde{\varepsilon}}=-\rho\bar{A}\bar{\tilde{p}},$$

(4.1)
$$\bar{N}\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\tilde{w}}+(\bar{N}+\bar{M})\frac{d\bar{\tilde{\varepsilon}}}{dz}=\bar{A}\frac{d\bar{\tilde{p}}}{dz},$$

$$\frac{k}{\gamma}\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\tilde{p}}=\frac{3n}{\alpha_w}s\bar{\tilde{p}}+s\bar{\tilde{\varepsilon}},$$

where

$$\bar{\tilde{u}}(\rho,z,s)=\int_0^\infty\int_0^\infty u(r,z,t)rJ_1(\rho r)e^{-st}\,dr\,dt,$$

$$\begin{bmatrix}\bar{\tilde{w}}(\rho,z,s)\\ \bar{\tilde{p}}(\rho,z,s)\end{bmatrix}=\int_0^\infty\int_0^\infty\begin{bmatrix}w(r,z,t)\\ p(r,z,t)\end{bmatrix}rJ_0(\rho r)e^{-st}\,dr\,dt,$$

$$\bar{\tilde{\varepsilon}}=\rho\bar{\tilde{u}}+\frac{d\bar{\tilde{w}}}{dz},$$

$$[\bar{N},\bar{M},\bar{A}]=\int_0^\infty[N,M,A)]e^{-st}\,dt.$$

We start the solution of the system (4.1) by performing the following preliminary operations: first we multiply the Eq. (4.1) by ρ and differentiate the Eq. $(4.1)_2$ with respect to the variable z, and after adding relations thus transformed we obtain:

(4.2)
$$(2\bar{N}+\bar{M})\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\tilde{\varepsilon}}=\bar{A}\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\tilde{p}}.$$

Taking into account the third of the Eqs. (4.1), we obtain the fourth order differential equation for the unknown function of the pressure in the pores in the form:

(4.3)
$$\left(\frac{d^2}{dz^2}-\rho^2\right)^2\bar{\tilde{p}}-s\bar{B}\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\tilde{p}}=0,$$

where the following notation has been introduced:

(4.4)
$$\bar{B}=\frac{\gamma}{k}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right).$$

The solutions of the Eq. (4.3), satisfying the conditions at infinity, has the form:

(4.5)
$$\bar{\tilde{p}}(\rho,z,s)=C_1(\rho,s)e^{-\rho z}+C_2(\rho,s)e^{-mz},$$

where

$$m^2=\rho^2+s\bar{B}.$$

Using the function $\bar{\bar{p}}$, we compute $\bar{\bar{\varepsilon}}$ from the Eq. $(4.1)_3$ and substituting it in to the two-remaining equations of the system (4.1), we obtain:

$$(4.6)\quad \begin{aligned} \left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\bar{u}} &= \frac{\bar{N}+\bar{M}}{s\bar{N}}\left\{\rho\left[\frac{k}{\gamma}\left(\frac{d^2}{dz^2}-\rho^2\right)-s\bar{B}_1\right]C_1 e^{-\rho z}+\right. \\ &\left. +\rho\left[\frac{k}{\gamma}\left(\frac{d^2}{dz^2}-\rho^2\right)-s\bar{B}_1\right]C_2 e^{-mz}\right\}, \\ \left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\bar{w}} &= \frac{\bar{N}+\bar{M}}{s\bar{N}}\left\{\rho\left[\frac{k}{\gamma}\left(\frac{d^2}{dz^2}-\rho^2\right)-s\bar{B}_1\right]C_1 e^{-\rho z}+\right. \\ &\left. +m\left[\frac{k}{\gamma}\left(\frac{d^2}{dz^2}-\rho^2\right)-s\bar{B}_1\right]C_2 e^{-mz}\right\}, \end{aligned}$$

where

$$\bar{B}_1=\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}.$$

The general solutions of the above differential equations have the form:

$$(4.7)\quad \begin{aligned} \bar{\bar{u}} &= C_3 e^{-\rho z}+C_1\frac{\bar{N}+\bar{M}}{2\bar{N}}\bar{B}_1 z e^{-\rho z}-\frac{\rho\bar{A}C_2}{s\bar{B}(2\bar{N}+\bar{M})}e^{-mz}, \\ \bar{\bar{w}} &= C_4 e^{-\rho z}+C_1\frac{\bar{N}+\bar{M}}{2\bar{N}}\bar{B}_1 z e^{-\rho z}-\frac{m\bar{A}C_2}{s\bar{B}(2\bar{N}+\bar{M})}e^{-mz}. \end{aligned}$$

