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INJECTION OF A MIXTURE OF SMALL PARTICLES INTO A FLOW, TREATED AS A PERTURBATION PROBLEM

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Small identical solid spherical particles are injected from a wall into a laminar flow of an incompressible viscous fluid. The concentration of particles is assumed to be low. The goal of this paper is to study the action of the particles on the velocity profile of the fluid.

The equations used for the fluid-particles mixture reduce the effect of each particle to a Stokes force acting on its center. A perturbation method is applied to solve these equations: the non-dimensional number $S=\tau_p/\tau_e$, where τ_p is a characteristic time for a particle and τ_e a characteristic time of the flow, is required to be small. It is shown that this condition is compatible with the other assumptions, but that the equations for the mixture are valid only within a limited region close to the wall. The results prove that the velocity profile of the fluid is affected by the particles if the mass concentration of the particles at the wall is of order $1/\mathrm{Re}\, S$ at least, where Re is the fluid flow Reynolds number. For the case where this condition is satisfied, velocity profiles are computed and written in closed form. Some typical profiles are shown for an uniform injection of particles.

1. Introduction

We consider a Newtonian viscous incompressible fluid flowing along a wall. A mixture of small identical solid particles is injected from the wall into the fluid with a finite velocity (Fig. 1). The particles are spherical and their volume concen-

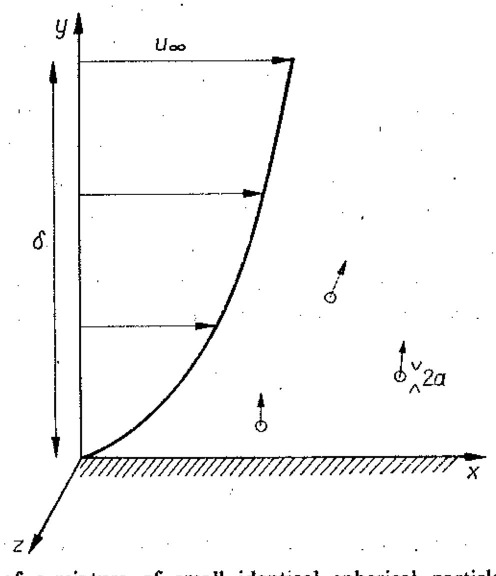

FIG. 1. Injection of a mixture of small identical spherical particles into a flow.

tration is assume to be low. We expect that under the influence of the particles the shape of the velocity profile of the fluid will change. We will study in this paper the conditions in which this change may occur and, in case it does occur, we will compute the shape of the deformed velocity profile.

Such a problem may arise whenever a fluid flows along a wall permeable to small solid particles, (e.g. in chemical engineering and bioengineering applications).

To find the velocity profile of the fluid, the general equations to be solved are the Navier-Stokes equations with the boundary conditions on particles and on the wall. As these equations cannot of course be solved directly, we will make a number of approximations tending to obtain a set of simplified equations for the mixture.

2. Assumptions

The assumptions we make are the following:

a) Small spherical particles of radius a are injected into a region of laminar flow of thickness δ (Fig. 1) so that:

$$a \ll \delta . \tag{2.1}$$

The flow being laminar, the Reynolds number

$$\mathrm{Re} = \frac{u_\infty \delta}{\nu}, \tag{2.2}$$

where u_∞ is the fluid velocity at the distance δ, is smaller than the Reynolds number corresponding to the laminar-turbulent transition, Re_t

$$\mathrm{Re} < \mathrm{Re}_t . \tag{2.3}$$

b) The concentration of particles is low. We neglect therefore: any Brownian effect, any shock between particles, any hydrodynamic interaction between particles, as computed for instance by BATCHELOR [1].

c) The Reynolds number of the fluid movement relative to a particle is low. To define this number we use the maximum velocity of the fluid relative to the particle. This maximum velocity occurs at the injection point, at the wall, as later the particle is entrained by the fluid and its relative velocity decreases. Let $v_{po}(x)$ be the injection velocity perpendicular to the wall. We assume $v_{po}(x)$ to be differentiable, and such that $\bar{v}_{p_0}$, the maximum injection velocity, be a quantity of the order of $v_{po}(x)$. The Reynolds number relative to the particle is defined by

$$\mathrm{Re}_v = \frac{\bar{v}_{p_0} a}{\nu}. \tag{2.4}$$

We require that

$$\mathrm{Re}_v \ll 1 . \tag{2.5}$$

The equations for the flow around a sphere are then the Stokes equations to the lowest order in Re_v.

It is known that:

1) The Stokes equations are valid in the entire flow field. The result when the sphere is far enough from any other solid body is the well-known Stokes force acting on the sphere:

$$\mathscr{F} = -6\pi a\mu(\mathbf{v}_p - \mathbf{v}), \tag{2.6}$$

where μ is the dynamic viscosity of the fluid, $\mathbf{v}$ is the fluid velocity of the "imperturbed" flow field (i.e. if the sphere were absent), v_p is the sphere velocity.

2) At a distance of the order a/Re_v away from the sphere (Oseen distance), the sphere appears, according to the outer equations to the lowest order in Re_v, to act on the outer flow only as a point force of magnitude $\mathscr{F}$.

Now, we precise the assumption b) by requiring that the distance between two particles be larger than the order of a/Re_v, so that there is no Stokes interaction between them. The effect of different particles on the outer flow appears then only to be the addition of the point forces $\mathscr{F}$.

