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THREE-DIMENSIONAL STABILITY PROBLEMS OF COMPOSITE MATERIALS AND COMPOSITE CONSTRUCTION COMPONENTS

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Thtee-dimensional static and dynamic linearized problems of stability were posed both for compressible and incompressible bodies. In the case of homogeneous sub-critical deformations the general solutions to these problems were constructed. Corresponding variational principles were formulated and their validity was proved. In the framework of the three-dimensional linearized theory, plane and spatial problems of instability of deformation of layered and fibre materials both at small and at large elastic deformations were studied. The elastic stability of bars, plates and rolled shells made of composite materials was studied and the regions of applicability of classical and improved theories applied here were determined.

Introduction

The problem of stability in mechanics of deformable bodies is a typical problem for thin-walled structural components of traditional materials and thus, in most cases, it has been posed in terms of one- and two-dimensional theories of bars, plates and shells, basing on the hypotheses of Euler, Bernoulli, Kirchhoff and Laval. These studies yielded many important results used later for designing different types of reinforcement and structures.

In various branches of modern technology, the wide application of reinforced materials, with their specific properties (an essential anisotropy in their deformability, low displacement resistance etc.), required universally valid approaches to the stability calculations for structural components made of new composite materials. For these materials limits of application of classical theories should be especially found. Currently, three main directions can be outlined in the studies of stability of composite structural components.

The first direction is related to studies carried out within the framework of classical applied theories of bars, plates and shells. The foundations of applied stability theories, methods of study and solutions to particular problems and also an analysis of stability studies of elastic and inelastic systems are given in the well--known monographies of S. P. Timoshenko, A. N. Dinnik, A. R. Ržhanicyn, V. V. Bolotin, A. S. Volmir, H. M. Muštari and K. Z. Galimov, A. F. Smirnov, P. I. Ogibalov and also in review papers by V. V. Bolotin and E. I. Grigoluk, E. I. Grigoluk, E. I. Grigoluk and V. V. Kabanov, V. Hatchinson and V. Koiter, I. I. Vorovič and N. I. Minakova and others.

The second direction includes studies based on more accurate applied theories (of the type of S. P. Timoshenko, E. Reissner, S. A. Ambartsumian and others) of bars, plates and shells. These theories are formed by introducing hypothesse, which are less restrictive than the classical ones, or by using other means of reducing three-dimensional problems to two-dimensional ones.

The third direction involes works which pose three-dimensional problems without using any hypotheses. Such an approach makes it possible to solve problems with an essentially three-dimensional stress state (problems of mathematical tectonics in rock mechanics, theory of surface phenomena and others), problems of mechanics of polymer and reinforced materials and to calculate the structural components of these materials. This approach provides also the possibility of error estimation and the determination of ranges of applicability of the theories as dependent on physical-mechanical characteristics of the composite materials. In its application to stability investigations of composites and composite structural components (anisotropic bodies) such an approach was developed in recent years in the USSR mainly at the Institute of Mechanics of the Academy of Sciences of the Ukrainian SSR.

The present paper presents the results of a study of the elastic stability of composites and composite structural components in the framework of a three-dimensional linearized theory, including a study of certain general problems of the three-dimensional theory of elastic stability [5, 6, 16–20, 26, 29, 32, 33, 36, 41, 44], the elucidation of possible mechanisms of loss of stability in the structure of layered and fibre materials [8, 9, 10, 13, 17, 21, 22, 30, 34, 36, 38, 40, 44] and stability studies of bars [14, 36, 44], plates [7, 11, 12, 29, 36, 44, 47, 48] and shells [2, 3, 7, 15, 23, 24, 25, 36, 44, 45, 46, 50] made of composites, with the determination of ranges of applicability of thə theories included. The results given in what follows have been essentially obtained by the authors.

1. Fundamental relations

We consider linearized problems and give fundamental relations for generalized stress σ^{*ij} (or the non-symmetric Kirchhoff stress tensor t^{in}) and components of the Green deformation tensor $2\varepsilon_{ij}$.

Compressible bodies

The linearized equations of motion and boundary conditions in Lagrangian coordinates, which before deformation coincide with curvilinear coordinates θ_i ($x_m = x_m(\theta_1, \theta_2, \theta_3)$) in the case of finite initial deformations, are of the form

$$\nabla_i \left[\sigma^{*in} (\delta_n^m + \nabla_n u_0^m) + \sigma_0^{*in} \nabla_n u^m\right] + X^{*m} = 0, \tag{1.1}$$

and

$$\left[\sigma^{*in} (\delta_n^m + \nabla_n u_0^m) + \sigma_0^{*in} \nabla_n u^m\right] N_i|_{S_1} = P^{*m}, \tag{1.2}$$

respectively.

The boundary conditions in displacements on a part of the surface S_2 are

$$u_n|_{S_2}=0. \tag{1.3}$$

The boundary conditions in the case of dynamic boundary problems are

$$u_m|_{t=0}=0, \quad u_m|_{t=T}=0. \tag{1.4}$$

The initial conditions in the case of mixed dynamic problems are

$$u_m|_{t=0}=0, \quad \dot{u}_m|_{t=0}=0. \tag{1.5}$$

Note that inertial forces are not isolated from body forces in the relations (1.1) and (1.2) and below. The relations for the components of the deformation tensor are as follows:

$$2\varepsilon_{ij}=(\delta_j^m+\nabla_j u_0^m)\nabla_i u_m+(\delta_i^m+\nabla_i u_0^m)\nabla_j u_m. \tag{1.6}$$

In terms of the Kirchhoff stress tensor the relations (1.1) and (1.2) take the following form:

$$\nabla_i t^{im}+X^{*m}=0, \tag{1.7}$$

$$t^{im}N_i|_{S_1}=P^{*m}, \tag{1.8}$$

where

$$t^{mn}=\sigma^{*mp}(\delta_p^n+\nabla_p u_0^n)+\sigma_0^{*mp}\nabla_p u^n. \tag{1.9}$$

The linearized state relations for the generalized stress tensor σ^{*in} and for the Kirchhoff stress tensor t^{im} in the case of a hyper-elastic medium can be written in the following form:

$$\sigma^{*in}=\lambda^{in\alpha\beta}\nabla_\beta u_\alpha, \tag{1.10}$$

$$t^{im}=\omega^{im\alpha\beta}\nabla_\beta u_\alpha, \tag{1.11}$$

where

$$\lambda^{in\alpha\beta}=\frac{1}{4}(\delta_i^\alpha+\nabla_i u_0^\alpha)\left(\frac{\partial}{\partial\varepsilon_{i\beta}^0}+\frac{\partial}{\partial\varepsilon_{\beta i}^0}\right)\left(\frac{\partial}{\partial\varepsilon_{in}^0}+\frac{\partial}{\partial\varepsilon_{in}^0}\right)\Phi^0, \tag{1.12}$$

$$\omega^{im\alpha\beta}=(\delta_n^m+\nabla_n u_0^m)\lambda^{in\alpha\beta}+\frac{1}{2}g^{m\alpha}\left(\frac{\partial}{\partial\varepsilon_{i\beta}^0}+\frac{\partial}{\partial\varepsilon_{\beta i}^0}\right)\Phi^0. \tag{1.13}$$