Using the constitutive equations and the geometrical relations, we may obtain the transforms for the stresses in which we are interested:

$$(4.8)\quad \begin{aligned} \bar{\bar{\sigma}}_r &= C_1\left[(\bar{N}+\bar{M})\bar{B}_1 z\rho+\frac{\bar{M}(\bar{N}+\bar{M})}{2\bar{N}}\bar{B}_1-\bar{A}\right]e^{-\rho z}- \\ &\quad -C_2\frac{2\bar{A}\bar{N}m^2}{s\bar{B}(2\bar{N}+\bar{M})}e^{-mz}+C_3\rho(2\bar{N}+\bar{M})e^{-\rho z}-\bar{M}C_4\rho e^{-\rho z}, \\ \bar{\bar{\sigma}}_z &= C_1\left[-(\bar{N}+\bar{M})\rho z\bar{B}_1+\frac{\bar{M}\bar{A}}{2\bar{N}}+\frac{3n(2\bar{N}+\bar{M})(\bar{N}+\bar{M})}{\alpha_w 2\bar{N}}\right]e^{-\rho z}+ \\ &\quad +C_2\frac{2\bar{A}\bar{N}\rho^2}{s\bar{B}(2\bar{N}+\bar{M})}e^{-mz}+C_3\bar{M}\rho e^{-\rho z}-C_4(2\bar{N}+\bar{M})\rho e^{-\rho z}, \\ \sigma_{rz} &= C_1(\bar{N}+\bar{M})\bar{B}_1\frac{1-\rho z}{2}e^{-\rho z}+\frac{2\bar{A}\bar{N}m\rho}{s\bar{B}(2\bar{N}+\bar{M})}C_2 e^{-mz}- \\ &\quad -C_3\bar{N}\rho e^{-\rho z}-C_4\bar{N}\rho e^{-\rho z}. \end{aligned}$$

The formulae (4.7) and (4.8) contain four coefficients C_1, C_2, C_3, C_4 for evaluation of which we need four relations. Three of them are the boundary conditions (3.2) while the fourth relation is obtained from the fact that as a result of the assumed method of decoupling of the system of equations, the order of the filtration equation has increased from the second to the fourth. The solution obtained (4.5) must then satisfy the Eq. $(4.1)_3$. From here the relation sought is obtained:

$$2\bar{N}\rho\,(C_3-C_4)+\left[\frac{3n}{\alpha_w}(3\bar{N}+\bar{M})+\bar{A}\right]C_1=0. \tag{4.9}$$

Before the boundary conditions are taken into account, we perform on them the Laplace transform, to obtain:

$$\begin{gathered}
\int\limits_0^\infty \left\{\frac{\bar{N}+\bar{M}}{2}C_1\bar{B}_1+2C_2\bar{N}\rho m\frac{\bar{A}}{s\bar{B}(2\bar{N}+\bar{M})}-C_3\rho\bar{N}-C_4\rho\bar{N}\right\}\rho J_0(\rho r)\,d\rho=0,\\
\int\limits_0^\infty C_1\left\{\left[\frac{\bar{M}\bar{A}}{2\bar{N}}+\frac{3n}{\alpha_w}\,\frac{(2\bar{N}+\bar{M})(\bar{N}+\bar{M})}{2\bar{N}}\right]+\frac{2\rho^2\bar{A}\bar{N}}{s\bar{B}(2\bar{N}+\bar{M})}C_2+C_3\bar{M}\rho-\right.\\
\left.-(2\bar{N}+\bar{M})\rho C_4\right\}\rho J_0(\rho r)\,d\rho=\int\limits_0^\infty \bar{\bar{Q}}(\rho s)\rho J_0(\rho r)\,d\rho,\\
\int\limits_0^\infty (-\rho C_1-mC_2)\rho J_0(\rho r)\,d\rho=0 \quad \text{for} \quad r<R,\\
\int\limits_0^\infty (C_1+C_2)\rho J_0(\rho r)\,d\rho=0 \quad \text{for} \quad r>R.
\end{gathered} \tag{4.10}$$

The condition $(4.10)_1$, $(4.10)_2$ and (4.9) enable us to express the parameters C_1, C_3, C_4 in terms of C_2 in the following manner:

$$\begin{gathered}
C_1=\frac{-1}{(\bar{N}+\bar{M})\bar{B}_1}\left[\frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}\,\frac{\rho}{m+\rho}C_2+\bar{\bar{Q}}(\rho,s)\right],\\
C_3=\frac{3n}{\alpha_w}\,\frac{\bar{\bar{Q}}(\rho s)}{2(\bar{N}+\bar{M})\rho\bar{B}_1}+\frac{\bar{A}\bar{N}}{(\bar{N}+\bar{M})(2\bar{N}+\bar{M})\bar{B}_1(m+\rho)}\left[\frac{3n}{\alpha_w}-\frac{m\bar{B}_1(\bar{N}+\bar{M})}{\bar{N}(\rho-m)}\right],\\
C_4=-\frac{k(2\bar{N}+\bar{M})\bar{B}}{\gamma(\bar{N}+\bar{M})\bar{B}_1}\,\frac{\bar{\bar{Q}}(\rho,s)}{2\bar{N}\rho}+\frac{\bar{A}\left[(2\bar{N}+\bar{M})\frac{\gamma}{k}\bar{B}\rho-\frac{3n}{\alpha_w}\bar{N}_m\right]C_2}{s\bar{B}(2\bar{N}+\bar{M})(\bar{N}+\bar{M})\bar{B}_1}.
\end{gathered} \tag{4.11}$$