As there must be only one particle in a sphere of volume $\frac{4}{3}\pi\left(\frac{a}{\mathrm{Re}_v}\right)^3$, the number of particles per unit volume, n, must satisfy the condition

$$n < \frac{3}{4\pi}\left(\frac{\mathrm{Re}_v}{a}\right)^3. \tag{2.7}$$

The effect of the wall on the Stokes force will not be taken into account here. The solution of the Stokes equations for a sphere in the presence of a wall has been found by FAXEN [2] and WAKIYA [3] who are quoted by HAPPEL and BRENNER [4]. It may be shown that the effect of the wall becomes negligible when the sphere is only a few diameters away from the wall. Here we assume the sphere to be small enough to neglect this region of interactions.

d) We require that the transverse force on a sphere due to the velocity gradient be small. We define the Reynolds number as follows:

$$\mathrm{Re}_\gamma = \frac{\gamma a^2}{\nu}, \tag{2.8}$$

where γ is the velocity gradient of the order of u_∞/δ. Such a transverse force of the order $\sqrt{\mathrm{Re}_\gamma}$ was computed by SAFFMAN [5] for $\mathrm{Re}_v^2 \ll \mathrm{Re}_\gamma \ll 1$. We require here that

$$\mathrm{Re}_\gamma \ll \mathrm{Re}_v^2 \tag{2.9}$$

so that the Stokes force is then the only force to be considered to the order Re_v.

e) We require the specific mass of particles ρ_p to be much larger than that of the fluid ρ

$$\rho_p \gg \rho \tag{2.10}$$

As follows from the non-dimensional transitory Navier-Stokes equations around a sphere in this case (Appendix 1), we may neglect all transitory forces acting on a particle, such as the hereditary force computed by BASSET [6] or VILLAT [7].

3. Equations

Taking into consideration all preceding assumptions and supposing, moreover, that there exists a small volume where locally all particles have the same velocity, we write down the following equations for permanent flow: Continuity of the fluid:

$$\nabla \cdot \mathbf{v} = 0 . \tag{3.1}$$

Momentum of the fluid:

$$\rho (v \cdot \nabla) \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + n \left[6\pi a \mu (\mathbf{v}_p - \mathbf{v}) \right] . \tag{3.2}$$

Continuity of the particles:

$$\nabla \cdot (n \mathbf{v}_p) = 0 . \tag{3.3}$$

Momentum of the particles:

$$m_p \frac{d\mathbf{v}_p}{dt_p} = m_p (\mathbf{v}_p \cdot \nabla) \mathbf{v}_p = 6\pi a \mu (\mathbf{v} - \mathbf{v}_p) , \tag{3.4}$$

where m_p is the mass of a particle, d/dt_p is the time derivative when following a given particle, p is the pressure.

Such a set of equations may be found in Soo [8].

The boundary conditions at the wall are the no-slip condition for the fluid and a given injection velocity and concentration for the particles (which may be variable with the distance):

$$Y = 0 : \left| \begin{array}{lll} u = v = 0 , & & \\ u_p = 0 , & v_p = v_{p_0}(x) , & \\ n = n_0(x) \quad \text{for} & x \geqslant 0 , \ 0 \quad \text{for} & x < 0 . \end{array} \right. \tag{3.5}$$

Now, we have to point out a limitation of the valicity of these equations for the present problem. Taking the velocity of the fluid perpendicular to the wall to be zero as a first approximation

$$v \cong 0 ,$$

which will be shown to be the case later on, we integrate the momentum equation of a particle (3.4) perpendicular to the wall:

$$v_p = v_{p_0}(x) \, e^{-t/\tau_p} ,$$

where t is the time; and

$$\tau_p = \frac{m_p}{6\pi a \mu} = \frac{\frac{4}{3} \pi a^3 \rho_p}{6\pi a \mu} \tag{3.6}$$

is a characteristic time of translation of the particle.

Integrating again we get the distance from the wall, y_p, reached by the particle:

$$y_p=\tau_p v_{p_0}(x)\,[1-e^{-t/\tau_p}]. \tag{3.7}$$

All particles leaving a given point, $x\geqslant 0$, at $y=0$, reach the limiting height $\tau_p v_{p_0}(x)$, thus constituting a high concentration layer where our equations are no more valid.

We will not consider this concentrated layer here and will restrict our study to the region

$$y<\tau_p v_{p_0}(x) \qquad (\text{for all } x\geqslant 0) \tag{3.8}$$

and to x small enough so that the concentrated layer of particles is not expanding down to the wall.

Let us now write the equations in a non-dimensional form. All velocities are set non-dimensional by taking as a reference quantity u_∞, the fluid velocity at the distance δ. We define u_∞, and thus δ, by requiring u_∞ to be of the order of $v_{p_0}(x)$

$$u_\infty=\bar{v}_{p_0},$$

which does not diminish the generality of the problem. Let

$$\begin{aligned}
&\mathbf{V}=\frac{\mathbf{v}}{u_\infty}, \qquad \mathbf{V}_p=\frac{\mathbf{v}_p}{u_\infty}, \qquad V_{p_0}(x)=\frac{v_{p_0}(x)}{u_\infty},\\
&X=\frac{x}{\delta}, \qquad Y=\frac{y}{\delta}, \qquad P=\frac{p}{\rho u_\infty^2},\\
&f=\frac{m_p n}{\rho}, \qquad f_0(x)=\frac{m_p n_0(x)}{\rho},\\
&\mathrm{Re}=\frac{u_\infty \delta}{\nu}.
\end{aligned} \tag{3.9}$$

$f(X, Y)$ is the mass concentration of particles per unit mass of liquid. $f_0(X)$ is the same quantity at the wall.

Another important parameter appears from non-dimensionalising.

$$S=\frac{\tau_p}{\tau_e}, \tag{3.10}$$

where τ_p, a characteristic time of the particle, has been defined in Eq. (3.6), $\tau_e=\delta/u_\infty$ is a characteristic time of the flow.

This parameter is characteristic of the comportment of the particle in the flow. If $S\gg 1$, the particle is going its own way and is little affected by the fluid flow. If $S\ll 1$, the particle is quickly carried away by the fluid flow. The non-dimensional equations and boundary conditions are written as

$$\nabla\cdot\mathbf{V}=0, \tag{3.11}$$

$$(\mathbf{V}\cdot\nabla\mathbf{V})=-\nabla P+\frac{1}{\mathrm{Re}}\nabla^2\mathbf{V}+\frac{f(\mathbf{V}_p-\mathbf{V})}{S}, \tag{3.12}$$

(3.13) $$\nabla\cdot(f\mathbf{V}_p)=0\,,$$

(3.14) $$(\mathbf{V}_p\cdot\nabla)\mathbf{V}_p=\frac{\mathbf{V}-\mathbf{V}_p}{S}\,,$$

(3.15) $$Y=0\left|\begin{array}{l} U=V=0\,,\\ U_p=0\,,\qquad V_r=V_{p_0}(x)\,,\\ f=f_0(x)\quad\text{for}\quad x\geqslant 0\,,\qquad 0\ \text{for}\ x<0. \end{array}\right.$$

where ∇ has now the components $\partial/\partial X$, $\partial/\partial Y$.

