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K O M U N I K A T Y

EVALUATION OF THE RELIABILITY OF STRUCTURAL SYSTEMS WITH CRITICAL SETS HAVING JOINT ELEMENTS IN COMMON

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Presented are herein two methods for evaluation the reliability of complex rigid-plastic structural systems, in which occur many minimum critical sets dependent on one another. The first method is based on the Bonferroni's inequality and allows to evaluate both from below and from above the reliability of the system. The second method developed by means of the separation of dependent critical sets is more accurate than the previous one, in particular, for small values of the reliability, however, by this method the evaluation from below is only possible. In the case of large values of the reliability occuring in building structural systems the two evaluations from below are sufficiently accurate.

1. Introduction

We shall consider structural systems composed of a finite number of elements and critical sets.

We assume that all critical minimum sets of elements are known, i.e. such sets in which the whole structure is efficient if one element at least is efficient. The condition of failure for the part or the whole structure is the exhaustion of load bearing capacity in all elements for any critical minimum set. In models of exhaustion of the load bearing capacity like these, the elements with random properties change their shape as a consequence of strains due to the action of the loading S according to the relation shown in Fig. 1 [10].

The real structures are often characterized by a quite large number of critical minimum sets as well as by the occurrence of the same elements in different critical sets. The weights for these elements are determined on the basis of the conditions of equilibrium of forces for the limit state of the structure. The exact determination of the reliability of structures is only possible in the case of the occurrence of three critical minimum sets with joint elements in common at the most [3], as well as for certain types with a greater number of critical sets, for example, for those sets in which the joint elements are common in pairs [2].

The exact determination of the reliability of the structure composed of elements satisfying the relation shown in Fig. 1 and whose mathematical model is presented in Fig. 2 is not as yet possible because of the lack of suitably general mathematical solutions.

For this reason we present methods allowing to evaluate the reliability from below and from above. The evealuation from below of the reliability is of essential importance in view of the safety of the structure. It also makes the lower-bound

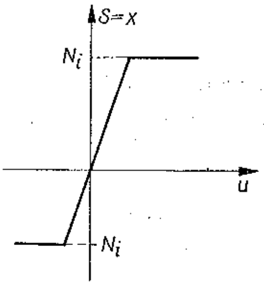

FIG. 1. Load-strain dependence for elements of structural systems.

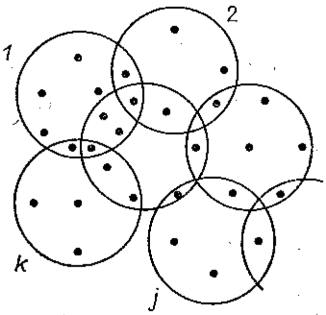

FIG. 2. Mathematical model of the structural system with minimum critical sets having joint elements in common.

evaluation possible for the ultimate load bearing capacity of the structure in question. The problem of evaluating the reliability of systems is the subject of interest of many mathematicians and, accordingly, various methods have been developed which allow to find that evaluation [4].

In the present paper we make use of two methods allowing to achieve a simple and efficient evaluation of the reliability for the model of structures as that shown in Fig. 2.

2. Evaluation of the reliability of the model of structure with application of Bonferroni's inequalities

Let $N_1, N_2, \ldots, N_n$ be independent random variables which correspond to the ultimate bearing capacities of elements and for which probability distributions are known. Let us assume that the k critical minimum sets of elements can be distinguished for that structure where their ultimate bearing capacities X_j are random

$$X_j = \sum_{i=1}^{n} a_{ij} N_i, \quad j=1, 2, \ldots, k. \tag{2.1}$$

variables where $a_{ij} \geqslant 0$ is the weight of the i-th element in the j-th critical set.

The distribution of the random variable X_j is indicated by $F_j(x)$

$$F_j(x) = \Pr(X_j < x), \quad j=1, 2, \ldots, k. \tag{2.2}$$

For our model the ultimate bearing capacity N is

$$N = \min(X_1, X_2, \ldots, X_j, \ldots, X_k) \tag{2.3}$$

Let us note that the random variables $X_1, X_2, ..., X_k$ are dependent random variables. The reliability of the structure is called the function $R(x)$ given by

$$R(x)=\Pr(N \geqslant x) \tag{2.4}$$

The reliability $R(x)$ of the structure can be determined on the basis of Poincare's [6], [7], as follows:

$$R(x)=1-\sum_{j=1}^{k} \Pr(X_j<x)+\sum_{\substack{j_1, j_2=1\\ j_1<j_2}}^{k} \Pr(X_{j_1}<x, X_{j_2}<x)- \tag{2.5}$$
$$-\sum_{\substack{j_1, j_2, j_3=1\\ j_1<j_2<j_3}}^{k} \Pr(X_{j_1}<x, X_{j_2}<x, X_{j_3}<x)+...+(-1)^k \Pr(X_1<x, X_2<x, ... X_k<x).$$

The exact determination of all terms occurring in the formula (2.5) is not possible. Let us note that in the Bonferroni's inequality [1], [6]

$$1-\sum_{j-1}^{k} \Pr(X_j<x) \leqslant R(x) \leqslant 1-\sum_{j=1}^{k} \Pr(X_j<x)+\sum_{\substack{j_1, j_2=1\\ j_1<j_2}}^{k} \Pr(X_{j_1}<x, X_{j_2}<x) \tag{2.6}$$

occur only these terms of the formula (2.5) which can be determined in a simple way. The application of the inequalities (2.6) for evaluation from below of the reliability of structural systems is dealt with, among others, in the papers [2], [3], and [8]. It should be mentioned that the inequality (2.6) is also useful to evaluate from above the reliability of the structure.