The remaining boundary conditions $(4.10)_3$ and $(4.10)_4$ lead to a system of dual integral equations with the unknown quantity C_2:

$$\begin{gathered}
\int\limits_0^\infty \left\{\frac{-\bar{\bar{Q}}(\rho,s)}{\bar{B}_1(\bar{N}+\bar{M})}+\left[1+\frac{2\rho\bar{A}\bar{N}}{s\bar{B}(2\bar{N}+\bar{M})\bar{B}_1(\bar{N}+\bar{M})}\right]C_2\right\}\rho J_0(\rho r)\,d\rho=0 \\
\text{for} \quad r>R,\\
\int\limits_0^\infty \left\{\frac{-\rho\bar{\bar{Q}}(\rho s)}{\bar{B}_1(\bar{N}+\bar{M})}+\left[\frac{2\rho^2\bar{A}\bar{N}(m-\rho)}{s\bar{B}(2\bar{N}+\bar{M})\bar{B}_1(\bar{N}+\bar{M})}+m\right]C_2\right\}\rho J_0(\rho r)\,d\rho=0\\
\text{for} \quad r<R.
\end{gathered} \tag{4.12}$$

Introducing the new variables

$$r = Ru, \qquad \rho = R^{-1} v,$$

and substituting

$$\Phi(v,s) = vR^{-1}\left\{\bar{\tilde{Q}}(v,s) - C_2\left[\bar{B}_1(\bar{N}+\bar{M}) + \frac{2\bar{A}\bar{N}R^{-2}v(m-v)}{sB(2\bar{N}+\bar{M})}\right]\right\},$$

$$f(u,s) = \frac{1}{R}\int_0^\infty \bar{\tilde{Q}}(v,s)G(v,s)J_0(uv)\,dv,$$

(4.13)
$$G(u,s) = \frac{s\bar{B}\bar{B}_1(\bar{N}+\bar{M})}{2\bar{B}_1(\bar{N}+\bar{M}) + \dfrac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}} \times \left[1 - \frac{s\bar{B}\bar{B}_1(\bar{N}+\bar{M})}{\bar{B}_1(\bar{N}+\bar{M})(m+v)^2 + \dfrac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}v(m+v)}\right],$$

we obtain the system of equations (4.12) in the form:

(4.14)
$$\frac{1}{R^2}\int_0^\infty \Phi(v,s)\left[v + \frac{R^2}{v}G(v,s)\right]J_0(u,v)\,dv = f(u,s), \quad u<1,$$

$$\frac{1}{R}\int_0^\infty \Phi(v,s)J_0(u,v)\,dv = 0, \quad u>1.$$

The mixed boundary condition $(3.2)_3$ is then reduced to the system of dual integral equations with zero order Bessel kernels, where the weight function has the following form:

$$v + \frac{1}{v}R^2 G(v,s).$$

5. Solution of the system of dual integral equations

To solve the system of equations (4.14), the Ufland-Lebedev method is applied. The unknown function is sought in the form:

(5.1)
$$\Phi(v,s) = v\int_0^1 \varphi(\zeta,s)\cos v\zeta\,d\zeta.$$

Introducing (5.1) into $(4.14)_2$, after simple transformations we obtain:

$$\frac{1}{R}\int_0^\infty v\int_0^1 \varphi(\zeta,s)\cos v\zeta\,d\zeta J_0(uv)\,dv = \frac{1}{R}\int_0^\infty [\varphi(1,s)\sin v]J_0 s\,dv - \frac{1}{R}\int_0^1 \varphi'\zeta(\zeta,s)\times$$

$$\times \sin v\zeta\,d\zeta J_0(uv)\,dv = \frac{\varphi(1,s)}{R}\int_0^\infty \sin v J_0(uv)\,dv - \frac{1}{R}\int_0^1 \varphi'(\zeta,s)\int_0^\infty \sin v\zeta J_0 \times$$

$$\times (uv)\,dv\,d\zeta \quad \text{for} \quad u>1.$$

Using the property of the Weber-Schafteitlin integral

$$(5.2)\qquad \int_0^\infty \sin v\zeta J_0(uv)\,dv = \begin{cases} \dfrac{1}{\sqrt{\zeta^2-u^2}}, & \zeta>u, \\ 0, & \zeta<u \end{cases}$$

confirm that the Eq. $(4.14)_2$ is satisfied identically. Next, substituting the Eq. (5.1) in to $(4.14)_1$, we obtain:

$$\frac{1}{R^2}\int_0^\infty v\int_0^1 \varphi(\zeta,s)\cos v\zeta\,d\zeta\left[v+\frac{R^2}{v}G(v,s)\right]J_0(uv)\,dv = \frac{1}{R^2}\int_0^1 \varphi(\zeta,s)\times$$

$$\times d\zeta\int_0^\infty v^2\cos\zeta v J_0(uv)\,dv + \int_0^1 d\zeta\int_0^\infty G(v,s)J_0(uv)\,dv = f(u,s)\quad \text{for}\quad u<1.$$