Taking into account the relations (1.9) and (1.11), the Eqs. (1.1) and boundary conditions (1.2) can be written as

$$\nabla_i(\omega^{im\alpha\beta}\nabla_\beta u_\alpha)+X^{*m}=0, \tag{1.14}$$

$$(\omega^{im\alpha\beta}\nabla_\beta u_\alpha)N_i|_{S_1}=P^{*m}. \tag{1.15}$$

Assuming the extensions and shearing strains to be small as compared to unity (changes in areas and volumes are neglected), we obtain the basic relations for the first version of the theory of small subcritical deformations. All the linearized relations and equations remain valid if we dispose of the index and neglect changes in size of the body before and after the deformation. If, however, elongations and shears are small as compared to unity and the initial deformed state is determined by means of the geometrical linear theory, then we get the second version of the theory of small subcritical deformations. In that case, for components of the deformations tensor we have

$$2\varepsilon_{ij}^{0}=\nabla_j u_i^0+\nabla_i u_j^0, \quad 2\varepsilon_{ij}=\nabla_j u_i+\nabla_i u_j \tag{1.16}$$

and all the preceding relations remain valid if we take into account

$$\sigma^{*ij}\approx\sigma^{ij}, \quad X^{*m}\approx X^m, \quad P^{*m}\approx P^m, \quad \delta_i^j+\nabla_i u_0^j u\approx\delta_i^j. \tag{1.17}$$

For example, the linearized equations of motion (1.1) and boundary conditions (1.2) have the form

$$\nabla_i(\sigma^{im}+\sigma_0^{in}\nabla_n u^m)+X^m=0, \tag{1.18}$$

$$(\sigma^{im}+\sigma_0^{in}\nabla_n u^m)N_i|_{S_1}=P^m, \tag{1.19}$$

where σ^{ij} — components of the tensor of real stress.

Incompressible bodies

The linearized equations of motion, incompressibility conditions and boundary conditions for stress can be represented in the following form:

$$\nabla_i[\varkappa^{im\alpha\beta}\nabla_\beta u_\alpha+G_0^{in}(\delta_n^m+\nabla_n u_0^m)p]+X^{*m}=0, \tag{1.20}$$

$$G_0^{ij}(\delta_j^n+\nabla_j u_0^n)\nabla_i u_n=0, \tag{1.21}$$

$$[\varkappa^{im\alpha\beta}\nabla_\beta u_\alpha+G_0^{in}(\delta_n^m+\nabla_n u_0^m)p]N_i|_{S_1}=P^{*m}. \tag{1.22}$$

The linearized state relations for the symmetric tensor of generalized stress σ^{*in} and the non-symmetric Kirchhoff stress tensor t^{im} are written in the form

$$\sigma^{*in}=\mu^{in\alpha\beta}\nabla_\beta u_\alpha+pG_0^{in}, \tag{1.23}$$

$$t^{im}=\varkappa^{im\alpha\beta}\nabla_\beta u_\alpha+G_0^{in}(\delta_n^m+\nabla_n u_0^m)p, \tag{1.24}$$

where, for a hyper-elastic body, the following holds:

$$\mu^{in\alpha\beta}=\lambda^{in\alpha\beta}-p^0(G_0^{i\beta}G_0^{tn}+G_0^{it}G_0^{\beta n}), \tag{1.25}$$

$$\varkappa^{im\alpha\beta}=(\delta_n^m+\nabla_n u_0^m)\mu^{in\alpha\beta}+g^{\alpha m}\left[\frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{i\beta}^0}+\frac{\partial}{\partial\varepsilon_{\beta i}^0}\right)\Phi^0+pG_0^{i\beta}\right]. \tag{1.26}$$

It is necessary to note that in the case of an incompressible body, arguments of the elastic potential Φ^0, in view of the incompressibility condition, follows the relation

$$\det\|\delta_r^s+2\varepsilon_r^{s0}\|=I_3^0=1. \tag{1.27}$$

For incompressible bodies the relations (1.20)–(1.26) can be written also for the case of small subcritical deformations. For example, for the first version of the theory of small subcritical deformations the linearized condition of incompressibility takes the form

$$g^{im}(\delta_i^n+\nabla_i u_0^n)\nabla_m u_n=0. \tag{1.28}$$

We note that in the case of an incompressible body the relations (1.28) and other relations of the first version on the theory of small subcritical deformations do not follow directly from the relations (1.20)–(1.26). This is so since in the case of small deformations the change of volume is determined by the first algebraic invariant, and for finite deformations by the third invariant of the Green deformation tensor.

Thus the relations given above exhaust the versions of posing dynamic and static linearized problems of the theory of finite and small subcritical deformations.

2. Variational principles

In formulating variational principles we shall assume that perturbations of volume and surface forces do not depend on perturbations of displacement and the functions under variation are continuously differentiable as many times as required. We consider the variational principles of the Chu-Washizu type for static linearized problems and we use the formulation of the latter in terms of the non-symmetric Kirchhoff stress tensor. In that case, for the theory of finite subcritical deformations, static problems are reduced to the relations (1.7)–(1.9) and (1.3), and for an incompressible body — to relations (1.20)–(1.22) and (1.3). We introduce the following functionals:

$$I_1(t,v,u)=\int_V\left[\frac{1}{2}\omega^{im\alpha\beta}v_{\alpha\beta},v_{mi}-t^{im}(v_{mi}-\nabla_i u_m)-X^{*m}u_m\right]dV-\\-\int_{S_1}P^{*m}u_m\,ds-\int_{S_2}N_i t^{im}u_m\,ds; \tag{2.1}$$

$$I_2(t,v,u,p)=\int_V\left[\frac{1}{2}\kappa^{im\alpha\beta}v_{\alpha\beta}v_{mi}-t^{im}(v_{mi}-\nabla_i u_m)+\right.\\\left.+pG_0^{ln}(\delta_n^m+\nabla_n u_0^m)v_{mi}-X^{*m}u_m\right]dv-\int_{S_1}P^{*m}u_m\,ds-\int_{S_2}N_i t^{im}u_m\,ds; \tag{2.2}$$

where

$$v_{\alpha\beta}=\nabla_\beta u_\alpha. \tag{2.3}$$

If the requirements listed at the beginning of this paragraph are fulfilled, then from the stationarity condition of the functional (2.1) (t^{im}, v_{im}, u_m are subject to variation) we obtain the relations (1.7), (2.3), (1.8), (1.11) and (1.3). In the same way, from the stationarity condition of the functional (2.2) (t^{im}, v_{im}, u_m, p are subject to variation) the relations (1.20), (1.21), (1.22), (2.3), (1.24) and (1.3) follow.

