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A DIRECT APPROACH TO SHAPE OPTIMISATION OF STRUCTURES

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Shape optimisation of structures constitutes — mathematically — a non-standard problem of the calculus of variations consisting in searching for the conditional extremum of a functional which does not possess, as a rule, a "localization property". The solution of this problem using classical methods of the calculus of variations, as well as dynamic programing methods, creates some difficulties. The paper presents a direct method of solving such problems — analogous to the Ritz-method in the calculus of variations — by assuming that the function constituting the solution is expressed in an analytical form (e.g. a polynomial) and by finding the coefficients of this expression using the methods of nonlinear programming.

1. Introduction

The optimisation of structural dimensions when the topology and shape are fixed is, in mathematical terms, the finding of the conditional extremum of the objective function. Shape optimisation of structures constitutes a problem of higher order of difficulty; it results, however, in higher savings. It consists in finding the conditional extremum of a functional, which is a far more complicated task than finding the extremum of a function.

The paper presents a direct method of solving such problems by assuming that the function describing the shape is expressed in an analytical form (e.g. a polynomial) and by finding the coefficients of this expression using the methods of nonlinear programming.

To illustrate the problem, we shall consider examples of structures, to which the method can be applied.

As a first example let ue consider a roof consisting of so-called shell-beams of an arc-shaped cross section (Fig. 1), which can be used to cover factory sheds, stores, etc. Assuming that roof shells are supported on longitudinal beams (e.g. along the line BC on Fig. 2) and stiffened at the ends by perpendicular webs, they may be considered as parts of these beams by carrying loads to the supporting planes AB and CD (Fig. 2). Such structures, called shell-beams, work therefore as beams in bending which have a span L (Fig. 2) and not as shells. A question can be asked, how can the optimum shape of the roof be found and, first of all, how is the arc-shaped cross-section of the shell-beam formed. This problem consists in finding the shape of a plane curve, that is in finding the formula for the function of a single variable

$$y=y(x).$$

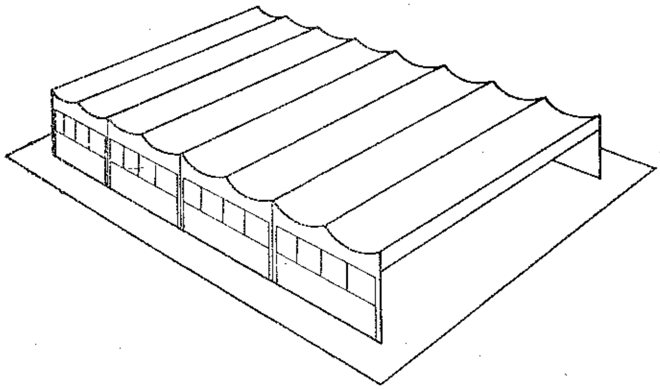


FIG. 1. Factory roof consisting of arc-shaped shell-beams [2].

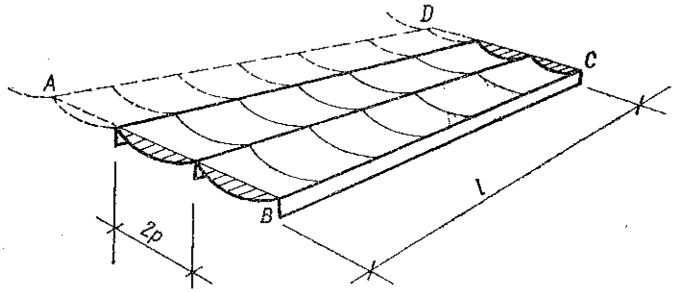


FIG. 2. A shell-beam [2].