The equations have to be limited to the region

(3.16) $$Y<SV_{p_0}(X)$$

with X not too large.

Note that $V_{p_0}(X)$ is of the order of 1 by definition.

4. Small parameters. Compatibility with other conditions

The system of Eqs. (3.11)–(3.16) is still difficult to solve directly. We will restrict our study to the case

(4.1) $$S\ll 1$$

and will consider the perturbation problem for $S\to 0$.

Physically, we thus consider only the case where particles are quickly carried away by the fluid. But, by the inequality (3.16), we restrict our attention to the very region where they are carried away.

First, we have to check if the condition (4.1) is compatible:

with the assumption we made earlier,

with the condition that the region in the inequality (3.16) we are considering is large enough to treat the particle mixture as a continuum.

The assumptions are:

a) with $A=a/\delta$ (2.1), (2.3)

$$A\ll 1\,,$$

$$\mathrm{Re}<\mathrm{Re}_t\,;$$

b) the inequality (2.7) may be written in terms of f:

(4.2) $$f<\frac{m_p}{\rho}\,\frac{3}{4\pi}\left(\frac{\mathrm{Re}_v}{a}\right)^3,$$
$$f<\frac{\rho_p}{\rho}(\mathrm{Re}_v)^3=\frac{\rho_p}{\rho}(A\,\mathrm{Re})^3\,.$$

c) (2.5)

$$\mathrm{Re}_v=\mathrm{Re}\,A\ll 1\,.$$

d) (2.9)

$$\mathrm{Re}_\gamma = \mathrm{Re}\,A^2 \ll \mathrm{Re}_v^2 = (\mathrm{Re}\,A)^2$$

or

(4.3) $$\mathrm{Re} \gg 1 .$$

e) (2.10)

$$\rho_p/\rho \gg 1 .$$

The condition (4.1) with Eqs. (3.6) and (3.10) is written as

(4.4) $$\frac{2}{9}\,\frac{\rho_p}{\rho}\,\mathrm{Re}\,A^2 \ll 1 .$$

Now to be sure that the region in the inequality (3.16) or (3.8) contains many particles, we have to write that it is much larger than a/Re, as (by assumption b), each fluid sphere of the radius a/Re_v contains only one particle:

$$\frac{a}{\mathrm{Re}_v} \ll \tau_p v_{p_0}(v) ,$$

$$\frac{A}{\mathrm{Re}_v} \ll S V_{p_0}(X) \sim S$$

or

(4.5) $$\mathrm{Re}^2 A^2 \gg \frac{9}{2}\,\frac{\rho}{\rho_p} .$$

The conditions (2.1), (2.3), (2.5), (4.3), (2.10), (4.4) and (4.5) are compatible in a limited region, as may be seen from Fig. 2, drawn for $\rho_p/\rho = 4.5 \times 10^3$ as an example.

The transition Reynolds number has been taken as $Re_t = 5 \times 10^5$ after SCHLICHTING [9], assuming a boundary layer situation on a flat plate, the distance to the leading edge being of the order of δ (or eventually less than δ).

An alternate situation may arise where particles are injected into the laminar sublayer of a turbulent flow. The thickness of the laminar sublayer δ^* is given, after SCHLICHTING [9], by

$$\frac{u^* \delta^*}{\nu} = 5 ,$$

u^* being the friction velocity, $u^* = \sqrt{\tau/\rho}$, τ being the shear stress at the wall. The condition that the particle layer $\tau_p\, v_{p_0}$ is less than the laminar sublayer δ^* is written as

$$\frac{\sqrt{\gamma \dfrac{u_\infty}{\delta} \tau_p \bar{v}_{p_0}}}{\nu} < 5$$

or, as $u_\infty = \bar{v}_p$

$$\frac{\tau_p}{(\delta/u_\infty)}\sqrt{\frac{u_\infty \delta}{\nu}} < 5,$$

$$S\sqrt{\mathrm{Re}} < 5.$$

This condition, replacing the condition $Re < Re_t$ is shown as a dashed line in Fig. 2.

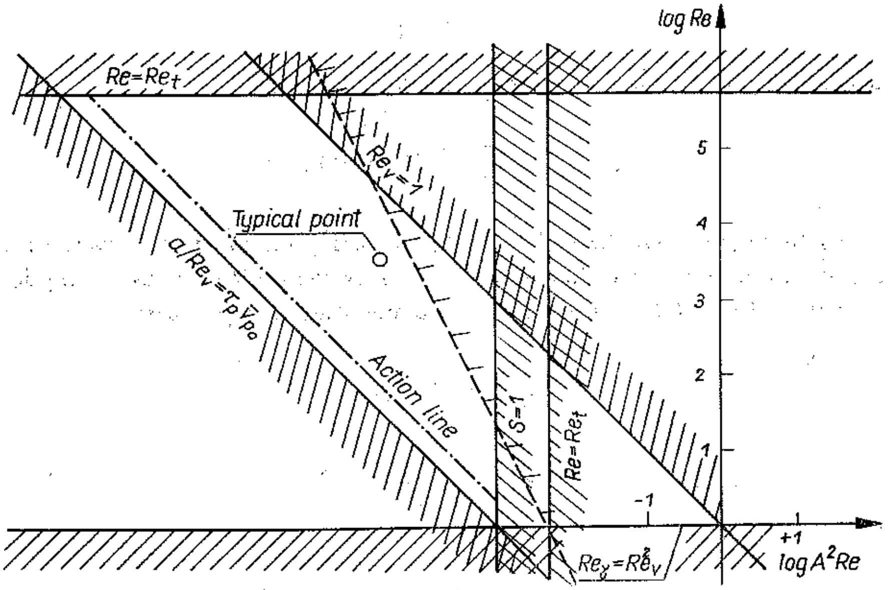

FIG. 2. Region of validity.