Thus it is characteristic of the formulated variational principles that all the relations of linearized problems including the boundary conditions for displacements can be obtained from the stationarity condition of respective functionals. In the same way, by introducing suitable functionals, variational principles can be formulated for various versions of the theory of small subcritical deformations.

3. General solutions of static linearized problems in the case of uniform subcritical deformations

In what follows we assume that the θ_i coordinates coincide with the rectangular Cartesian coordinates x_i. Below, all the considerations are carried out in Lagrange coordinates x_i of the non-deformed body.

Linearized relations between stress and derivatives of displacements for a transversally isotropic compressible body in the case of a uniform initial state

$$\begin{aligned} &u_m^0=\delta_{im}(\lambda_i-1)x_i, \quad \sigma_{ij}^{*0}=\delta_{ij}p_j^{*0}, \quad \sigma_{ij}^0=\delta_{ij}p_j, \quad \lambda_1=\lambda_2=\lambda, \\ &\lambda_i=\text{const}, \quad p_1^*=p_2^*, \quad p_1=p_2 \end{aligned} \tag{3.1}$$

can be written in the following form:

For a compressible body with finite initial deformations

$$\sigma_{ij}^*=\delta_{ij}a_{ik}\lambda_k u_{k,k}+(1-\delta_{ij})G_{ij}(\lambda_i u_{i,j}+\lambda_j u_{j,i}). \tag{3.2}$$

For the first version of the theory of small initial deformations it is necessary to take into account the fact that $\sigma_{ij}^*\approx\sigma_{ij}$.

We note that for finite initial deformations and the first version of small initial deformations the linearized relations (3.2) do not coincide with the relations for a linear transversally-isotropic body

$$\begin{aligned} &a_{11}\equiv a_{22}, \quad a_{11}-a_{12}\equiv 2G_{12}, \quad a_{13}\equiv a_{23}, \quad G_{23}\equiv G_{13}, \\ &a_{ij}\equiv a_{ji}, \quad G_{ij}\equiv G_{ji}, \quad \lambda_j a_{ij}\neq\lambda_i a_{ji}, \end{aligned} \tag{3.3}$$

For the second version of equations of the theory of small initial deformations the relations between the stress and derivatives of displacements are of form

$$\sigma_{ij}=\delta_{ij}a_{ik}u_{k,k}+(1-\delta_{ij})G_{ij}(u_{i,j}+u_{j,i}) \tag{3.4}$$

and coincide with the relations for a linear elastic anisotropic body.

We consider now the construction of general solutions of equations for finite and various versions of small initial deformations in the case of static problems. Solutions to equations of the type (1.1), (1.18) in the case of a uniform initial state, (3.1) for a cylindrical transversally isotropic body whose axis of isotropy coincides with the axis Ox_3 and with a curvilinear cross-section, can be represented in the form

$$\begin{aligned} &u_n=\frac{\partial}{\partial s}\Psi-\frac{\partial^2}{\partial n\,\partial x_3}\chi, \quad u_s=-\frac{\partial}{\partial n}\Psi-\frac{\partial^2}{\partial s\,\partial x_3}\chi, \\ &u_3=A\left(\Delta+B\frac{\partial^2}{\partial x_3^2}\right)\chi, \quad \Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}, \end{aligned} \tag{3.5}$$

where n and s are a normal and a tangent to the contour, respectively, and the functions Ψ and χ fulfill the equations

$$(3.6)\qquad \begin{gathered}\left(\Delta+\zeta_1^2\frac{\partial^2}{\partial x_3^2}\right)\Psi=0,\\ \left[\Delta^2+(\zeta_2^2+\zeta_3^2)\,\Delta\frac{\partial^2}{\partial x_3^2}+\zeta_2^2\,\zeta_3^2\,\frac{\partial^4}{\partial x_3^4}\right]\chi=0.\end{gathered}$$

The solution (3.5) and (3.6) are general solutions for finite initial deformations and different variants of small initial deformations. We write down the expressions for the constants A, B and ζ^2 for all the versions of the problems.

For finite initial deformations

$$A=\frac{\lambda_1}{\lambda_3}\,\frac{a_{11}+p_1^*\lambda^{-2}}{a_{13}+G_{13}},\qquad B=\frac{G_{13}+p_3^*\lambda^{-2}}{a_{11}+p_1^*\lambda^{-2}},\qquad \zeta_1^2=\frac{G_{13}+p_3^*\lambda^{-2}}{G_{12}+p_1^*\lambda^{-2}},$$

$$(3.7)\quad \zeta_{2,3}^2=C\pm\left[C^2-\frac{(a_{33}+p_3^*\lambda_3^{-2})\,(G_{13}+p_3^*\lambda^{-2})}{(a_{11}+p_1^*\lambda^{-2})\,(G_{13}+p_1^*\lambda_3^{-2})}\right]^{\frac{1}{2}},$$

$$2C=\frac{(a_{11}+p_1^*\lambda^{-2})\,(a_{33}+p_3^*\lambda_2^{-2})+(G_{13}+p_1^*\lambda_2^{-2})\,(G_{13}+p_3^*\lambda^{-2})-(a_{13}+G_{13})^2}{(a_{11}+p_1^*\lambda^{-2})\,(G_{13}+p_1^*\lambda_2^{-2})}.$$

For the first version of the theory of small initial deformations in the formulae (3.7), it should be that $p_i\approx p_i$.

For the second version of the theory of small initial deformations in the expressions (3.7), it should be assumed that $p_i^*\simeq p_i$, $\lambda_3=\lambda=1$.

It is necessary to note that constants a_{ik}, G_{ij} entering the relations (3.2)–(3.7) have different form for different versions of the theory. The general solutions constructed here enable one to obtain, for different versions of the theory, characteristic determinants of a large group of stability problems of reinforced materials and structural components. Those characteristic determinants can be built for an arbitrary form of an elastic potential and it is only for numerical calculations that it is necessary to set a specific form of the potential for calculating the constants a_{ik} and G_{ij}.