Integrating the above expression twice, and taking into account the values of the Weber-Schafteitlin integral

$$(5.3)\qquad \int_0^\infty \cos v\zeta J_0(uv)\,dv = \begin{cases} \dfrac{1}{\sqrt{u^2-\zeta^2}}, & \zeta<u, \\ 0, & \zeta>u, \end{cases}$$

arrive at:

$$(5.4)\qquad \begin{aligned} &-\frac{1}{R^2}\int_0^u \frac{\varphi''(\zeta,s)}{\sqrt{u^2-\zeta^2}}\,d\zeta + \frac{s\bar{B}\bar{B}_1(\bar{N}+\bar{M})}{2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}}\int_0^u \frac{\varphi(u,s)}{\sqrt{u^2-\zeta^2}}\,d\zeta - \\ &-\frac{s^2\bar{B}^2\bar{B}_1^2(\bar{N}+\bar{M})^2R^2}{2\bar{B}_1(\bar{N}+\bar{M})+\dfrac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}}\int_0^1 \varphi(\zeta,s)\int_0^\infty H(v,s)\cos\zeta v J_0(uv)\,dv\,d\zeta = f(u,s), \end{aligned}$$

with the conditions

$$(5.5)\qquad \varphi(1,s)=0,\qquad \varphi'(0,s)=0,$$

where

$$(5.6)\qquad H(v,s)=\frac{1}{\bar{B}_1(\bar{N}+\bar{M})(m+v)^2+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}v(m+v)}.$$

Assuming $\zeta=u\sin\theta$ in the first two integrals, and using in the third the representation of the Bessel function

$$(5.7)\qquad J_0(u,v)=\frac{2}{\pi}\int_0^{\pi/2}\cos(uv\sin\theta)\,d\theta,$$

we obtain after transformations:

$$(5.8)\qquad \int_0^{\pi/2}\left\{-\frac{1}{R^2}\varphi''(u\sin\theta,s)+\frac{\bar{B}_1(\bar{N}+\bar{M})s\bar{B}\varphi(u\sin\theta,s)}{2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}}-\right.$$
$$-\frac{2s^2\bar{B}^2\bar{B}_1^2(\bar{N}+\bar{M})^2R^2}{\pi[2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}]}\int_0^{\infty}H(v,s)[\cos(\zeta+u\sin\theta)v+$$
$$\left.+\cos(\zeta-u\sin\theta)v]\,dv\int_0^1\varphi(\zeta,s)\,d\zeta\right\}d\theta=f(u,s).$$

The compact form of this expression may be obtained after introducing the notation:

$$(5.9)\qquad F(u\sin\theta,s)=-\frac{1}{R^2}\varphi''(u\sin\theta,s)+\frac{\bar{B}_1(\bar{N}+\bar{M})s\bar{B}\varphi(u\sin\theta,s)}{2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}}+$$
$$+\frac{-2s^2\bar{B}^2\bar{B}_1^2(\bar{N}+\bar{M})^2R^2}{\pi[2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}+(2\bar{N}+\bar{M})^{-1}]}\int_0^1\varphi(\zeta,s)[H(\zeta+u\sin\theta,s)+$$
$$+\tilde{H}(\zeta-u\sin\theta,s)]\,d\zeta,$$

where

$$(5.10)\qquad \begin{aligned}\tilde{H}(\zeta+u\sin\theta,s)&=\int_0^{\infty}H(v,s)\cos v(\zeta+u\sin\theta)\,dv,\\ \tilde{H}(\zeta-u\sin\theta,s)&=\int_0^{\infty}H(v,s)\cos v(\zeta-u\sin\theta)\,dv.\end{aligned}$$

We obtain, then, the Schlömilch integral equation

$$(5.11)\qquad \int_0^{\pi/2}F(u\sin\theta,s)\,d\theta=f(u,s),$$

the solution of which has the form:

$$(5.12)\qquad F(u,s)=\frac{2}{\pi}\left[f(0,s)+u\int_0^{\pi/2}f'(u\sin\theta,s)\,d\theta\right].$$