A typical point is shown in this figure:

$$\mathrm{Re} = 3.16 \times 10^3,$$

$$\mathrm{A} = 10^{-4}$$

corresponding to the injection of small solid particles of the radius $a = 1\ \mu m$ and the density $\rho_p = 5.8 \times 10^3$ kg/m^3, with injection velocity $\bar{v}_{p_0} = 50.5$ cm/s into air (density $\rho = 1.3$ kg/m^3, kinematic viscosity $\nu = 1.6 \times 10^{-5}$m^2/s) flowing along the wall with a velocity gradient 5.05 s^{-1}.

It may be checked that all conditons are valid within a 10^{-2} precision, except

$$\mathrm{Re}_v = 0.316.$$

But it is known from the experiment [8] that the Stokes force is still valid for Reynolds numbers fairly close to unity.

The only condition not shown in Fig. 2 is the condition (4.2) on the mass concentration of particles. For our numerical example,

$$f < 1.42 \times 10^2.$$

The region we study is of the order

$$\tau_p \bar{v}_{p_0} = 3.16 \text{ mm}.$$

5. Singular perturbation problem. Condition for particle action

If we let $S \to 0$ in Eqs. (3.11) to (3.14), we get for the velocities

(5.1) $$\mathbf{V}_p = \mathbf{V},$$

so that the boundary condition (3.15) at the wall

$$Y=0: \ V=0; \quad V_p = V_{p_0}(X) \neq 0$$

cannot be applied. This is a singular perturbation problem. Let us define the interior coordinate

(5.2) $$\tilde{Y} = \frac{Y}{S^\alpha},$$

α being a positive real constant that we have to find. Y being of the order of S^α, thus small, the velocity of the fluid parallel to the wall will be (in laminar flow) of the order of $U \sim Y \sim S^\alpha$. Therefore, let

(5.3) $$\tilde{U} = \frac{U}{S^\alpha}.$$

The velocity of particles parallel to the wall is smaller than that of the fluid as the fluid entrains them. Thus: $U_p \sim U \sim S^\alpha$.

Let

(5.4) $$\tilde{U}_p = \frac{U_p}{S^\alpha}.$$

The velocity of particles perpendicular to the wall is of the order of 1 by definition:

(5.5) $$V_p \sim V_{p_0} \sim 1.$$

Let also

(5.6) $$f = \bar{f}_0 \tilde{f},$$

$\bar{f}_0$ being of the order of f_0, e.g. $\bar{f}_0$ being the maximum of $f_0(x)$. Let also

(5.7) $$\tilde{V} = \frac{V}{S^{2\alpha}},$$

so that the equation of continuity for the fluid (3.11) is still valid.

The momentum equation for particles (3.14) may be written as

(5.8) $$S^{2\alpha} \tilde{U}_p \frac{\partial \tilde{U}_p}{\partial X} + V_p \frac{\partial \tilde{U}_p}{\partial \tilde{Y}} = S^{\alpha-1}(\tilde{U} - \tilde{U}_p),$$

(5.9) $$S^\alpha \tilde{U}_p \frac{\partial V_p}{\partial X} + \frac{1}{S^\alpha} V_p \frac{\partial V_p}{\partial \tilde{Y}} = \frac{S^{2\alpha} \tilde{V} - V_p}{S}.$$

We require the equations to be the less degenerated possible (VAN DYKE [10]). This gives

$$\alpha=1\,.$$

The interior coordinate (5.2) is

$$\tilde{Y}=\frac{Y}{S}.$$

Thus the region in the inequality (3.16) where the equations are valid is of the order of magnitude of the interior region.

Now let us derive expressions to the first order for flow quantities in this region.

Equation (5.9) is, to the order of S,

$$V_p\frac{\partial V_p}{\partial \tilde{Y}}=-V_p\,.$$

So that for $\tilde{Y}<V_{p_0}(X)$

$$V_p=V_{p_0}(X)-\tilde{Y} \tag{5.10}$$

and for $\tilde{Y}>V_{p_0}(X)$

$$V_p=0\,.$$

The equation of continuity for particles (3.13), is, to the order of S,

$$\frac{\partial \tilde{f}\,V_p}{\partial \tilde{Y}}=0\,,$$

which can be integrated

$$f\,V_p=f_0(X)\,V_{p_0}(X)\,. \tag{5.11}$$

The mass flow rate of particles is perpendicular to the wall, to the order of S.

The concentration of particles is, by Eqs. (5.10) and (5.11),

$$f=\frac{f_0(X)}{1-\dfrac{\tilde{Y}}{V_p(X)}}\,, \tag{5.12}$$

which, as we expected, through Eq. (3.7) gets infinite for $\tilde{Y}\to V_p\ (X)$. This equation was derived for $V\cong 0$. We proved here that the expression (5.7)

$$V\sim S^2,$$

which gives the same result for the order of magnitude we consider.

The momentum equation for the fluid in the y direction (2.12) reduces to

$$\frac{\partial P}{\partial \tilde{Y}}=f\,V_p=f_0(X)\,V_p(X),$$

where Eq. (5.11) has been applied. This is a pressure gradient due to the momentum of particles.

Integrating

$$P = f_0(X)\, V_{p_0}(X)\, \tilde{Y} + P_1(X), \tag{5.13}$$

$P_1(X)$ is an unknown function.