For an incompressible transversally-isotropic body, the linearized equations of state for the case of finite initial deformations can be represented in the form

$$(3.8)\qquad \sigma_{ij}^*=\delta_{in}\,a_{ik}\,u_{k,k}+(1-\delta_{in})\,\mu_{in}\,(\lambda_i\,u_{i,n}+\lambda_n\,u_{n,i})+\delta_{in}\,\lambda_2^{-2}\,p.$$

The general solutions of the static equations (1.20) in the case of a uniform initial state have the form

$$(3.9)\qquad \begin{gathered} u_n=\frac{\partial}{\partial s}\Psi-\frac{\partial^2}{\partial n\,\partial x_3}\chi,\qquad u_s=-\frac{\partial}{\partial n}\Psi-\frac{\partial^2}{\partial n\,\partial x_3}\chi,\\ u_3=\lambda^{\frac{3}{2}}\Delta\chi,\qquad p=\lambda^{-1}\left[\left(2\mu_{12}\lambda^{-\frac{1}{2}}-a_{13}\lambda^{\frac{5}{2}}+a_{12}\lambda^{-\frac{1}{2}}-\right.\right.\\ \left.\left.-\mu_{13}\lambda^{\frac{5}{2}}+p_1^*\lambda^{\frac{1}{2}}\right)\Delta+\left(\mu_{13}\lambda^{-\frac{1}{2}}+p_3^*\lambda^{\frac{1}{2}}\right)\frac{\partial^2}{\partial x_3^2}\right]\frac{\partial}{\partial x_3}\chi,\\ \lambda_1=\lambda_2=\lambda^{-\frac{1}{2}},\qquad \lambda_3=\lambda,\qquad \sigma_{ij}^{*0}=\delta_{ij}\,p_j^*,\qquad p_1^*=p_2^*,\end{gathered}$$

where Ψ and χ satisfy Eqs. (3.6), and the quantities ζ^2 can be determined by means of the formulae

$$\zeta_1^2 = \frac{\mu_{13}\,\lambda^{-1} + p_3^*}{\mu_{12}\,\lambda^{-1} + p_1^*}, \qquad \zeta_{2,3}^2 = C \pm \left[C^2 - \frac{\lambda^{-2}\,(\lambda^{-\frac{1}{2}}\,\mu_{13} + p_3^*\,\lambda^{\frac{1}{2}})}{\lambda^{\frac{3}{2}}\,(\lambda^2\,\mu_{13} + p_1^*)} \right]^{\frac{1}{2}},$$

$$2C = \{[\lambda^2\,a_{33} - \lambda^{-1}\,(a_{13} + \mu_{13}) + p_3^*]\,\lambda^{\frac{3}{2}} - [\lambda^{\frac{3}{2}}\,(a_{13} + \mu_{13}) -$$

$$- \lambda^{-\frac{5}{2}}\,(a_{12} + 2\mu_{12}) - \lambda^{-\frac{3}{2}}\,p_1^*]\}\,[2\lambda^{\frac{3}{2}}\,(\mu_{13}\,\lambda^2 + p_1^*)]^{-1}.$$

The constants a_{ik} and μ_{in} can be calculated from the respective relations; the latter depend on the form of the elastic potential.

4. Stability of reinforced layered and fibrous materials

Layered materials

Basing on the general solutions given above, plane and three-dimensional problems of stability of deformation of layered materials at small and highly elastic subcritical deformations are considered, i.e. the loss of stability of structure of the material is studied for the case when the wave-length of the form of loss of stability is not determined by the length of the sample or the shape of the structure component, but by the relations between the geometric and mechanical characteristics of the layers. The reinforced materials are assumed to be piece-wise homogeneous, that is the problems are posed as exact ones. The material is composed of subsequent layers of the bond and the filler. In the case of small subcritical deformations the materials of the layers are assumed to be orthotropic, and for finite initial deformations problems with transversally isotropic layers are considered. In order to solve the considered problems, it is necessary to construct such solutions of the equations (3.6) and (3.7) for a compressible body or of the equations (3.6) and (3.10) for an incompressible one, which satisfy suitable periodicity and symmetry conditions accordingly to the type of stability loss (bending, symmetric and others). Then, the continuity conditions of the vectors of stress and displacement at the line separating the layers

$$(4.1)\quad P_{1i}^*|_{x_{3i}=-h} = P_{1i}^{*(1)}|_{x_{3i}^{(1)}=h^{(1)}},\ P_{2i}^*|_{x_{3i}=-h} = P_{2i}^{*(1)}|_{x_{3i}^{(1)}=h^{(1)}},\ P_{3i}^{*(1)}|_{x_{3i}=-h} = P_{3i}^{x(1)}|_{x_{3i}^{(1)}=h^{(1)}},$$

$$u_{1i}|_{x_{3i}=-h} = u_{1i}^{(1)}|_{x_{3i}^{(1)}=-h^{(1)}}, \qquad u_{2i}|_{x_{3i}=-h} = u_{\prime}^{(1)}|_{x_{3i}^{(1)}=h^{(1)}}, \qquad u_{3i}|_{x_{3i}=h} = u_{3i}^{(1)}|_{x_{3i}^{(1)}=h^{(1)}},$$

result in a homogeneous system of equations which, through related conditions of the existence of non-trivial solutions, yield characteristic equations determining the critical loadings. All the quantities related to layers of filler are marked by the index "1".

As for particular cases we obtain and study characteristic equations for the following problems: stability of a strip (layer) bounded by two half-planes (half-spaces); stability of the interface, between a compressible and an incompressible medium, in the form of half-planes (half-spaces); surface instability of a half-plane (half-space) of a compressible and an incompressible material.

It has been found that an internal loss of stability does not occur at each grouping of components in a composite. It does occur when the less rigid layers of the bond are substantially thicker than the layers of the filler (at low concentrations of the filler). The critical values of the parameters of wave excitation when there is loss of stability in the structure of the material at $h/h^{(1)} > 10$ can be determined with sufficient precision by replacing layers of the filler with half-planes (half-spaces). In

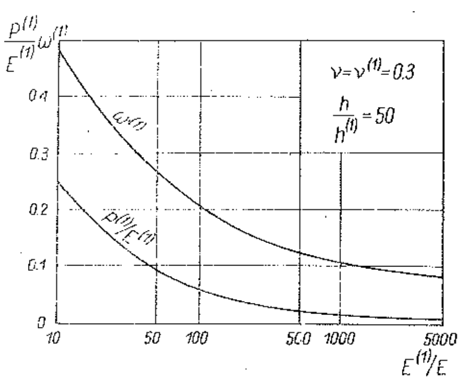

FIG. 1.

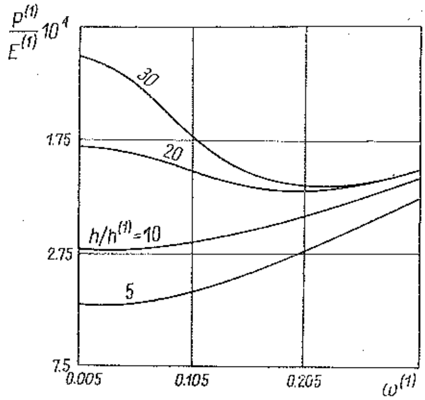

FIG. 2.

