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DYNAMIC BUCKLING OF A SANDWICH BAR COMPRESSED BY PERIODICALLY VARIABLE FORCE

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The paper presents an analysis of dynamic buckling of a sandwich bar compressed by a periodically variable force. In order to determine the stability of the bar transverse motion equations of its transverse vibration were formulated. From the equations of motion, differential equations interrelating of the bar dynamic deflection with space and time were derived. The partial differential equations were solved using the method of separation of variables (Fourier's method). Then an ordinary differential equation (Hill's equation) describing the bar vibration was solved. An analysis of the solution became the basis for determining the regions of sandwich bar motion instability. Finally, the critical damping coefficient values at which parametric resonance occurs have been calculated.

Key words: sandwich bars, stability.

1. Introduction

Sandwich constructions are characterized by light weight and high strength. Such features are highly valuable in aviation, building engineering and automotive applications. The primary aim of using sandwich constructions is to obtain properly strong and rigid structures with vibration damping capacity and good insulating properties. Figure 1 shows a scheme of a sandwich construction which is composed of two thin facing plates and a relatively thick core [4, 5]. The core, made of plastic and metal sheet or foil, transfers transverse forces and maintains a constant distance between the plates. Sandwich constructions are classified into bars, plates and beams. A major problem in the design of sandwich constructions is the assessment of their stability under axial loads causing their buckling or folding. The existing methods of calculating such structures are limited to the assessment of their stability under loads constant in time [3, 5].

There are no studies dealing with the analysis of parametric vibration and dynamic stability (dynamic buckling). This paper presents a dynamic analysis of a sandwich bar compressed by a periodically variable force, assuming that the core is linearly viscoelastic. Differential equations describing the dynamic flexural buckling of bars are derived and regions of instability are identified. The dynamic analysis of sandwich constructions is of great importance for automotive vehicles and aeroplanes, since most of the loads which occur in them have the form of time-dependent forces.

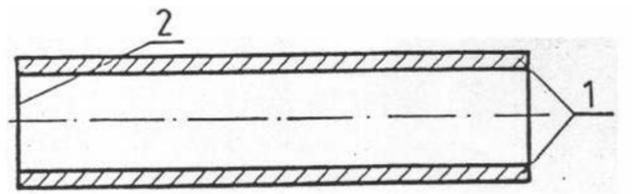

FIG. 1. Scheme of sandwich construction 1 – plates , 2 – core.

2. Dynamic buckling of a sandwich bar

A simply-supported sandwich bar compressed by time-dependent force F is shown in Fig. 2. Force F can be expressed as follows:

$$F = F_1 + F_2 \cos(pt), \tag{2.1}$$

where F_1 – constant component of the compressive force, F_2 – amplitude of the variable component of the compressive force, p – frequency of variable component F_2, t – time.

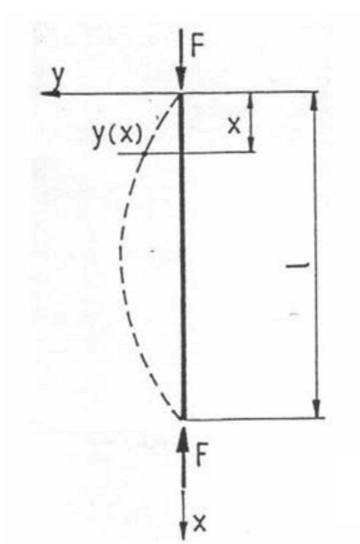

FIG. 2. Sandwich bar compressed by force F.

The cross-section of the sandwich bar is shown in Fig. 3. The basis for describing the dynamic buckling of a sandwich bar is the differential equation of sandwich beam centre line. The equation can be written as

$$B\frac{\partial^4 y}{\partial x^4} = q - k\frac{B}{S}\frac{\partial^2 q}{\partial x^2}, \tag{2.2}$$

where B – flexural rigidity of the bar, q – load intensity, k – a coefficient representing the influence of the transverse force on the deflection of the bar, S – transverse rigidity of the bar.