The momentum equation for the fluid in the X direction (2.12) reduces, to the order of S, after using Eqs. (5.12) and (5.13), to

$$0 = -\tilde{Y}\frac{d f_0 V_{p_0}}{dX} - \frac{dP_1}{dX} + \frac{1}{\mathrm{Re}\,S}\frac{\partial^2 \tilde{U}}{\partial \tilde{Y}^2} + f_0(X)\frac{\tilde{U}_p - \tilde{U}}{1 - \dfrac{\tilde{Y}}{V_{p_0}(X)}}. \tag{5.14}$$

The slowing down of the fluid by particles is represented by the last term. This term is of the same order as the others if

$$f_0(X) < \frac{1}{\mathrm{Re}\,S} \tag{5.15}$$

or, as $V_{p_0}(X)$ is of the order of 1,

$$f_0(X)\, V_{p_0}(X) \not\ll \frac{1}{\mathrm{Re}\,S},$$

$$f_0(X)\,[V_{p_0}(X)]^2 \not\ll \frac{1}{\mathrm{Re}\,S}.$$

The particles must have a minimum mass concentration, or momentum, or kinetic energy in order to modify the fluid velocity profile.

We can show that the condition (5.15) is compatible with the assumption of low concentration, Eq. (4.2).

This last equation is written as

$$f_0 < \frac{\rho_p}{\rho}(A\,\mathrm{Re})^3 = f_{0\,\max}\ \text{(say)}.$$

The compatibility between the inequalities (5.15) and (4.2) may be written as

$$f_{0\,\max} < \frac{1}{\mathrm{Re}\,S}$$

or

$$(A\,\mathrm{Re})^5 > \frac{9}{2}\left(\frac{\rho}{\rho_p}\right)^2, \tag{5.16}$$

this is shown in Fig. 2 as the "action line". The typical point is in the region where particle action occurs.

6. Velocity profiles and friction coefficient at the wall

Assume that the condition (5.15) applies and let us then compute the velocity profile of the fluid. The momentum equation of the fluid (5.14) has to be coupled with the momentum equation of the particles (5.8), giving a system for the unknown variables $\tilde{U}$, $\tilde{U}_p$.

Combining both equations we find

$$(6.1)\qquad \frac{V_{p_0}-\tilde{Y}}{\mathrm{Re}\,S}\frac{\partial^2\tilde{U}}{\partial\tilde{Y}^2}+\frac{1}{\mathrm{Re}\,S}\frac{\partial\tilde{U}}{\partial\tilde{Y}}-f_0 V_{p_0}\tilde{U}=\frac{df_0 V_{p_0}}{dX}\tilde{Y}\left(V_{p_0}-\frac{\tilde{Y}}{2}\right)+V_{p_0}\frac{dP_1}{dX}+C,$$

C being a function of X to be determined.

The solution of Eq. (6.1) may be found in a closed form (Appendix 2):

$$(6.2)\qquad \frac{1}{2}\frac{d\ln K_0}{dX}\tilde{Y}^2-V_{p_0}\frac{d\ln K_0}{dX}-\frac{C}{f_0 V_{p_0}}-\frac{\dfrac{dP_1}{dX}}{f_0}+a(V_{p_0}-\tilde{Y})\,I_2(\eta)+\frac{b}{I_2(\eta)},$$

where

$$K_0(X)=\mathrm{Re}\,Sf_0(X)\,V_{p_0}(X),$$

$$\eta=2\sqrt{K_0(V_{p_0}-\tilde{Y})};$$

I_2 is the Bessel function with imaginary argument of order 2; a, b, are functions of X to be determined. $\tilde{U}_p$ may then be found from Eq. (5.14) to be

$$(6.3)\qquad \tilde{U}_p=-\frac{1}{2}\frac{d\ln K_0}{dX}\tilde{Y}^2+\left(\frac{1}{K_0^2}\frac{dK_0}{dX}-\frac{1}{f_0 V_{p_0}}\frac{dP_1}{dX}\right)\tilde{Y}-$$
$$-\left(\frac{C}{f_0 V_{p_0}}+\frac{V_{p_0}}{K_0^2}\frac{dK_0}{dX}\right)-\frac{a}{K_0}I_2(\eta)-\frac{2}{\eta}\left[\frac{a\eta^2}{4K_0}-\frac{b}{[I_2(\eta)]^2}\right]\left[I_1(\eta)-\frac{2}{\eta}I_2(\eta)\right].$$

The functions $a(X)$, $b(X)$, $c(X)$ have to be found using the boundary conditions. Here a problem arises, as we have only two conditions (3.15) at the wall

$$Y=0:\quad \tilde{U}=0;\quad \tilde{U}_p=0.$$

As this is a singular perturbation problem, it is logical to supply a third condition by matching with the outer flow the expression (5.1). This matching condition is written as

$$(6.4)\qquad \tilde{U}-\tilde{U}_p\to 0\quad \text{for}\quad \tilde{Y}\to\infty.$$

But as we have seen the equations are no more valid for $\tilde{Y}\geqslant V_{p_0}(X)$. Thus we have to abandon the matching condition and substitute an information about the concentrated layer of particles at

$$\tilde{Y}\sim V_{p_0}(X).$$

We do not know much about the particles, except that their concentration is high. We may assume that they are travelling at the same speed as the fluid.

(6.5) $$\tilde{U}-\tilde{U}_p \to 0 \quad \text{for} \quad \tilde{Y} \to V_{p_0}(X).$$

This physical condition may be mathematically accepted as it can be regarded as an approximate form of the matching condition (6.4) sometimes used in the boundary layer theory.

The boundary conditions at the wall give

$$a=\frac{1}{\dfrac{dI_2}{d\eta_0}+I_1}\left[\operatorname{Re}SC\frac{2}{\eta_0}\left(\frac{2}{\eta_0}\frac{d\ln I_2}{d\eta_0}-1\right)+\frac{1}{f_0V_{p_0}}\frac{dP_1}{dX}\frac{d\ln I_2}{d\eta_0}-\frac{\eta_0}{2}\frac{1}{K_0^2}\frac{dK_0}{dX}\right],$$

$$b=\frac{1}{\dfrac{dI_2}{d\eta_0}+I_1}\left[\frac{C}{f_0V_{p_0}}I_2\left(\frac{\eta_0}{2}I_2+I_1\right)+\right.$$
$$\left.+\frac{1}{K_0f_0V_{p_0}}\frac{dP_1}{dX}\frac{\eta_0^2}{4}I_1I_2+\frac{1}{K_0^3}\frac{dK_0}{dX}\frac{\eta_0^3}{8}I_2^2\right],$$

where I_1 and I_2 are the Bessel functions with imaginary arguments taken for

$$\eta=\eta_0=2\sqrt{K_0V_{p_0}}.$$

The third condition (6.5) gives

$$b=0$$

and thus C as a function of $\dfrac{dP_1}{dX}$.