Fig. 1 we display value of the minimum quantities $\frac{p^{(1)}}{E^{(1)}}$ $(\sigma_{11i}^{0} = -p;\ \sigma_{1i}^{0(1)} = -p^{(1)})$ and the corresponding quantities $\omega^{(1)}$ as dependent on the parameter $E^{(1)}/E$ for isotropic layers. In Fig. 2 the dependence of the quantities $p^{(1)}/E^{(1)}$ on the parameter of wave-excitation $\omega^{(1)}$ is displayed for the bending form of stability loss for orthotropic layers.

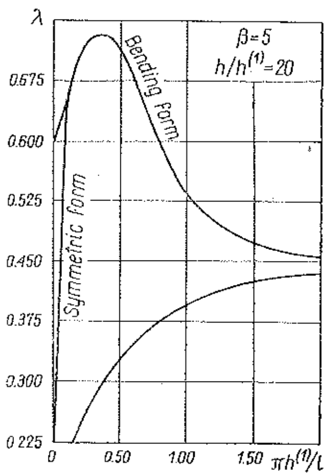

FIG. 3.

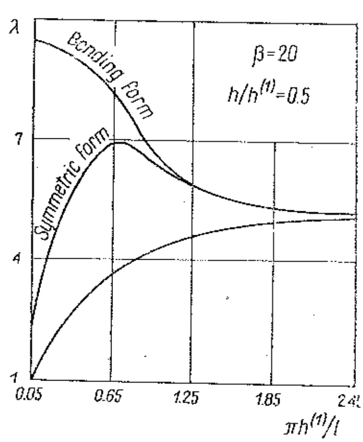

FIG. 4.

It is necessary to note that all the numerical results obtained here, including those displayed in Figs. 1 and 2, are valid for the following problems in the case of small deformations; the three-dimensional non-axially-symmetric problem $\omega^{(1)}=\sqrt{\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}}\,h^{(1)}$ (transversally isotropic layers); the three-dimensional axially-symmetric problem $\omega^{(1)}=(h^{(1)}/R)\,\kappa_k\,J_0'(\kappa_k)=0$ and the plane problem (orthotropic layers) $\omega^{(1)}=\pi h^{(1)}/l$.

In Figs. 3 and 4 the relations are displayed between the parameters of the critical shortening and the parameter of wave excitation ($\omega^{(1)}=\pi h^{(1)}/l$ — for a plane problem, $\omega^{(1)}=\kappa_k(h^{(1)}/R)$ for a three-dimensional axially-symmetric problem) in the case of finite initial deformations (the elastic potential was chosen in the form of Treloar and Muni).

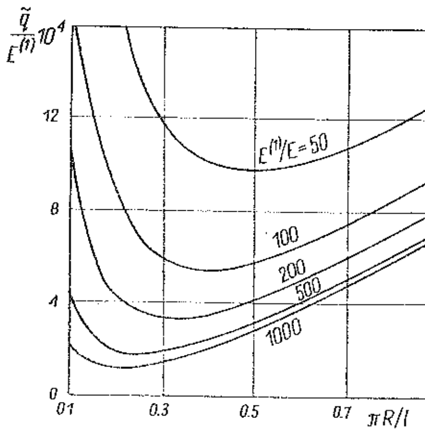

FIG. 5.

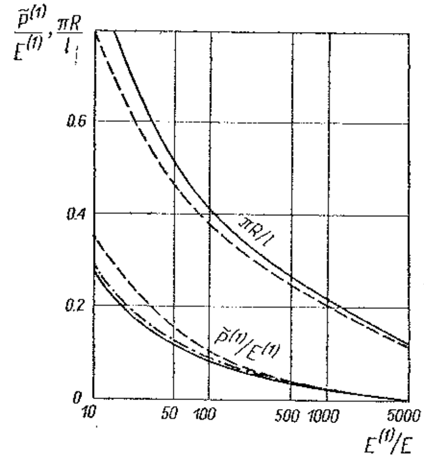

FIG. 6.

Fibrous materials

We consider the stability loss of a hollow (continuous) fibre having a circular cross-section, placed in an infinite elastic medium, under compression of intensity $\bar{q}$ acting along the fibre. For a composite with a low volume-percentage of fibres the assumed model is fully valid and the results obtained below present first approximation. These results describe completely enough the mechanisms of stability loss of a fibre material when the concentration of the filler is small and the wavelength of the form of the instability loss is substantially smaller than the distance between the fibres.

Assuming that a full coupling between a fibre and the filler is achieved on the cylindrical interfaces between the fibre and the filler and on the free internal surface of the fibre, we set the following conditions in terms of strain and displacements:

$$\text{(4.2)}\qquad \begin{gathered} P_r|_{r=R}=P_r^{(1)}|_{r=R}, \quad P_\theta|_{r=R}=P_\theta^{(1)}|_{r=R}, \quad P_3|_{r=R}=P_3^{(1)}|_{r=R}, \\ u_r|_{r=R}=u_r^{(1)}|_{r=R}, \quad u_\theta|_{r=R}=u_\theta^{(1)}|_{r=R}, \quad u_3|_{r=R}=u_3^{(1)}|_{r=R}, \\ P_r^{(1)}|_{r=R}=0, \quad P_\theta^{(1)}|_{r=R}=0, \quad P_3^{(1)}|_{r=R}=0. \end{gathered}$$

In Fig. 5 the dependence of the quantity $\frac{\tilde{q}}{E^{(1)}} \cdot 10^4$ on the wave excitation parameter $\pi R/l$ is displayed for various ratios $E^{(1)}/E$ at $S^{(1)}=0.05$ (S and $S^{(1)}$ — concentrations of the bond and the filler respectively) and $\nu=\nu^{(1)}=0.3$ in the case an isotropic filler and continuous fibre. In Fig. 6 the dependence of the value of critical deformation $\tilde{p}^{(1)}/E^{(1)}$ ($\sigma_{33}^{0(1)}=-\tilde{p}^{(1)}$) and of the wave excitation parameter $\pi R/l$ on the ratio $E^{(1)}/E$ is given for $\nu=\nu^{(1)}=0.4$ (a logarithmic scale is assumed for the abscissa). The curves drawn in dashed lines were taken from the paper by M. A. SADOVSKY, S. L. PU, M. A. HUSSAIN (*Buckling of microfibres*, J. Appl. Mech., **34**, 4, 1967). For the dash-point curve the value $\nu=\nu^{(1)}=0.3$ was assumed.