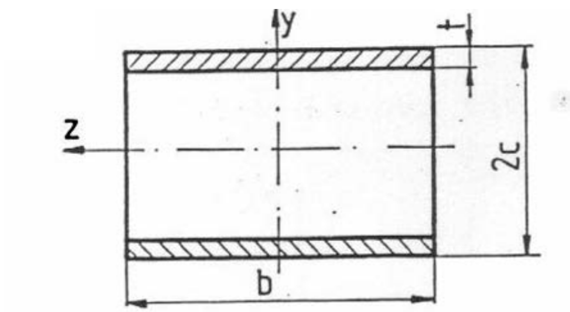

FIG. 3. Cross section of sandwich bar.

In sandwich constructions the core is merely sheared and does not transfer normal stresses whereby coefficient k is equal to one ($k = 1$).

$$S = 2bcG_c, \tag{2.3}$$

where b, c – dimensions of the core (Fig. 3), G_c – modulus of rigidity of the core material.

Load intensity q can be written in the form:

$$q = q_1 + q_2 + q_3, \tag{2.4}$$

$$q_1 = -F\frac{\partial^2 y}{\partial x^2}, \qquad q_2 = -\mu\frac{\partial^2 y}{\partial t^2}, \qquad q_3 = -\eta_r\frac{\partial y}{\partial t}, \tag{2.5}$$

where μ – unit mass of the sandwich bar, η_r – damping coefficient of the core material.

After substituting Eqs. (2.5) into differential Eq. (2.2), the following differential equation is obtained:

$$B\left(1 - \frac{F}{S}\right)\frac{\partial^4 y}{\partial x^4} + F\frac{\partial^2 y}{\partial x^2} - \frac{B}{S}\mu\frac{\partial^4 y}{\partial x^2 \partial t^2} + \mu\frac{\partial^2 y}{\partial t^2} + \eta_r\frac{\partial y}{\partial t} - \frac{B}{S}\eta_r\frac{\partial^3 y}{\partial x^2 \partial t} = 0. \tag{2.6}$$

The above equation is a fourth-order homogeneous equation with time-dependent coefficients. It was solved by the method of separation of variables (Fourier‘s method). The solution can be presented in the form of an infinite series:

$$y = \sum_{n=1}^{\infty} X_n\,(x) T_n\,(t)\,. \tag{2.7}$$

Eigenfunctions $X_n(x)$, satisfying the boundary conditions at the supports of the bar at its ends, have the following form:

$$X_n\,(x) = A_n \sin\left(\frac{\pi n x}{l}\right). \tag{2.8}$$

Having substituted Eqs. (2.7) and (2.8) into the differential Eq. (2.6), one gets the following ordinary differential equation describing functions $T_n(t)$:

$$\ddot{T}_n + 2h\dot{T}_n + \omega_{on}^2\,(1 - 2\psi_n \cos pt)\,T_n = 0, \tag{2.9}$$

where

$$2h = \frac{\eta_r}{\mu}, \qquad 2\psi_n = \frac{F_2\left(\dfrac{\pi n}{l}\right)^2}{\mu\omega_{on}^2}. \tag{2.10}$$

The square of frequency ω_{on} can be expressed as follows:

$$\omega_{on}^2 = \omega_o^2 - \frac{F_1\left(\dfrac{\pi n}{l}\right)^2}{\mu}, \tag{2.11}$$

where ω_o – natural frequency of vibration of the bar when $F_1 = 0$, $\eta_r = 0$.

The square of frequency ω_o can be expressed as follows:

$$\omega_o^2 = \frac{B\left(\dfrac{\pi n}{l}\right)^2}{\mu\left[1 + \frac{B}{S}\left(\dfrac{\pi n}{l}\right)^2\right]}. \tag{2.12}$$

Differential equation (2.9) is Hill’s equation in the form [1, 2]:

$$\ddot{T}_n + 2h\,\dot{T}_n + \Omega_n^2\,[1 - f\,(t)]\,T = 0. \tag{2.13}$$

If there is no damping in the core ($h = 0$) and assuming $f(t) = 2\psi$ cospt, one gets the following classical Mathieu equation:

$$\ddot{T}_n + \omega_{on}^2\,(1 - 2\psi_n \cos pt)\,T_n = 0. \tag{2.14}$$

In order to solve Eq. (2.13), a change of variable was made and the solution was expressed in the form:

$$T_n(t) = e^{-ht}\varphi_n(t). \tag{2.15}$$

In this way, a new differential equation for function $\phi_n(t)$ was obtained:

$$\ddot{\varphi}_n + \omega_n^2 [1 - f_1(t)]\varphi_n = 0, \tag{2.16}$$

where

$$\omega_n^2 = \Omega_n^2 - h^2, \tag{2.17}$$

$$f_1(t) = \frac{\Omega_n^2}{\omega_n^2} f(t). \tag{2.18}$$

Equation (2.16) is the Mathieu equation without damping. Therefore for the analysis of this equation one can use the solution of Eq. (2.14), substituting $f_1(t)$ for $f(t)$ and $\Omega_n^2 - h^2$ for ω_n^2.