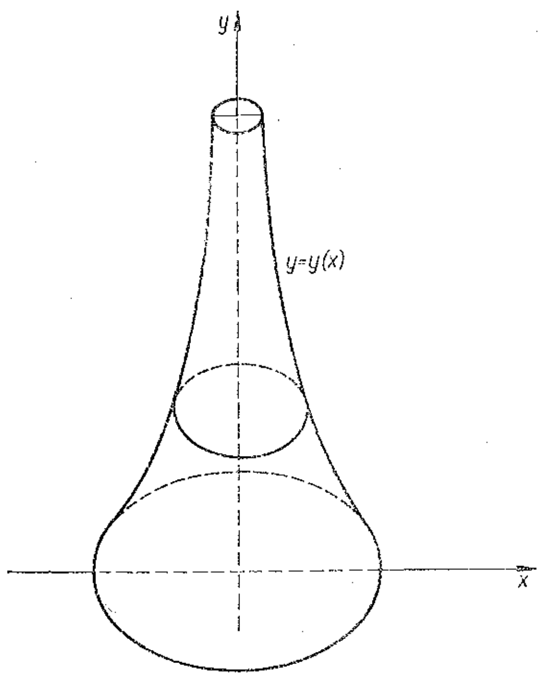


FIG. 3. Shape of industrial chimney — surface of revolution generated by a plane curve.

Another example of seeking the shape of a plane curve generating a surface of revolution, is the optimisation of the shape of an industrial chimney (Fig. 3) [4]. A further example is provided by the optimisation of the shape of a shell of double curvature constituting the roof of a building (Fig. 4). The optimisation of the shape involves in this case finding the function of two variables $z=z(x, y)$ defining the shape of the shell.

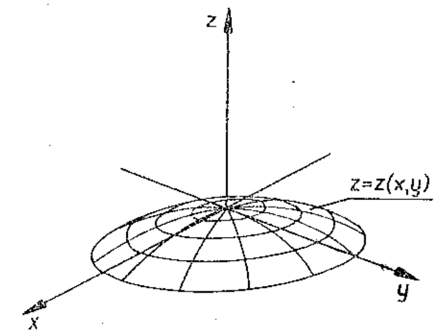


FIG. 4. Shell of double curvature.

Economy criteria are mostly used as objective functions in problems of this kind. They represent construction costs, (less often maintenance costs) expressed, among others, as a function of the form of the structure. In every case strength, construction-, erection-, maintenance- and other constraints must be fulfilled assuring that the produced design can be erected. The optimisation problems specified in this manner consist in optimising functionals.

In mathematical terms the problem can be formulated as follows:

$$(1.1)\qquad \begin{aligned} &\max_{y} \{F(y) | y \in M\}, \\ &M := \{y | g(y) \geqslant 0,\ y \in X\}, \end{aligned}$$

where $F(y)$ — the functional whose conditional extremum is sought, $y=y(x_i)$, the function sought for which the functional $F(y)$ reaches maximum value and which constitutes the solution to the problem, $x_i \in [a_i, b_i]$, $i=1, ..., n$

$$g(y)=\begin{bmatrix} g_1(y) \\ \vdots \\ g_m(y) \end{bmatrix},$$

where m is the dimension of the vector of the functionals of the variable y.

The functionals $g(y)$ constitute the constraints which must be adhered to by the admissible solution; M — domain (set) of admissible solution, X — normed linear space.

The problem, expressed mathematically by the formula (1.1), can be verbally formulated as follows: To find such an element of normed, linear space X (i.e. such a function $y(x)$, $x \in [a, b]$) for which the functional $F(y)$ attains conditional maximum while M is a set of admissible solutions, i.e. a set of functions y within the constraints $g(y)$.

The optimum shape design problems presented above can be reckoned among the class of the calculus of variations for conditional extrema. These are as a rule non-standard problems of the calculus of variations. As we shall see later (see Sects. 2 and 3 below) in the detailed formulation of the numerical example 1, nonlocal functionals [1] occur.

It is known as yet that there exist no general methods for the solution of tasks of the calculus of variations. Taking into account the fact that shape optimisation problems represent one of the more difficult tasks of finding conditional extrema of functionals (worse still the functionals are often nonlocal), the number of problem that can be solved using the methods of the calculus of variations is severely restricted. Dynamic programming, which is rather a computing technique and is limited practically to the function of a single variable, is not an advantageous method for this type of problems. The amount of computations is large, even for fast computers. A direct method for the solution of shape optimisation problems is presented in this paper. It consists in reducing the calculus of the variations problem to the nonlinear programming problem.