Figure 7 shows in a logarithmic scale the relations between the wavelength of buckling waves and the diameter of a fibres which is situated in an elastic matrix and is subjected to compression at different ratios $E^{(1)}/E$ and $\nu=\nu^{(1)}=0.4$. Also, experimental points are indicated there for various diameters of the glass fibre. Those points were obtained at the setting of the epoxy matrix. The experimental data was obtained from works of the Institute of Mechanics of the Ukrainian SSR

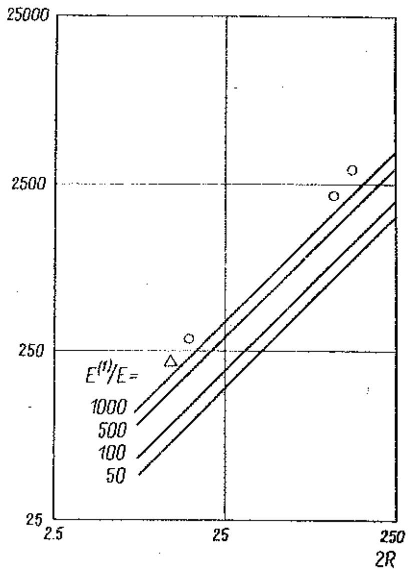

FIG. 7.

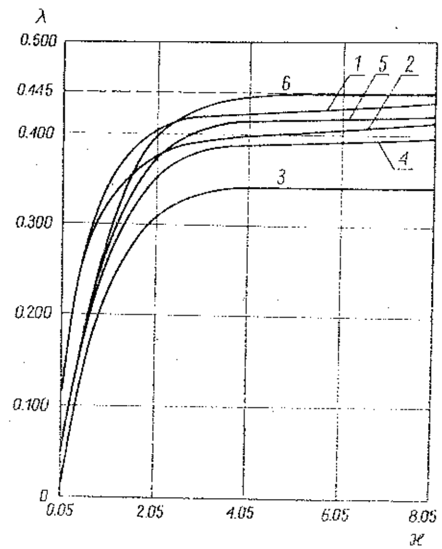

FIG. 8.

and from a paper by B. Rosen. A linear dependence between the wavelength and the fibre diameter follows from the cited results. The coincidence of theoretical and experimental data allows us to interpret the appearance of wave-like shapes of fibres at cuts of composites as a result of an elastic stability loss at a high temperature.

In Figs. 8 to 10 numerical results are given for the Treloar potential $\Phi=C(I_1-3)$ in the case of finite subcritical deformations (the materials of the matrix and the fibre are assumed to be incompressible and isotropic). In Fig. 8 the relation between critical shortening λ and the wave-excitation parameter $\kappa=\pi R/l$ is displayed. The curves 1, 2, ..., 6 are related to the following values of the parameter $\beta = C^{(1)}/C = 0$; 0,05; 5; 10; 20; 500, respectively. Ue note that the curve "1" corresponds to the problem of surface instability of a cylindrical cavity and for $\kappa \to \infty$ the critical extension λ tends to its asymptotic value $\lambda^* = 0.44$. No internal loss of stability was discovered within a wide range of change of parameters β $(0 \leqslant \beta \leqslant 500)$ and κ $(0.005 \leqslant \leqslant \kappa \leqslant 10.2)$.

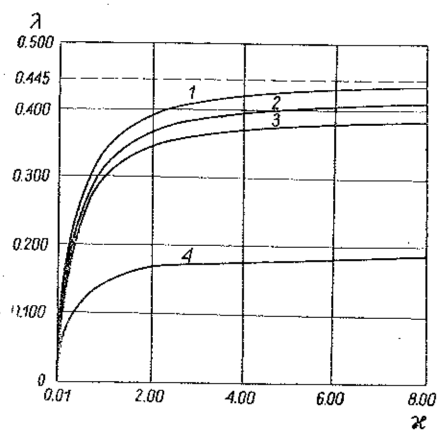

FIG. 9.

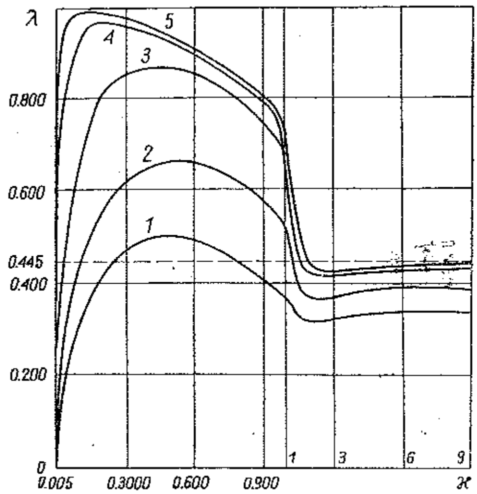

FIG. 10.

In Figs. 9 to 10 some results are given for the case of a non-axially-symmetric form of loss of stability. For $\beta < 1$ (Fig. 9) no internal loss of stability is observed (indices 1, 2, 3 and 4 correspond to $\beta = 0.00005$; 0.05; 0.1; 0.5). At $\beta > 1$ (Fig. 10) the loss of stability occurs in the structure of the material in a non-axially-symmetric form and the wavelength of the form of loss of stability grows with β growing (indices 1, 2, ..., 5 correspond to $\beta = 5$, 10, 50, 500, 5000) and the results obtained in the cases of bending and torsion coincide.

5. Stability of bars, plates and cylindrical shells

We consider first the stability of a transversally isotropic bar of a circular cross--section, which is compressed along its axis by stresses of intensity p. In Fig. 11 relations are given between the parameter $p^* = p_{cr}/p_e$ (p_e — critical stress obtained basing on the hypothesis of plane cross-sections) and the quantity $\alpha = \pi R/l$. The indices 1, 2, 3, ..., 7 correspond to the values $E_3/G = 6$, 10, 20, 30, 40, 60, 100, respectively. The dotted curves were obtained in accordance with the theory of Timoshenko. Figure 12 presents results on the stability of a glass-plastic bar which was reinforced mostly in the longitudinal direction.

As can be seen from the diagram, the theoretical curve agrees well with the experimental data.

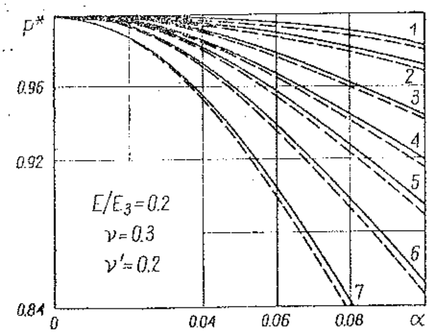

FIG. 11.