3. Evaluation of the reliability of the model of the structure on the basis of separation of dependent critical sets

Let us separate the critical minimum sets of elements for the structure shown in Fig. 2 in such a way as to maintain all elements in separated sets may differ by a nonrandom factor a_{ij} resulting from different weights for elements in different, stochastically independent by now, critical sets. Such a separated model of the structure is shown in Fig. 3.

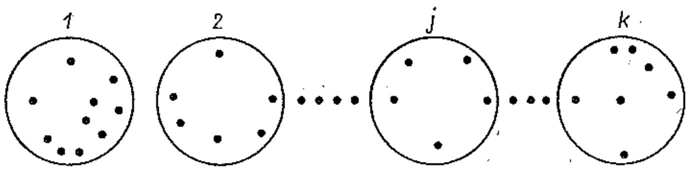

Fig. 3. Separated model of the structure shown in Fig. 2.

The ultimate bearing capacity $\bar{N}$ for the separated model of the structure is

$$\bar{N}=\min(\bar{X}_1, \bar{X}_2, ..., \bar{X}_k), \tag{3.1}$$

where $\bar{X}_j$ are the independent random variables with the distributions $F_j(x)$.

The reliability for the separated model of structure is

$$\bar{R}(x)=\Pr(\bar{N}\geqslant x)=\prod_{j=1}^{k}\Pr(\bar{X}_j\geqslant x)=\prod_{j=1}^{k}[1-F_j(x)]. \tag{3.2}$$

In further considerations we make use of lemma 1, which is the particular case of the theorem 5.1 presented in [5].

LEMMA 1. Let $T_1, T_2, \ldots, T_k$ be the independent random variables $s_j=f_j(T_1, T_2, \ldots \ldots, T_k)$ is the non-decreasing function, and $j=1, 2, \ldots., p$, then

$$\Pr(S_1\geqslant s_1, \ldots S_k\geqslant s_k)\geqslant\prod_{j=1}^{k}\Pr(S_j\geqslant s_j). \tag{3.3}$$

THEOREM 1. *The following inequality holds true*:

$$R(x)\geqslant\bar{R}(x) \tag{3.4}$$

i.e. the reliability of the separated model of the structure is lower than the reliability of the unseparated model.

Proof. Let us note that the random variables X_j from the relation (2.1) may be written in the form $X_j=f_j(N_1, N_2, \ldots, N_n)$, where $f_j(x_1, \ldots, x_n)=\sum_{i=1}^{n} a_{ij}x_i$ is a non-decreasing function, and $N_1, N_2, \ldots, N_n$ are the independent random variables. Since the reliability which is evaluated for the model of structure in question is

$$R(x)=\Pr\{\min(X_1, X_2, \ldots, X_k)\geqslant x\}=\Pr\{X_1\geqslant x, \ldots, X_k\geqslant x\}, \tag{3.5}$$

then using the lemma 1, we obtain

$$R(x)\geqslant\prod_{j=1}^{k}\Pr(X_j\geqslant x)=\prod_{j=1}^{k}[1-F_j(x)]=\bar{R}(x) \tag{3.6}$$

QED.

From the theorem such a note follows: if the inequality (3.4) holds true, then we may say, on the basis of [9], that the ultimate bearing capacity N cf the structure is stochastically gieater than the ultimate beasring capacity $\bar{N}$ for the separated structure.

Basing on [1], [4] and [5], the assumption that the ultimate bearing capacities N_j are independent of one another can be weakened in order to evaluate the reliability of complex structural systems.

From the Bonferroni's inequality we have

$$\bar{R}(x)\geqslant 1-\sum_{j=1}^{k}F_j(x). \tag{3.7}$$

Combining the inequalities (2.6), (3.4) and (3.7), we can see that the following sequence of inequalities holds true:

$$1-\sum_{j=1}^{k}F_j(x)\leqslant\bar{R}(x)\leqslant R(x)\leqslant 1-\sum_{j=1}^{k}F_j(x)+\sum_{\substack{j_1, j_2=1\\ j_1<j_2}}^{k}\Pr(X_{j1}<x, X_{j2}<x). \tag{3.8}$$

From the inequality (3.8) it follows that the evaluation from below of the reliability of the structure which is based on the model with separated critical sets is more accurate than that from the Bonferroni's inequality, called also the Boolean inequality.