Let us now analyze the stability of solutions of the differential equation (2.16), limiting the analysis to the first (most important) region of instability. First the Mathieu equation without damping, i.e. Eq. (2.14), should be solved. Here the author's original method has been used for this purpose. The solution of Eq. (2.14), containing the first region of instability, can be presented in the form:

$$T_n(t) = A(t)\cos\frac{pt}{2} + B(t)\sin\frac{pt}{2}, \tag{2.19}$$

where $A(t), B(t)$ – slowly variable functions of time t, satisfying the following conditions:

$$\ddot{A} \ll \dot{A} \ll A, \qquad \ddot{B} \ll \dot{B} \ll B. \tag{2.20}$$

Expression (2.19) was differentiated and substituted into (2.14), whereby the following differential equation describing function $A(t)$ has been obtained:

$$\ddot{A}(t) + \frac{1}{p^2}A(t)\left(\frac{1}{4}p^2 - \omega_{on}^2 - \psi_n\omega_{on}^2\right)\left(\frac{1}{4}p^2 - \omega_{on}^2 + \psi_n\omega_{on}^2\right) = 0. \tag{2.21}$$

The solution of the above equation has the form:

$$A(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{2.22}$$

where λ_1, λ_2 – the roots of the characteristic equation:

$$\lambda^2 + \frac{1}{p^2}\left(\frac{1}{4}p^2 - \omega_{on}^2 - \psi_n\omega_{on}^2\right)\left(\frac{1}{4}p^2 - \omega_{on}^2 + \psi_n\omega_{on}^2\right) = 0. \tag{2.23}$$

Unstable solutions are obtained when $\lambda^2 > 0$, i.e. when

$$\left(\frac{1}{4}p^2 - \omega_{on}^2 - \psi_n \omega_{on}^2\right)\left(-\frac{1}{4}p^2 + \omega_{on}^2 - \psi_n \omega_{on}^2\right) > 0. \tag{2.24}$$

Introducing

$$\left(\frac{\omega_{on}}{p}\right)^2 = z, \tag{2.25}$$

one obtains condition (2.24) in the following form:

$$\left(\frac{1}{4} - z - \psi_n z\right)\left(-\frac{1}{4} + z - \psi_n z\right) > 0. \tag{2.26}$$

The relevant quadratic equation which results from the above inequality has the following roots:

$$z_1 = \frac{1}{4\,(1+\psi_n)}, \qquad z_2 = \frac{1}{4\,(1-\psi_n)}. \tag{2.27}$$

The solution of inequality (2.26) has been presented graphically in Fig. 4 and expressed as:

$$z_1 < z < z_2. \tag{2.28}$$

When substitution (2.25) was taken into account, the following relations de scribing the boundary lines of the first region of instability have been obtained:

$$2\sqrt{1-\psi_n} < \frac{p}{\omega_{on}} < 2\sqrt{1+\psi_n}. \tag{2.29}$$

The above relations are identical with the generally known expressions which can be found, for example, in [1].

Relation (2.29) is illustrated graphically in Fig. 4. On the basis of the solution of differential Eq. (2.14), differential equation (2.13) was solved and reduced to Eq. (2.16). Finally, the following condition of instability was obtained:

$$\lambda^2 > h^2. \tag{2.30}$$

When solving the above inequality, the relation for λ^2 (formula (2.23)) was used, substituting $\Omega_n^2 - h^2$ for ω_{on}^2, and the following inequality has been obtained:

$$\frac{1}{p^2}\left[\frac{1}{4}p^2 - \left(\Omega_n^2 - h^2\right)\left(1 + \frac{\Omega_n^2}{\Omega_n^2 - h^2}\psi_n\right)\right] \cdot \left[-\frac{1}{4}p^2 - \left(\Omega_n^2 - h^2\right)\left(\frac{\Omega_n^2}{\Omega_n^2 - h^2}\psi_n - 1\right)\right] > h^2. \tag{2.31}$$