For simplicity, let us write the expressions of C, $\tilde{U}$, $\tilde{U}_p$ for uniform injection of particles:

$$\frac{dP_0}{dX}=0; \qquad \frac{dV_{p_0}}{dX}=0.$$

We get:

(6.6) $$C=\frac{\eta_0}{2K_0}\frac{I_1(\eta_0)}{I_0(\eta_0)}\left(-\frac{dP_1}{dX}\right),$$

where I_0 is the Bessel function with imaginary argument of the 0^{th} order:

(6.7) $$\tilde{U}=\frac{1}{f_0}\frac{I_2(\eta_0)}{I_0(\eta_0)}\left[1-\frac{\eta^2I^2(\eta)}{\eta_0^2I_2(\eta_0)}\right]\left(-\frac{dP_1}{dX}\right),$$

(6.8) $$\tilde{U}_p=\frac{1}{f_0}\left\{1-\left(\frac{\eta}{\eta_0}\right)^2+\frac{2I_1(\eta_0)}{\eta_0I_0(\eta_0)}\left[\frac{\eta I_1(\eta)}{\eta_0I_1(\eta_0)}-1\right]\right\}\left(-\frac{dP_1}{dX}\right).$$

Both profiles for the fluid velocity $U=S\tilde{U}$ and for the particle velocity $U_p=S\tilde{U}_p$ are plotted in Fig. 3.

It can be found that C has a physical interpretation. If the solution $\tilde{U}$ of Eq. (6.1) is searched as a series,

$$\tilde{U}=A_1\tilde{Y}+A_2\tilde{Y}^2+\dots+A_n\tilde{Y}^n+\dots$$

it can be found that $A_1=\mathrm{Re}SC$ and thus C is related to the friction coefficient at the wall, C_f:

$$(6.9)\qquad C_f=\frac{\hat{\mu}\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho u_\infty^2}=\frac{\mu}{\rho u_\infty^2}\frac{u_\infty}{\delta}\left(\frac{\partial U}{\partial Y}\right)_{Y=0}=\frac{1}{\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)_{Y=0}=\frac{1}{\mathrm{Re}}\left(\frac{\partial \tilde{U}}{\partial \tilde{Y}}\right)_{\tilde{Y}=0}=SC\,.$$

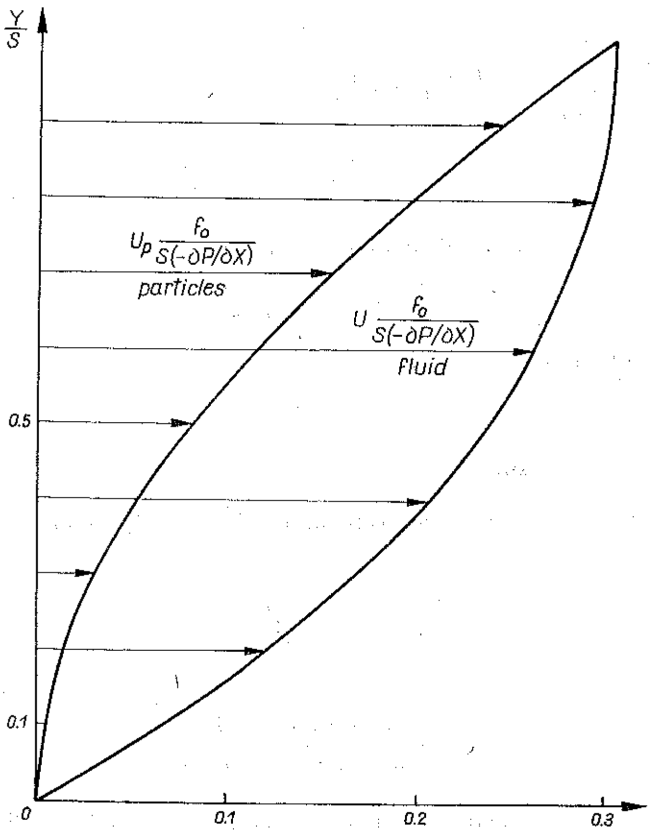

FIG. 3. Velocity profiles for uniform injection of particles $f_0=1/\mathrm{Re}\,S$.

Thus, for constant injection of particles the friction coefficient at the wall is related to the pressure gradient $\frac{dP}{dX}$ by Eqs. (6.6) and (6.9):

$$(6.10)\qquad C_f=\frac{S}{\sqrt{\mathrm{Re}\,Sf_0}}\,\frac{I_1(2V_{p_0}\sqrt{\mathrm{Re}\,Sf_0})}{I_0(2V_{p_0}\sqrt{\mathrm{Re}\,Sf_0})}\left(-\frac{dP}{dX}\right).$$

7. Conclusion

The main results of this paper are the following:

1) A disperse suspension of small identical spherical particles injected from a wall will modify the velocity profile of a fluid flowing along this wall if the mass concentration of injected particles, f_0, is at least of the order 1/Re S which we write as

$$f_0 \not\ll \frac{1}{\mathrm{Re}\, S},$$

where

$$\mathrm{Re} = \frac{\bar{v}_{p_0}\delta}{\nu},$$

$\bar{v}_{p_0}$ being of the order of the injection velocity of particles, δ being the distance from the wall where the fluid velocity is of the order $\bar{v}_{p_0}$. S, a number characteristic of the comportment of a particle relative to the fluid, is defined in the text (Formula (3.10)).