2. The approach proposed

The shape optimisation problem formulated in Sect. 1 or, in mathematical terms, the problem of finding the conditional extremum of a functional was appro ached in 1967 by the author [2, 3] analogously to the Ritz method in the calculus of variation in the following way: It is assumed that the function constituting the solution of the problem can be represented analytically (e.g. by a polynomial with unknown coefficients). For the purpose of finding the optimum shape of a plane curve, it is assumed that this function can be represented by

$$y=\sum_{i=0}^{n} a_i x^i \tag{2.1}$$

and for finding the optimum shape of a surface (see Fig. 4), which is the function of the two independent variables x, y, the equation of this surface can be assumed to be represented by the following polynomial:

$$z=\sum_{i,j=0}^{n} a_{ij} x^i y^j . \tag{2.2}$$

The unknowns in the above functions y and z are the coefficients of the polynomial, i.e. a_i or a_{ij}, respectively. These unknowns are treated as decision variables in the following optimisation problem:

$$\max_a \{f(a)|a \in A\}, \tag{2.3}$$

where $a=\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ in case of one-dimensional problems, $a=\begin{bmatrix} a_{00} \\ \vdots \\ a_{nn} \end{bmatrix}$ in case of two-dimensional problems, where f is the objective function (replacing the functional F in the original formulation (1.1)). Next, the optimum values of the decision variables are sought, for which the objective function f reaches a conditional extremum (denoted here as a maximum). In this manner, the problem of finding the conditional extremum of a functional has been reduced to the much simpler problem of finding the conditional extremum of a function. Thus reduced, the problem can be solved be using the methods of mathematical programming.

The way to solve this problem can be presented best by the example [2, 3] mentioned in Sect. 1 above. The problem consists in finding the optimum shape of the cross-section of shell-beams which constitute the roof of a building (Fig. 1). As mentioned in Sect. 1, these beams are considered as elements in bending which have a span L, supported in planes AB and CD (Fig. 2). The task for the optimisation of the shell-beam can be formulated in various ways, depending on the choice of the optimisation criterion. For example, one can minimize the weight of the beam keeping its strength at the required level, or one can search for the maximum strenth using no more than the specified amount of material. This second criterion will be used below.

In order to formulate mathematically the problem, let us consider the cross-section of one element, that is an arc of the width $2p$ and height y_{max} (Fig. 5). Let us assume the thickness of element to be constant and small in comparison with its width. In further considerations we shall therefore restrict our attention to the neutral axis of the arc. The strength of the shell-beam is proportional to the moment of inertia of its cross-section. We can therefore assume that the moment of inertia stands for the strength of the shell-beam. The moment of inertia is a functional of the function expressing the cross-section shape. We seek such a function defining the shape of the arc so that the functional reaches the maximum, while complying with the constraints, namely that the arc length l_d and the arc height y_{max} are not exceeded. Taking symmetry into account, only half of the cross-section needs to be considered (Fig. 6).

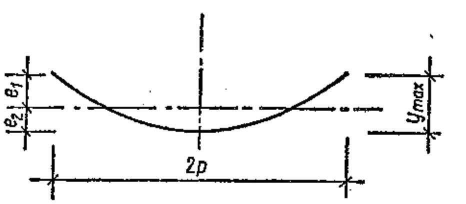


FIG. 5. Arc-shaped cross-section of a shell-beam [2].

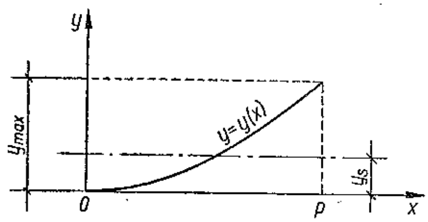


FIG. 6. Optimised arc [2].

The mathematical model of the problem has therefore the following form:

$$\max_{y} \{I(y) \mid y \in M\}, \tag{2.4}$$

$$M := \{y \mid l(y) \leqslant l_d,\ y \leqslant y_{max},\ y \in X\},$$

where $y = y(x)$, $0 \leqslant x \leqslant p$, $I(y)$ denotes the moment of inertia of the arc cross-section about its own centroidal axis, l_y — the legnth of the arc, l_d — the adopted constraints on arc length, y_{max} — the adopted constraints on arc height and X — normed linear space.