Coming back to the Eq. (5.9), we obtain the following expression for the function $\varphi(u,s)$ which is sought:

$$(5.13)\qquad -\frac{1}{R^2}\varphi''(u,s)+\frac{s\bar{B}_1\bar{B}(\bar{N}+\bar{M})}{2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}}\varphi(u,s)-$$
$$-\frac{s^2\bar{B}^2\bar{B}_1^2(\bar{N}+\bar{M})^2R^2}{\pi[\bar{B}_1(\bar{N}+\bar{M})+\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}]}\int_0^1\varphi(\zeta,s)K(u,\zeta,s)\,d\zeta=F(u,s),$$

where

$$K(u,\zeta,s)=\tilde{H}(\zeta+u,s)+\tilde{H}(\zeta-u,s).$$

This is a Fredholm integro-differential equation of the second kind with continuous and bounded kernel, as is well seen from the shape of the expression (5.6). By means of the symmetry of the kernel $K(u, \zeta, s)$, the Eq. (5.13) may be presented in the form:

$$(5.14)\quad -\frac{1}{R^2}\varphi''(u,s)+\frac{s\bar{B}\bar{B}_1(\bar{N}+\bar{M})}{2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}}\varphi(u,s)-$$

$$-\frac{s^2\bar{B}^2\bar{B}_1^2(\bar{N}+\bar{M})^2R^2}{\pi[\bar{B}_1(\bar{N}+\bar{M})+\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}]}\int_{-1}^{1}H(\zeta-u,s)\varphi(\zeta,s)\,d\zeta=F(u,s)$$

with the conditions (5.5).

Determining from this equation the function $\varphi(s,s)$ and then from the Eq. (5.1) $\Phi(u,s)$, the system of dual equations (4.14) may be solved. Because of the combined form of the kernel of the Eq. (5.14), the exact closed form solution is not possible to obtain. The existence in the kernel of a complex parameter also constitutes a barrier to the application of numerical techniques. In this situation, it seems that the orthogonalization method is the most effective for solution of the Eq. (5.14). The solution is then sought in the form:

$$(5.15)\quad \varphi(u,s)=\sum_{n=0}^{N}a_n(s)P_n(u)+\varphi_0,$$

where $P_n(u)$ is a complete set of the orthogonal functions in the interval $\langle -1, 1\rangle$. For the set of the functions $P_n(u)$, a system of Legendre polinoms, which possesses all properties mentioned above is assumed.

Taking into account the conditions (5.5)

$$\varphi(u,s)|_{u=1}=\sum_{n=0}^{N_0}a_n(s)P_n(1)+\varphi_0=0,$$

$$\varphi'(u,s)|_{u=0}=\sum_{n=0}^{N_0}a_n(s)P_n'(u)|_{u=0}=0 \quad \text{fulfil } \hat{u},$$

we obtain:

$$(5.16)\quad \varphi'=-\sum_{n=0}^{N_0}a_n(s)P_n(1),$$

and finally:

$$(5.17)\quad \varphi=\sum_{n=0}^{N_0}a_n[P_n(u)-P_n(1)].$$

Substituting (5.17) into (5.14), multiplying the Eq. (5.14) subsequently by $P_i(u)$, $i=1,2,\ldots,N_0$ and integrating with respect to u in the interval $\langle -1, 1\rangle$, we obtain the following system of linear algebraic equations with unknown coefficients $a_n(s)$:

$$(5.18)\quad \sum_{n=0}^{N_0}A_{mn}(s)a_n(s)=B_m(s),$$

where

$$B_m(s)=\int_{-1}^{1} F(u,s)P_m(u)\,du,$$

$$A_{mn}(s)=-\frac{1}{R^2}\int_{-1}^{1}\int_{-1}^{1} P'_m(u)P_n(\zeta)\,du\,d\zeta+\frac{\bar{B}_1(\bar{N}+\bar{M})s\bar{B}}{2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})}\int_{-1}^{1}\int_{-1}^{1} P_m(u)\times$$
$$\times P_n(\zeta)\,du\,d\zeta-\frac{2s^2\bar{B}^2\bar{B}_1^2(\bar{N}+\bar{M})^2R^2}{\pi[2\bar{B}_1(\bar{N}+\bar{M})+2\bar{A}\bar{N}(2\bar{N}+\bar{M})^{-1}]}\int_{-1}^{1}\int_{-1}^{1} H(\xi-u,s)P_m(u)P_n(\zeta)\,du\,d\zeta.$$

Cramer's formulae enable effective evaluation of the coefficients $a_n(s)$.

6. Computations of pressure in the pores and displacement of the boundary half-space

From the Eq. (4.5), for $z=0$

$$\tilde{\bar{p}}=C_1+C_2,$$

and the inverse Hankel transform is:

$$\bar{p}=\frac{1}{R}\int_0^\infty (C_1+C_2)vJ_0(uv)\,dv. \tag{6.1}$$

From the Eq. $(4.14)_2$, it results that for $u>1$:

$$\frac{-1}{\bar{B}_1(\bar{N}+\bar{M})R^2}\int_0^\infty \Phi(v,s)J_0(uv)\,dv=0.$$