4. Example

Let us consider an example shown in Fig. 4 [3]. The distribution of ultimate moments is normal here. The expected ultimate bearing capacities of plastic hinges are the same and are equal to $E(M_i)=3.600$ kGm, the standard deviation is $s_i = 366$ kGm. The length L of the bars in the frame is 6 m.

The reason why this example is chosen is that we know the exact solution for a model like that. We shall determine here the reliability in the way given in [3] as well as evaluate its bounds basing on the Bonferroni's inequality and by means of the separation of dependent critical sets.

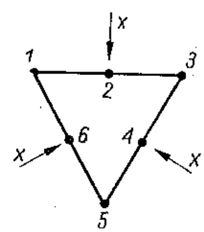

Fig. 4. Example of the structure with three dependent minimum critical sets.

For the example in question the following minimum critical sets of elements can be distinguished: the set A_1-elements 1, 2, and 3; A_2-3, 4, 5; and A_3-5, 6, 1. The expected ultimate bearing capacities of critical minimum sets are the same and equal to

$$E(X)=E(X_1)=E(X_2)=E(X_3)=\frac{4}{6}[0.5E(M_1)+E(M_2)+ +0.5E(M_3)]=E(N_1)+E(N_2)+E(N_3)=4.800 \text{ kG}.$$

The standard deviations of the ultimate bearing capacities A_1, A_2 and A_3 are the same and equal to

$$\sigma_x=\frac{4}{6}\sqrt{2\cdot 183^2+366^2}=299 \text{ kG}.$$

The reliability of the structure will be determined from Table 1 given in [3] on the basis of the arguments z and a

$$z=\frac{x-E(X)}{\sigma_1}, \quad a=\frac{\sigma_2}{\sigma_1}=\frac{244}{122}=2.$$

The evaluation from below will be calculated from the Bonferroni's inequality on the basis of the argument u_1 and the tables of normal distribution $\varphi(u_1)$ [11]

$$u_1=\frac{x-E(X)}{\sigma_x}.$$

The evaluation from below of the reliability using the inequality (3.7) will be determined from the formula

$$R(x)\geqslant 1-3\Phi(u_1)=1-3F_j(x).$$

The evaluation of the reliability of the structure by means of the separation of dependent critical sets will also be calculated by means of the argument u_1 and the tables [11] from the formula

$$\bar{R}(x)=\prod_{j=1}^{3}[1-\Phi_j(u_1)]=\prod_{i-1}^{3}[1-F_j(x)].$$

The results of calculations for different loads of x are listed in Table 1.

Table 1. Evaluation of the reliability of the structure shown in Fig. 4

No	Way of evaluation	Loading of structure in kG				
		4434	4312	4190	4068	3946
1	2	3	4	5	6	7
1	accurate	0.72119	0.86029	0.94107	0.97901	0.99366
2	to Boole (Bonferroni)	0.666304	0.845347	0.937975	0.978571	0.993646
3	separation of critical sets	0.702045	0.853183	0.939249	0.978724	0.993659

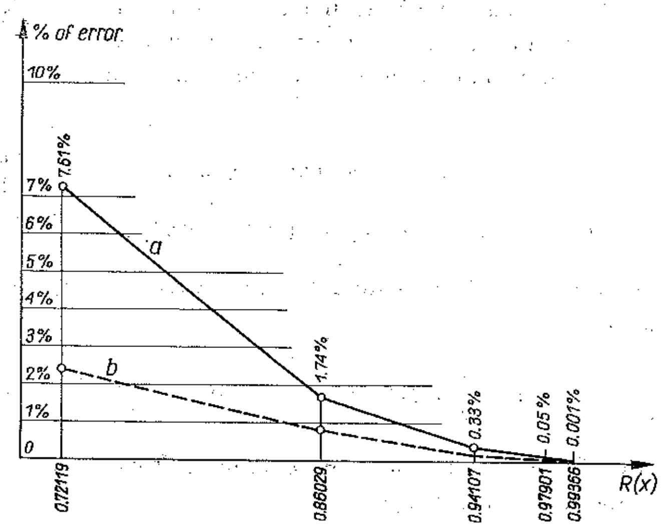

FIG. 5. Error of evaluation of the reliability: a) basing on the Bonferroni's inequality, b) basing on the separation of dependent critical sets.

In Fig. 5 a diagram of the error in % of evaluation of the reliability basing on the Bonferroni's inequality is drawn by a full line, and the corresponding diagram in the case of the separation of dependent critical sets—by a broken line.

5. REMARKS AND CONCLUSIONS

Basing on the Bonferroni's inequality (Boolean inequality) the evaluation both from below and from below and from above of the reliability of the structure can be calculated. For the evaluation of the reliability of building, structural systems are better fitted for the evaluations from below because, subsequently, basing on them the lower bound of the ultimate bearing capacity can be evaluated.

The evaluation from below of the reliability obtained by means of the separated model of structure (by way of separation of dependent minimum critical sets) is better than that obtained on the basis of the Bonferroni's inequality, in particular for small values of the reliability.

In the case of medium and large reliabilities, $R(x) > 0.99$, the difference between the two evaluations are not essential and they can be considered to be equivalent to one another.

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