In accordance with the approach presented above, the solution of the problem will be as follows:

Let us assume that the equation of arc shape has the form of a polynomial (see formula (2.1))

$$y=a_0+a_1 x+a_2 x^2+\dots+a_{n-1} x^{n-1}+a_n x^n \tag{2.5}$$

and let us assume that the coefficients a_i, $i=0, 1, \dots, n$ are decision variables and finding their values is the aim of optimisation inquiry.

The mathematical model of optimisation assumes for this particular example the following form (compare with the formula (2.4)):

$$\max \{I(a) | l(a) \leqslant l_d,\ y(a) \leqslant y_{\max}\}, \tag{2.6}$$

where

$$a=\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}, \qquad x \in [0, p].$$

The solution of the problem will therefore be the vector a, whose coordinates a_i (i.e. the coefficients of the polynomial), $i=0,1, \dots, n$, constitute decision variables. After having computed explicit values for I and l, we finally obtain

$$\max_y \int_0^p \left(y - \frac{\int_0^p y\sqrt{1+\dot{y}^2}\, dx}{\int_0^p \sqrt{1+\dot{y}^2}\, dx} \right)^2 \sqrt{1+y^2}\, dx, \tag{2.7}$$

$$y=y(x), \qquad x \in [0, p]$$

with the constraints

$$\int_0^p \sqrt{1+\dot{y}^2}\, dx \leqslant l_d \tag{2.8}$$

and

$$y \leqslant y_{\max}. \tag{2.9}$$

In the formula (2.7) a "nonlocal functional" appears. In the above example, once the class of function defining the shape of the arc is assumed to be the polynomial (2.5), the integrals in the formulae (2.7) and (2.8) cannot be presented in a closed form, nor calculated in an elementary way but must be computed approximately using numerical methods. To obtain a numerical solution of this problem, the author used the Monte Carlo method.

3. Numerical solution of the example

Taking into account the symmetry of the arc in relation to the vertical axis passing through the origin of the coordinates, it is sufficient to deal with a half-arc only (Fig. 6). For this reason, the terms a_0 as well as $a_1 x$ vanish. The equation of the arc has now the form

$$y=a_n x^n+a_{n-1} x^{n-1}+\dots+a_2 x^2. \tag{3.1}$$

In our example, let this polynomial be of the fourth degree (see Fig. 6):

$$y=ax^4+bx^3+cx^2, \tag{3.2}$$

the position of the centre of gravity:

$$y_s=\frac{\int\limits_0^p (ax^4+bx^3+cx^2)\sqrt{1+(4ax^3+3bx^2+2cx)^2}\,dx}{\int\limits_0^p \sqrt{1+(4ax^3+3bx^2+2cx)^2}\,dx} \tag{3.3}$$

and the moment of inertia of the arc with recpect to the axis through the centre of gravity:

$$I=\int\limits_0^p (ax^4+bx^3+cx^2-y_s)^2\sqrt{1+(4ax^3+3bx^2+2cx)^2}\,dx. \tag{3.4}$$

Optimisation of the arc shape consists in finding such values of the decision variables a, b and c, for which the maximum of the function I occurs, fulfilling simultaneously the following arbitrarily chosen, constraints:

Constraint 1:

$$l=\int\limits_0^p \sqrt{1+(4ax^3+3bx^2+2cx)^2}\,dx\leqslant l_a, \tag{3.5}$$

a quadrant of the circle of a radius p will be adopted for l_a. Therefore, $l_a=\pi p/2$.

Constraint 2:

$$y=ax^4+bx^3+cx^2\leqslant y_{max} \tag{3.6}$$

let $y_{max}=p$.