The Laplace transform for pressure in the pores at $z=0$ has the form

$$\bar{p}(u,s)=\frac{-1}{\bar{B}_1(\bar{N}+\bar{M})R}\int_0^\infty \Phi(u,s)J_0(uv)\,dv=$$
$$=\frac{-1}{\bar{B}_1(\bar{N}+\bar{M})R}\frac{\varphi(1,s)}{\sqrt{1-u^2}}+\frac{1}{\bar{B}_1R(\bar{N}+\bar{M})}\int_u^1\frac{\varphi'(\zeta,s)}{\sqrt{\xi^2-u^2}}\,d\zeta. \tag{6.2}$$

Using the conditions (5.5) and performing the inverse Laplace transform we finally arrive at:

$$p(u,t)=\frac{1}{2\pi i}\int_s \frac{1}{\bar{B}_1(\bar{N}+\bar{M})R}\int_u^1\frac{\varphi'(\zeta,s)}{\sqrt{\zeta^2-u^2}}\,d\zeta\,ds. \tag{6.3}$$

The value of the displacement $w(r,0,t)$ is obtained from the Eq. (4.7) for $z=0$:

$$\bar{\tilde{w}}=C_4-C_2\frac{\bar{A}m}{s\bar{B}(2\bar{N}+\bar{M})},$$

and evaluating C_4 from the Eq. (4.11), C_2 from the Eq. $(4.13)_1$ and performing the inverse Hankel and Laplace transforms, we obtain:

$$\bar{\tilde{w}}=\frac{k(2\bar{N}+\bar{M})\bar{B}\left[(\bar{N}+\bar{M})\bar{B}_1+\frac{v}{m+v}\frac{4\bar{A}\bar{N}}{2\bar{N}+\bar{M}}\right]R\bar{\tilde{Q}}(v,s)}{2\bar{N}v\gamma(\bar{N}+\bar{M})B_1\left[(\bar{N}+\bar{M})\bar{B}_1+\frac{v}{m+v}\frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}\right]}-\frac{k(2\bar{N}+\bar{M})\bar{B}\frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}\frac{v}{m+v}\Phi(v,s)R^2}{2\bar{N}v\gamma(\bar{N}+\bar{M})\bar{B}_1\left[\bar{B}_1(\bar{N}+\bar{M})+\frac{v}{m+v}\frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}\right]v}. \tag{6.4}$$

After transformation, we have:

$$w=\frac{1}{2\pi i}\int_0^\infty \tilde{w}J_0(uv)R^{-1}v\,dv\,ds. \tag{6.5}$$

7. Numerical example

The results obtained in Sect. 5 and 6 enable a partial quantitative analysis of the problem considered to be performed. More complete qualitative and quantitative analysis may be made after calculation of the numerical examples and presenting the solution in the form of diagrams.

The loading of the boundary in this example is assumed in the form

$$Q(r,t)=T(t)q(r),$$

where

$$T(t)=\begin{cases}\dfrac{t}{t_0}, & 0\leqslant t<t_0,\\ 1, & t_0\leqslant t\end{cases}$$

and is presented in Fig. 2.

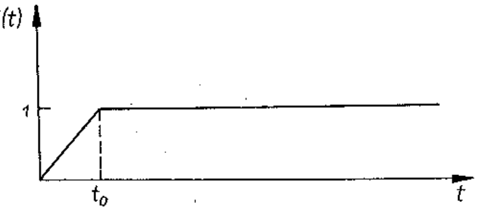


FIG. 2.

Using this function, it is possible, with accuracy sufficient for practical applications, to describe the real distribution of the loading applied to the soil; t_0 denotes here the time of duration of the total loading. In the computations, t_0 is assumed as 1, 15, 30 and 100 days, respectively. The parameters characterizing the properties of the medium were determined on the basis of the results presented in papers [13] and [15]. The numerical example was computed for an elastic skeleton. This assumption does not limit the drawning of conclusions for the kernels of regular and weakly singular integral operators. Nor is here any essential difficulty in taking into account the rheological effects of the medium. The assumed parameters of the medium are of orientation character and approximate solely the properties of a certain group of clay soils.

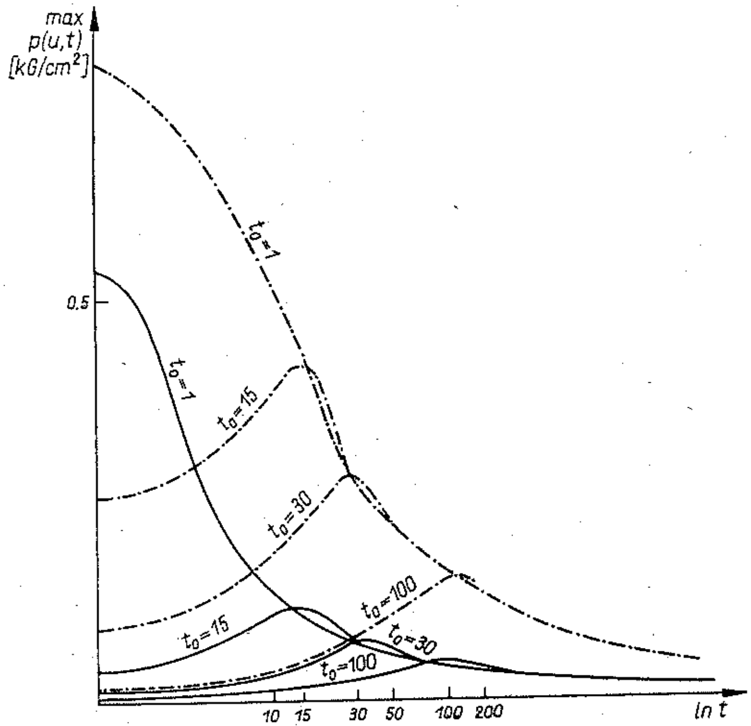


FIG. 3.