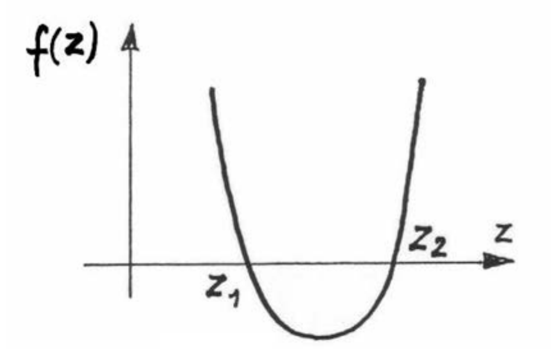

FIG. 4. Solutions z_1 and z_2 of inequality (2.28)

By solving the above inequality, the following condition for the occurrence of parametric resonance for a system with damping has been obtained:

$$\psi_n > 2\sqrt{\xi_n - 2\xi_n^2}, \tag{2.32}$$

where

$$\xi_n = \left(\frac{h}{\Omega_n}\right)^2.$$

Also the following condition for the relative damping coefficient ξ_n at which parametric resonance occurs was derived:

$$0 < \xi_n < \frac{1}{3}. \tag{2.33}$$

The roots of the appropriate quadratic equation which can be derived from inequality (2.32) were expressed as follows:

$$z_1 = \frac{1 - 3\xi_n - \sqrt{\psi_n^2 - 4\xi_n + 8\xi_n^2}}{4\left[(1-\xi_n)^2 - \psi_n^2\right]}, \tag{2.34}$$

$$z_2 = \frac{1 - 3\xi_n + \sqrt{\psi_n^2 - 4\xi_n + 8\xi_n^2}}{4\left[(1-\xi_n)^2 - \psi_n^2\right]}. \tag{2.35}$$

When $z_1 = z_2$, the coordinates of a 'wedge' of instability in system $\left(\psi_n, \dfrac{p}{\Omega_n}\right)$ are obtained. The equality of roots z_1 and z_2 occurs when the discriminant of inequality (2.31) is equal to zero.

After performing appropriate transformations, the following equation is obtained:

$$z = z_1 = z_2 = \frac{1}{4\,(1-\xi_n)}. \tag{2.36}$$

Substituting

$$z = \left(\frac{\Omega_n}{p}\right)^2, \tag{2.37}$$

the following result is obtained:

$$\frac{p}{\Omega_n} = 2\sqrt{1 - 3\xi_n}\ . \tag{2.38}$$

Hence the 'wedge' of the first region of instability has the coordinates:

$$\psi_n = 2\sqrt{\xi_n - 2\xi_n^2}\ , \qquad \frac{p}{\Omega_n} = 2\sqrt{1 - 3\xi_n}\ . \tag{2.39}$$

The boundary lines of the first region of instability are shown in Fig. 5. In a similar way as in the case without damping, the following relations for the boundary lines of the first region of instability are obtained:

$$\frac{p}{\Omega_n} < 2\sqrt{\frac{(1-\xi_n)^2 - \psi_n^2}{1 - 3\xi_n - \sqrt{\psi_n^2 - 4\xi_n + 8\xi_n^2}}}\ , \tag{2.40}$$

$$\frac{p}{\Omega_n} > 2\sqrt{\frac{(1-\xi_n)^2 - \psi_n^2}{1 - 3\xi_n + \sqrt{\psi_n^2 - 4\xi_n + 8\xi_n^2}}}\ . \tag{2.41}$$

Relations (2.40) and (2.41) describe the upper and lower boundary line, respectively.

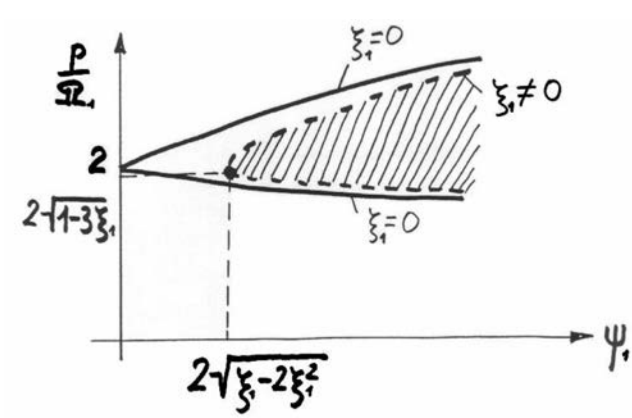

FIG. 5. First region of instability ($\xi_n = 0$, without damping; $\xi_n \neq 0$, with damping)