$p=1$ will be adopted as half-span of the arc (see Fig. 6). The Monte Carlo method was used to obtain the solution, using a random number generator which supplies uniformly distributed numbers in the range [0,1]. Each triple of numbers chosen by this generator and normed into the scale of the variables a, b, c, respectively, was treated as a combination of the values of these variables. The constraints (the formulae (3.5) and (3.6)) were checked for this combination of variables. Any combination of the values of the variables violating one or more constraints was disregarded. The random choice was repeated till a combination of values of the variables a, b, c, was found within the constraints. For this combination, the value of the functional (the formula (3.4)) was calculated and compared with the one previously obtained. A better result was stored and the random choice was repeated. If, after about ten thousand random choices, no improvement occured, the procedure was stopped and the last result was taken as the approximation of the optimum solution. A detailed description of the use of the Monte Carlo method in optimisation is given in [3]. In this example of arc shape optimisation, the integrals could not be calculated using closed formulae and were obtained by using numerical methods according to the formula

$$\int\limits_0^p f(x)\,dx=\frac{h}{2}(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n),$$

where

$$y_i=y(x_i), \quad i=0, 1, \ldots, n,$$

$$x_0=0, \quad x_i=x_{i-1}+h, \quad h=p/n,$$

i.e.

$$y_0=f(x_0)=f(0), \quad y_1=f(h), \quad y_2=f(2h), \ldots .$$

The choice of the suitable number n for the numerical integration intervals in the range $[0, p]$ was made experimentally. It was proved that smaller intervals than $h=p/40$ did not improve the accuracy of the results. Nevertheless, to secure the highest possible accuracy $h=p/100$ was finally used.

Substitution of random values for the variables a, b and c took place in two cases:

(a) in the range [0, 1],

(b) in the range [−6, 6](1).

Since the optimum in the range [−6, 6] differs from that in [0, 1], it is very probable that there exist local optima in the presented example.

Calculations on a digital computer produced the following expressions for the arc shape:

For case (a):

(3.7) $$y=0.6826x^4+0.2141x^3+0.1009x^2.$$

For case (b):

(3.8) $$y=-3x^4+4x^3.$$

For comparison, calculations have been made for the following cross-section shapes:

1) Straight, inclined line (Fig. 7) for which

$$l=p\sqrt{2}, \quad y_s=\frac{p^2\sqrt{2}}{2p\sqrt{2}}=\frac{p}{2},$$

$$I=\int_0^p (x-p/2)^2\sqrt{2}\,dx=0.1178p^3.$$

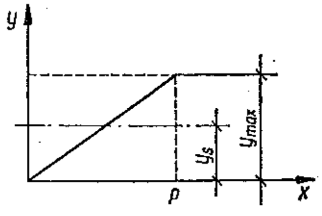


FIG. 7. Inclined straight [2].

(1) The author wishes to express his thanks to Professor Z. KĄCZKOWSKI for calling attention to this range of values of variables.

2) A quadrant of a circle (Fig. 8) for which

$$y=p-\sqrt{p^2-x^2}, \quad y'=\frac{x}{\sqrt{p^2-x^2}}, \quad l=\frac{\pi p}{2},$$

$$y_s=\frac{\int\limits_0^p \left(p-\sqrt{p^2-x^2}\,\frac{p}{\sqrt{p^2-x^2}}\right)dx}{\pi p/2}=\frac{p\,(\pi-2)}{\pi},$$

$$I=\int\limits_0^p \frac{(p^2-2p\sqrt{p^2-x^2}+p^2-x^2-2py+2y_s\sqrt{p^2-x^2}+y_s)\,p}{\sqrt{p^2-x^2}}\,dx=0.1488p^3.$$

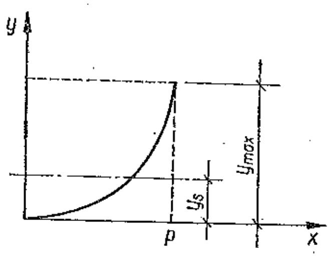


FIG. 8. Quadrant of a circle [2].