—— $k=2$ cm/day

–·–·– $k=0.2$ cm/day

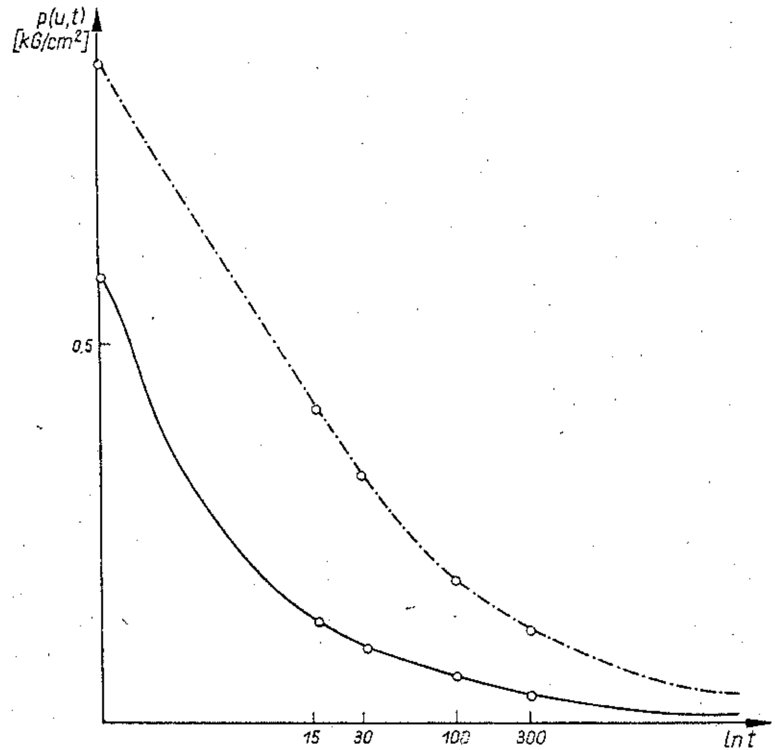


FIG. 4.

—— $k=2$ cm/day

–·–·– $k=0.2$ cm/day

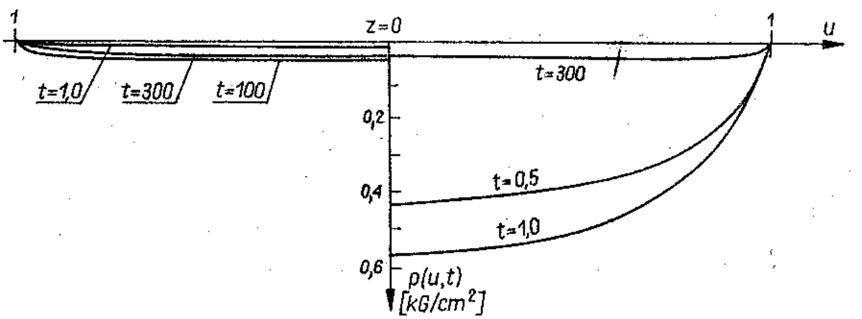
The solution of the integral equation is presented in the form of a combination of orthogonal systems of functions complete in the interval $\langle -1, 1\rangle$. These functions were assumed in the form of the Legendre polinoms. In, computations the following form of the polinoms was used:

$$P_0(u)=1,$$

$$P_1(u)=u,$$

$$P_2(u)=\frac{1}{2}(3u^2-1),$$

$$P_3(u)=\frac{1}{2}(5u^3-3u).$$



$t_0=100$ *days* $\quad k=2$ *cm/day* $\quad t_0=1$ *day*

FIG. 5.