3. Example of calculations

This section presents calculations of a simply-supported sandwich bar compressed by variable axial force F. The bar dimensions: $b = 25 \cdot 10^{-3}$ m, $c = 7.5 \cdot 10^{-3}$ m, $t = 0.5 \cdot 10^{-3}$ m, $l = 0.5$ m and the following data describing the physical properties of the plate and the core material were assumed.:

$$E_t = 68.67 \cdot 10^3 \text{ MPa}, \qquad G_c = 6.867 \cdot 10^3 \text{ MPa}, \qquad \mu = 14.5 \cdot 10^{-2} \frac{\text{Ns}^2}{\text{m}^2}.$$

The flexural rigidity of the plates is expressed by

$$B = E_t \cdot I = 96.6 \text{ Nm}^2,$$

where I – moment of inertia of the plates' cross-section with respect to the z axis (Fig. 3):

$$I = 2btc^2 = 14062.5 \cdot 10^{-13} \text{ m}^4,$$

The transverse rigidity of the core was calculated from relation (2.3):

$$S = 2bcG_c = 25751 \cdot 10^3 \text{ N}.$$

From relation (2.11), after equating ω_o to zero, the first critical force F_{1kr} was calculated:

$$F_{1kr} = 3809 \text{ N}.$$

It was assumed that constant compressive force component $F_1 = 2000$ N and the variable component amplitude $F_2 = 900$ N. From relations (2.11) and (2.12), ω_o and ω_{on} were calculated:

$$\omega_o = 1018 \text{ s}^{-1}, \quad \omega_{on} = 701.5 \text{ s}^{-1}. \quad (\Omega_n)$$

The corresponding damping coefficient $\xi_n = 0.01$ was assumed. From relation (2.10)

$$\psi_n = 0.25$$

was calculated.

From formula (2.40), the boundary value of coefficient ψ_n at which the parametric resonance occurs was calculated:

$$\psi_{ngr} = 0.198.$$

If $\Psi_n < \Psi_{\text{ngr}}$, no parametric resonance arises. It follows from the above that there exist compressive force components F_1 and F_2 at which the bar does not lose stability.

Assuming that there is no damping ($\zeta_n = 0$), from relation (2.30) the frequencies of the exciting force at which the bar loses stability are calculated:

$$p' = 1567 \text{ s}^{-1}, \qquad p'' = 1215 \text{ s}^{-1}, \qquad (\varDelta p = 352 \text{ s}^{-1}),$$

where p' and p'' are the frequencies corresponding to respectively the upper and lower boundary line.

For damping described by coefficient $\zeta_n = 0.01$ the following values were obtained from relations (2.41) and (2.42):

$$p' = 1487 \text{ s}^{-1}, \qquad p'' = 1270 \text{ s}^{-1}, \qquad (\varDelta p = 217 \text{ s}^{-1}).$$

The above calculations show that the frequency range $\varDelta p$ in which instability of the bar occurs is smaller when damping is present. Then the calculations for $F_1 = 2000$ N and $F_2 = 1500$ N were performed. From formula (2.10)

$$\psi_n = 0.4$$

was obtained

The following boundary frequency values were obtained:

$$p' = 1660 \text{ s}^{-1}, \qquad p'' = 1086 \text{ s}^{-1} \qquad (\text{if } \zeta_n = 0),$$

$$p' = 1606 \text{ s}^{-1}, \qquad p'' = 1108 \text{ s}^{-1} \qquad (\text{if } \zeta_n = 0.01).$$

The following frequency ranges Δp were obtained:

$$\varDelta p = 574 \text{ s}^{-1} \qquad \text{for } \xi_n = 0,$$

$$\varDelta p = 498 \text{ s}^{-1} \qquad \text{for } \xi_n = 0.01.$$

4. Conclusions

If the sandwich bar is compressed by a time-dependent force, there are several ranges of the frequency of its variation in which the bar loses stability.

Damping reduces the force variation frequency range in which instability of the sandwich bar occurs.

Owing to damping, there are certain values of compressive force components (F_1, F_2) at which the sandwich bar does not lose stability.

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