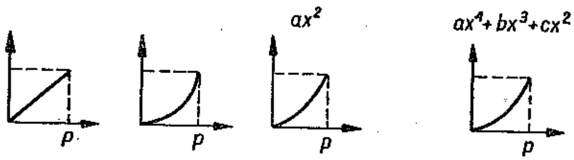
3) A parabola of the 2nd degree

$$y=x^2.$$

Table 1 shows the optimum results for all cases presented above. They are presented in the following order:

1) An inclined straight line.
2) A quadrant of a circle.
3) 2nd degree parabola.
4) 4th degree parabola for which two cases were investigated:
   a) for positive values of the coefficients in the range ($a, b, c, \in [0, 1]$)
   b) for values of coefficients in a larger range ($a, b, c, \in [-6, 6]$).

Table 1. Values of the moment of inertia for half-width of cross-sections (p=1)



	1	2	3	4	5
$y_{max}=p=1$	0.1178	0.1488	0.1415	[a] 0.159	[b] 0.164
	(79)	(100)	(95)	(107)	(110)

Among arc shapes proper, the 2nd degree parabola gives the worst result. The best result is obtained by using the 4th degree parabola, case (b). Figure 9 shows the comparison of four arc shapes. To facilitate the comparison, percentage values

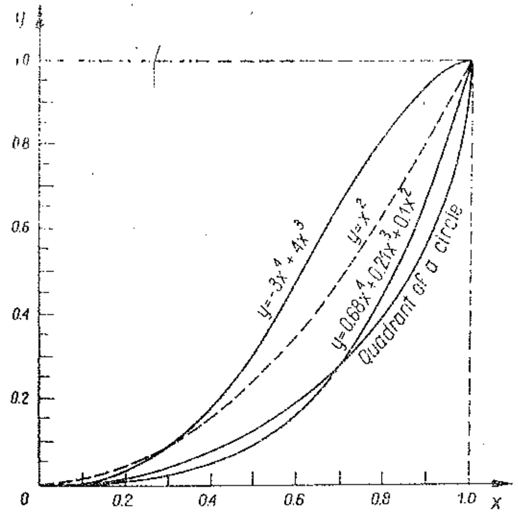


FIG. 9. Comparison of arc shapes.

of the moments of inertia are given in Table 1, taking the value obtained for the quadrant of a circle as 100. The best result is indicated by drawing a border around it and the second best one — by underlining it.

4. Conclusions

The direct approach to shape optimisation of structures presented above has the advantage of being extremely simple. Because of this, it lends itself to practical application of engineering structures. When using this method, the design engineer will not be frightened away by complicated mathematics and will be able to concentrate on building the mathematical model of the optimised structure that would represent the real structure adequately.

This is precisely the area of work where no one can replace him.

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STRESZCZENIE

BEZPOŚREDNIA METODA OPTYMALIZACJI KSZTAŁTU KONSTRUKCJI

Optymalizacja kształtu konstrukcji stanowi matematycznie nietypowy problem rachunku wariacyjnego polegający na znajdowaniu warunkowego extremum funkcjonału, przy czym funkcjonał z reguły nie ma własności lokalności. Rozwiązanie tego problemu metodami rachunku wariacyjnego lub programowania dynamicznego przedstawia duże trudności.

Artykuł opisuje bezpośrednią metodę rozwiązania, analogiczną do metody Ritza w rachunku wariacyjnym, polegającą na dobraniu analitycznego wyrażenia na poszukiwaaną funkcję (np. w postaci wielomianu) i na znalezieniu wartości współczynników tego wielomianu metodami programowania nieliniowego.

Резюме

НЕПОСРЕДСТВЕННЫЙ МЕТОД ОПТИМИЗАЦИИ ФОРМЫ КОНСТРУКЦИЙ

Оптимизация формы конструкции составляет математически нетипичную задачу вариационного исчисления, заключающуюся в нахождении условного экстремума функционала, причем функционал, как правило, не имеет свойств локальности. Решение этой задачи, при использовании методов вариационного исчисления или динамического программирования, представляет большие трудности. Статья описывает непосредственный метод решения — аналогичный методу Ритца в вариационном исчислении — заключающийся в предложении аналитического выражения для искомой функции (например в виде многочлена) и в нахождении значений коэффициентов этого выражения методами нелинейного программирования.

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