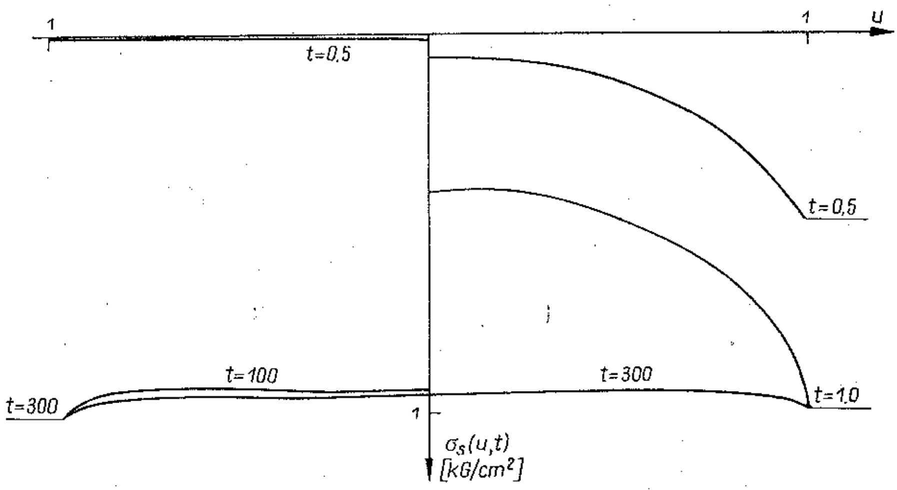


FIG. 6.

For evaluation of the inverse Laplace transforms, the method of interpolational polinoms, given in the paper [11], was applied. The following parameters characterizing the properties of the medium were assumed:

$$\mu=125\ \mathrm{kG/cm^2},$$

$$\alpha_v=500\ \mathrm{kG/cm^2},$$

$$\alpha_p=25\ \mathrm{kG/cm^2},$$

$$k=2\cdot10^{-1}\ \mathrm{cm/day},\qquad k=2\ \mathrm{cm/day},$$

$$n=0.4,$$

$$\alpha_w=6\cdot10^4\ \mathrm{kG/cm^2}$$

$$R=500\ \mathrm{cm},$$

$$q(r)=1\ \mathrm{kG/cm^2}.$$

The computations were performed on an computer Cyber 72. The results obtained are illustrated graphically. The diagrams of the distribution of pressure in the pores with time for different coefficients of filtration, are presented, together with the distribution of pressure in the pores for different rates of loading, the distribution of maximal values of the pressure with time for different rates of loading, and finally the diagrams of the stresses in the skeleton for $z=0$.

8. Final conclusions

The results presented and the formulae derived constitute a complete solution of the problem formulated. As has been shown, the states of strain and stress in a consolidating half-space having a visco-elastic skeleton may be effectively determined and the exact form of solution obtained.

The solution obtained and the numerical example presented justify the following conclusions:

1. From the analysis performed, it results that the rate of loading of the medium has a significant influence on the process of consolidation.

2. The distribution of pressure in time depends in an essential manner on the filtration coefficient.

3. The maximal value of pressure in the pores depends on the rate of loading and is reached (in our problem) under application of full loading. The longer the loading time the smaller this value. The largest magnitude of pressure in the pores is reached for $u=0$.

4. Form the analytical solution it results that pressure in the pores on the boundary is a continuous function of the time.

5. In the final phase of the consolidation process ($t\to\infty$). the solution tends asymptotically to the solution valid for the uniphase medium.

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Streszczenie

KONSOLIDACJA PÓŁPRZESTRZENI LEPKO-SPRĘŻYSTEJ PRZY MIESZANYCH WARUNKACH PRZEPUSZCZALNOŚCI BRZEGU

W pracy podano ścisłe rozwiązanie osiowo-symetrycznego zadania konsolidacji półprzestrzeni lepko-sprężystej przy mieszanych warunkach przepuszczalności brzegu. Proces konsolidacji wywołany jest zmiennym w czasie, normalnym obciążeniem działającym na brzeg półprzestrzeni. Brzeg półprzestrzeni wolny od obciążenia jest przepuszczalny, natomiast część brzegu pozostająca pod obciążeniem jest nieprzepuszczalna.

Na gruncie trójwymiarowej, sprzężonej teorii konsolidacji Biota z uwzględnieniem reologicznych własności szkieletu wyznaczono stan naprężeń i stan deformacji ośrodka uzyskując efektywne rozwiązanie. W szczególności dla ośrodka o szkielecie sprężystym podano wykresy ciśnienia porowego pod obciążeniem dla różnych chwil czasu i różnych prędkości przyrostu obciążenia.

Резюме

КОНСОЛИДАЦИЯ ВЯЗКО-УПРУГОГО ПОЛУПРОСТРАНСТВА ПРИ СМЕШАННЫХ УСЛОВИЯХ ПРОНИЦАЕМОСТИ ГРАНИЦ

В работе дается точное решение осесимметричной задачи консолидации вязко-упругого попупространства при смешанных условиях проницаемости границ. Процесс консолидации вызывается переменной во времени, нормальной нагрузкой действующей на границе полупространства. Свободная граница полупространства проницаемая, часть же границы находящаяся под нагрузкой непроницаемая. На грунте трехмерной, сопряженной теории консолидации Био, с учетом реологических свойств скелета, определены напряженное и деформационное состояния среды, получая эффективное решение. В частности, для среды с упругим скелетом приведены диаграммы пористого давления под нагрузкой для разных моментов времени и разных скоростей прироста нагрузки.

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