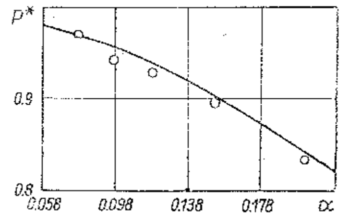

FIG. 12.

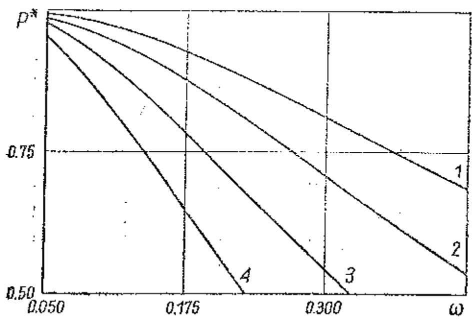

FIG. 13.

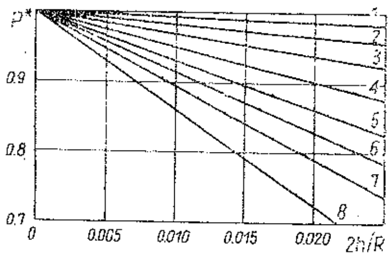

FIG. 14.

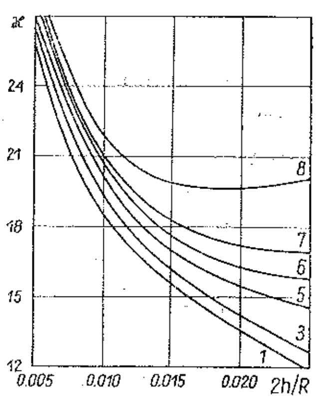

FIG. 15.

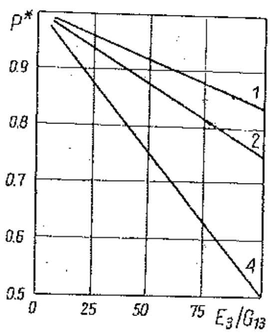

FIG. 16.

In Fig. 13 relations between the non-dimensional parameter p^* and the wave--excitation parameter ω are presented. The indices 1, 2, 3 and 4 correspond to $E_3/G=5$, 20, 50, 100, respectively. The results given in Fig. 13 are valid for a long orthotropic plate $(\omega=\pi h/l)$ and also for plates which are rectangular transversally-isotropic $\omega^2=((\pi m/a)^2+(\pi n/b^2))\,h$, circular $(\omega=\kappa_k\ (h/R))\ J_0(\kappa_k)=0)$ or annular $J_0'(\kappa_k)\,N_0'(\kappa_k\,R/R_0)=J_0'(\kappa_k\,(R)\,N_0'(\kappa_k)=0$.

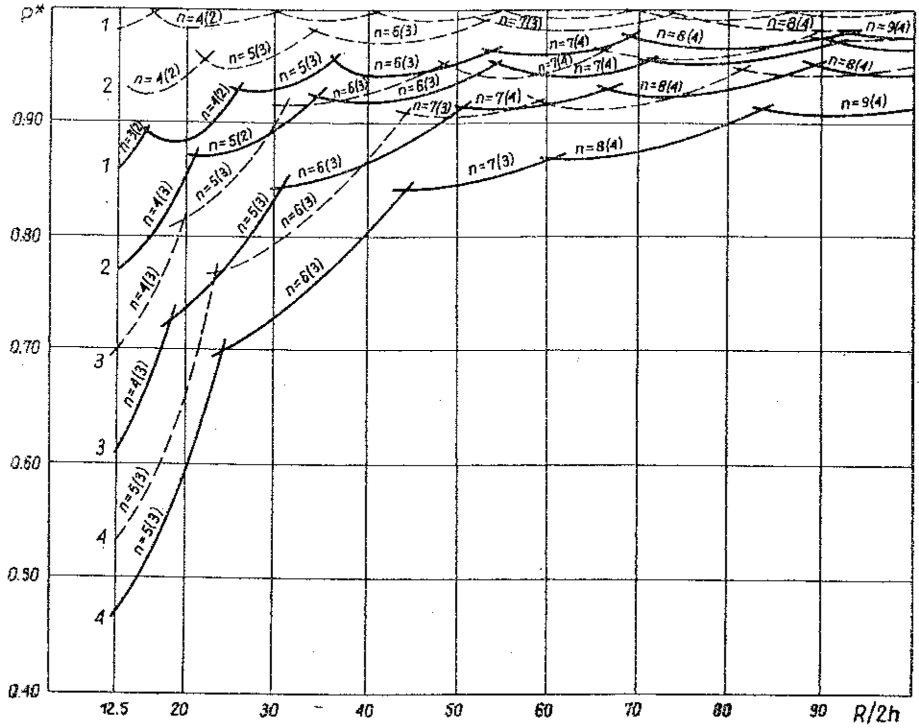

FIG. 17.

Figures 14 to 19 show results on the stability of cylindrical transversally-isotropic and orthotropic shells. For the case of axially-symmetric deformations, results are given in Figs. 14 to 16 $p^*=p_{cr}/p_e$, p_e critical loading calculated using the Kirchhoff--Laval hypothesis $\kappa=m\pi R/l$. In Figs. 14 to 15, the indices 1, 2, ..., 9 correspond to $E_3/G=E_3/G_{12}=E_3/G_{13}=4$, 10, 20, 30, 40, 50, 60, 80, 100, and $E_1/E_3=E_2/E_3=1$, $\nu_{12}=\nu_{21}=0.3$ $\nu_{13}=\nu_{23}=0.2$. In Fig. 16 the indices 1, 2, 3 and 4 correspond to $2h/R=1/150$, 1/100, 1/75, 1/50 and $E_1/E_2=0.3$, $E_1/E_3=0.5$, $\nu_{12}=0.25$, $\nu_{13}=0.30$, $\nu_{23}=0.10$.