2) For the case $f_0 \ll \frac{1}{\mathrm{Re}\, S}$ (compatible with the assumptions) and for uniform injection of particles, the velocity profiles of the fluid and the particles are computed in a closed form (Formulae (6.7) and (6.8)). The friction coefficient at the wall is also given (Formula (6.10)).

Appendix 1. Forces acting on a dense sphere

We consider here a sphere of the density ρ_p, much larger than the density ρ of the surrounding fluid. We assume that the Reynolds number of the flow around the sphere is small.

We want to show that the forces acting on the sphere are due only to the permanent fluid motion.

The Navier-Stokes equations for the flow around the sphere are written as

$$\nabla \cdot \mathbf{v} = 0,$$

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = - \nabla_p + \mu \nabla^2 \mathbf{v}$$

with the boundary conditions on the sphere, $r=a$, $\mathbf{v}=\mathbf{v}_p+\boldsymbol{\omega}_p \times \mathbf{r}$ at infinity, $r \to \infty$, $\mathbf{v} \to \mathbf{v}_\infty$ (given fluid velocity)

Notations are the usual ones. $\mathbf{r}$ is a vector originating from the sphere center; $r=|\mathbf{r}|$; a is the sphere radius. v_p is the sphere translational velocity, $\boldsymbol{\omega}_p$ is the sphere rotational velocity.

To get the non-dimensional equations we choose the following reference quantities: a for the length, u_∞ a characteristic fluid velocity, for the reference velocity, τ_p a characteristic time of the particle for the reference time. We choose this scale of time as we are mainly interested in the particle motion.

Let $\tau_v = a^2/\nu$ be a characteristic time for viscosity.

Let

$$\mathbf{R}=\frac{\mathbf{r}}{a}; \quad \mathbf{V}=\frac{\mathbf{v}}{u_\infty}; \quad T=\frac{t}{\tau_p},$$

$$\mathrm{Re}_v=\frac{u_\infty a}{\nu} \quad \text{(we assume } \mathrm{Re}_v \ll 1),$$

$$P=\frac{p}{\rho u_\infty^2}\,\mathrm{Re}_v,$$

$$\mathbf{V}_p=\frac{\mathbf{v}_p}{u_\infty}; \quad \mathbf{\Omega}_p=\frac{\boldsymbol{\omega}_p a}{u_\infty}; \quad \mathbf{V}_\infty=\frac{\mathbf{v}_\infty}{u_\infty}.$$

The non-dimensional equations and boundary conditions are written as

$$\nabla\cdot\mathbf{V}=0,$$

$$\frac{\tau_v}{\tau_p}\frac{\partial \mathbf{V}}{\partial T}+\mathrm{Re}_v(\mathbf{V}\cdot\nabla)\,\mathbf{V}=-\nabla P+\nabla^2\mathbf{V}.$$

∇ now being the vector of the components $\partial/\partial X$, $\partial/\partial Y$, $\partial/\partial Z$ $(X=x/a, \ldots)$

$$R=1 \quad \mathbf{V}=\mathbf{V}_p+\mathbf{\Omega}_p\times\mathbf{R},$$

$$R\to\infty \quad \mathbf{V}\to\mathbf{V}_\infty.$$

For small Re_v, we get as a first approximation the Stokes equations.

In the solution we get the Stokes force plus some other nonstationary forces acting on the sphere

$$\mathbf{F}=\underbrace{-6\pi a\mu(\mathbf{v}_p-\mathbf{v}_l)}_{\text{Stokes}}\underbrace{-6a^2\rho\sqrt{\pi\nu}\int_0^t\frac{\frac{d}{d\theta}(\mathbf{v}_p-\mathbf{v}_l)}{\sqrt{t-\theta}}\,d\theta}_{\text{Basset/Villat}}\underbrace{-\frac{2}{3}\pi a^3\rho\frac{d}{dt}(\mathbf{v}_p-\mathbf{v}_l)}_{\text{added mass}}+\underbrace{\frac{4}{3}\pi a^3\rho\frac{d\mathbf{v}_l}{dt}}_{\text{acceleration pressure field}},$$

$\mathbf{v}_l$ being the velocity of the unperturbed flow field.

$$\mathbf{F}=-6\pi a\mu\left\{(\mathbf{v}_p-\mathbf{v}_l)+\sqrt{\frac{\tau_v}{\pi\tau_p}}\int_0^T\frac{\frac{d}{d\Theta}(\mathbf{v}_p-\mathbf{v}_l)}{\sqrt{T-\Theta}}\,d\Theta+\frac{1}{9}\frac{\tau_v}{\tau_p}\frac{d}{dT}(\mathbf{v}_p-\mathbf{v}_l)-\frac{\tau_v}{\tau_p}\frac{d\mathbf{v}_l}{dT}\right\}.$$

For $\tau_p \gg \tau_v$, the Stokes force is the predominant one and the equation of dynamics for the sphere translation

$$\frac{4}{3}\pi a^3 \rho_p \frac{d\mathbf{v}_p}{dt} = -6\pi a\mu(\mathbf{v}_p - \mathbf{v}_l)$$

gives the characteristic time

$$\tau_p = \frac{4/3\,\pi a^3 \rho_p}{6\pi a\mu} = \frac{2}{3}\tau_v \frac{\rho_p}{\rho},$$

we see that the Basset force is of the order $\sqrt{\rho/\rho_p}$ as compared to the Stokes force, and the added mass and acceleration pressure field are of the order ρ/ρ_p. The action on the sphere is thus to the order $\sqrt{\rho/\rho_p}$ due only to permanent fluid motion.

Appendix 2. General solution of the differential equation

(1) $$(V_{p_0} - \tilde{Y})\frac{\partial^2 \tilde{U}}{\partial \tilde{Y}^2} + \frac{\partial \tilde{U}}{\partial \tilde{Y}} - K_0 \tilde{U} = 0 .$$

Let $Z = V_{p_0} - \tilde{Y}$. Equation (1) becomes

(2) $$Z\frac{\partial^2 \tilde{U}}{\partial Z^2} - \frac{\partial \tilde{U}}{\partial Z} - K_0 \tilde{U} = 0 .$$

We look for a solution in the form of a series:

(3) $$\tilde{U}_1 = Z^\alpha(1 + a_1 Z + a_2 Z^2 + \ldots + a_n Z^n + \ldots),$$

where α and the $a_n's$ are constants.