For non-axially-symmetric deformations, numerical results were obtained using the method of power series (1,7). In Fig. 17 relations are given between the non-dimensional parameter of loading p^* and the quantity $R/2h$, which were obtained

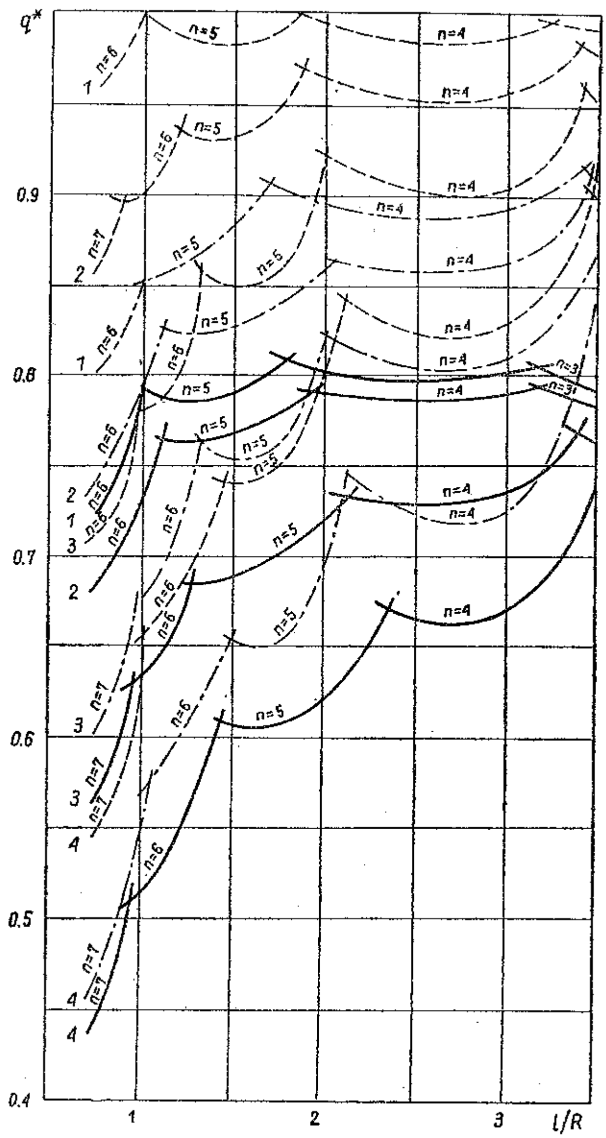

FIG. 18.

from the three-dimensional linearized equations (solid lines) and in accordance with a corrected kinematic theory of the Timoshenko type (dashed lines) in the case of an axial compression of a cylindrically-orthotropic shell, the material of which is characterized by the following mechanical parameters: $E_1/E_3=0.10$, $E_2/E_3=1.00$, $E_3/G_{23}=2.50$, $\nu_{12}=\nu_{13}=0.20$, $\nu_{23}=0.25$.

In Fig. 18 diagrams are presented for the dependence of the non-dimensional parameter of pressure q^* on the quantities l/R. Solid lines are drawn for a "dead" and "dash-dot" lines for a "tracing" uniform side loading, and dashed lines corres-

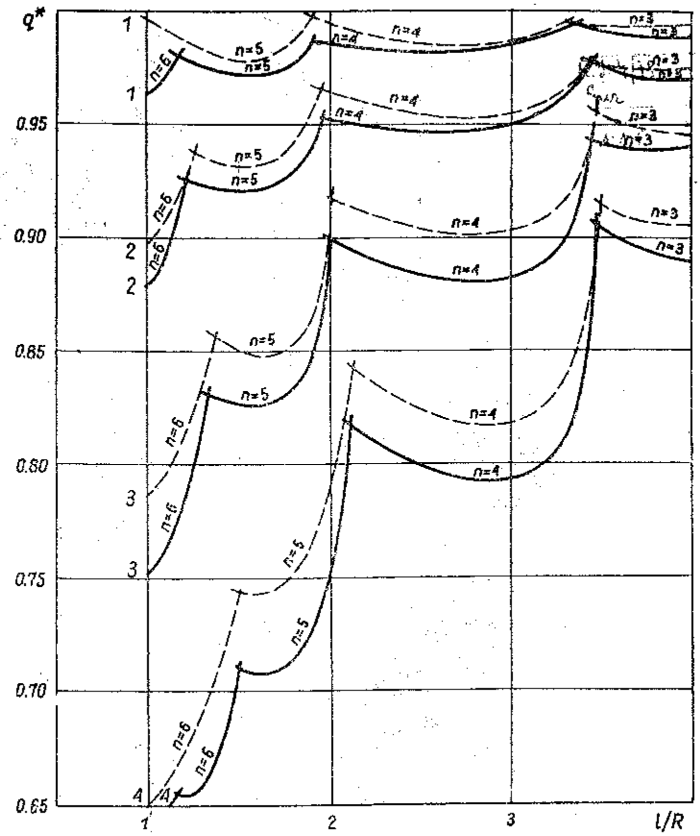

FIG. 19.

pond to the kinematic theory of the Timoshenko type. In Fig. 19 the solid curves are obtained in accordance with a static theory and the dashed curves in accordance with a kinematic theory of the Timoshenko type. Results are displayed for the following mechanical and geometrical characteristics: $E_1/E_3=0.1, E_2/E_3=1, E_3/G_{23}==2.5, \nu_{12}=\nu_{13}=0.2, \nu_{23}=0.25, 2h/R=0.04$.

In all the figures 17 to 19 the curves with the indices 1, 2, 3 and 4 correspond to $E_3/G=E_3/G_{12}=E_3/G_{13}=5, 20, 50, 100$; n — number of waves in the circular direction; the number of half-waves along the generatrix is given in parenthesis.

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Streszczenie

TRÓJWYMIAROWE ZAGADNIENIA STATECZNOŚCI MATERIAŁÓW KOMPOZYTOWYCH I WYKONANYCH Z NICH ELEMENTÓW KONSTRUKCJI

Sformułowano trójwymiarowe statyczne i dynamiczne zlinearyzowane zagadnienia stabilności ściśliwych i nieściśliwych ciał, a w przypadku jednorodnych podkrytycznych deformacji zbudowano ich ogólne rozwiązania. Sformułowano i udowodniono odpowiednie zasady wariacyjne. W ramach trójwymiarowej zlinearyzowanej teorii zbadano płaskie i przestrzenne zagadnienia o niestateczności deformacji warstwowych i włóknistych materiałów przy małych i dużych sprężystych deformacjach. Zbadano sprężystą stateczność prętów, płyt i walcowych powłok, wykonanych z materiałów kompozytowych oraz określono obszary stosowalności klasycznych i uściślonych stosowanych teorii.

Резюме

ТРЕХМЕРНЫЕ ЗАДАЧИ УСТОЙЧИВОСТИ КОНПОЗИТНЫХ МАТЕРИАЛОВ И ЭЛЕМЕНТОВ КОНСТРУКЦИЙ ИЗ НИХ

Дана постановка трехмерных статических и динамических линеаризированных задач сжимаемых и несжимаемых тел и, в случае однородных докритических деформаций, построены их общие решения. Сформулированы и доказаны соответствующие вариационные принципы. В рамках трехмерной линеаризированной теории исследованы плоские и пространственные задачи о неустойчивости деформирования слоистых и волокнистых материалов при малых и высокоэластических деформациях. Изучена упругая устойчивость стержней, пастин и цилиндрических оболочек, вьтполненых из композитных материалов, и определены области применимости классических и уточненных прикладных теорий.

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