Putting the expression (3) into Eq. (2) gives the following requirements for the constant α and the coefficients a_n:

(4) $$\alpha(\alpha - 2) = 0,$$

(5) $$a_1(\alpha+1)(\alpha-1) = K_0,$$

(6) $$a_2(\alpha+2)\alpha = K_0 a_1,$$

. . . .

(7) $$a_{n+1}(\alpha+n+1)(\alpha+n-1) = K_0 a_n .$$

Equation (4) gives $\alpha=0$ or $\alpha=2$. $\alpha=0$ gives with Eqs. (5) and (6) $K_0 = -a_1 = 0$ which is impossible, as $K_0 \neq 0$. Thus $\alpha=2$. Next, we get

$$a_1 = \frac{K_0}{3},$$

$$a_2 = \frac{K_0^2}{(4\times 2)\times 3},$$

. . . .

$$a_{n+1} = \frac{2K_0^{n+1}}{(n+3)!\,(n+1)!}.$$

A solution to Eq. (2) is therefore

$$\tilde{U}_1 = Z^2\left(1+\frac{K_0}{3}Z+\frac{K_0^2}{24}Z^2+\dots+\frac{2K_0^n}{(n+2)!\,n!}Z^n+\dots\right).$$

This series can be related to the Bessel function with an imaginary argument of order (2):

$$I_2(t)=\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+3)}\left(\frac{t}{2}\right)^{2+2m}=\sum_{m=0}^{\infty}\frac{1}{m!\,(m+2)!}\left[\left(\frac{t}{2}\right)^2\right]^{m+1},$$

which has an infinite radius of convergence. We then get

$$\tilde{U}_1=\left[\sum_{n=0}^{\infty}\frac{2}{n!\,(n+2)!}(K_0\,Z)^{n+1}\right]\frac{Z}{K_0}=\frac{2Z}{K_0}I_2(2\sqrt{K_0\,Z}). \tag{8}$$

Let us search the other solution to Eq. (2) as a function:

$$\tilde{U}_2=f(Z)\times\tilde{U}_1. \tag{9}$$

Putting back Eq. (9) into Eq. (2) we get

$$f(Z)\left[Z\frac{\partial^2\tilde{U}_1}{\partial Z^2}-\frac{\partial\tilde{U}_1}{\partial Z}-k_0\,\tilde{U}_1\right]+[Zf''(Z)-f'(Z)]\,\tilde{U}_1+ +2f'(Z)\,Z\frac{\partial\tilde{U}_1}{\partial Z}=0. \tag{10}$$

The first term of Eq. (10) is zero by Eq. (2). The next terms give, after integration,

$$f(Z)=\frac{Z}{\tilde{U}_1^2}.$$

Thus

$$\tilde{U}_2=\frac{Z}{\tilde{U}_1}.$$

The general solution of Eq. (2) is given by

$$\tilde{U}=C_1\,\tilde{U}_1+C_2\,\tilde{U}_2=C_1\frac{2Z}{k_0}I_2(2\sqrt{K_0\,Z})+\frac{C_2\,K_0}{2I_2(2\sqrt{K_0\,Z})},$$

where C_1, C_2 are constants.

The general solution of Eq. (1) is then

$$\tilde{U}=a(V_{p_0}-\tilde{Y})\,I_2(2\sqrt{k_0(V_{p_0}-\tilde{Y})})+\frac{b}{I_2(2\sqrt{K_0(V_{p_0}-\tilde{Y})})}.$$

where a, b are constants.

The following formula is used in the text:

$$Z\frac{d}{dZ}I_n(Z)+nI_n(Z)=ZI_{n-1}(Z)$$

see (GRADSHTEYN [11]).

I_n is the Bessel function with imaginary argument of order n.

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Streszczenie

WTRYSKIWANIE MIESZANINY MAŁYCH CZĄSTEK DO STRUMIENIA CIECZY JAKO PROBLEM PERTURBACYJNY

Małe, identyczne, twarde cząstki kuliste są wtryskiwane ze ścianki do strumienia nieściśliwej, lepkiej cieczy. Założono, że koncentracja cząstek jest mała. Celem jest zbadanie oddziaływania cząstek na profil prędkości cieczy.

Równania przyjęte do opisu mieszaniny ciecz-zawiesina redukują wpływ każdej cząstki do siły Stokesa, działającej w jej środku. Do rozwiązania równań wyjściowych zastosowano metodę perturbacji żądając, by bezwymiarowe liczby $S=\tau_p/\tau_e$, gdzie τ_p jest czasem charakterystycznym dla zawiesiny, a τ_e czasem charakterystycznym przepływu, były wielkościami małymi. Wykazano, że warunek ten jest zgodny z założeniami, lecz równania dla mieszaniny są poprawne tylko wewnątrz ograniczonego obszaru w pobliżu ścianki. Wyniki dowodzą, że cząstki oddziaływują na profil prędkości cieczy, jeśli koncentracja masy cząstek przy ściance jest co najmniej rzędu $1/\mathrm{Re}\, S$, gdzie Re jest liczbą Reynoldsa strumienia cieczy. Dla przypadku, gdy ten warunek jest spełniony, policzono profile prędkości, a rozwiązanie przedstawiono w postaci zamkniętej. Pokazano kilka typowych profili prędkości dla równomiernego wstrzyknięcia cząstek.

Резюме

ИНЖЕКЦИЯ СМЕСИ МАЛЫХ ЧАСТИЦ В ПОТОК ЖИДКОСТИ КАК ПЕРТУРБАЦИОННАЯ ЗАДАЧА

Малые, идентичные, жесткие сферические частицы подлежат инжекции из стенки в поток несжимаемой вязкой жидкости. Предположено, что концентрация частиц мала. Целью работы является исследование взаимодействия частиц с профилем скорости жидкости. Уравнения, принимаемые для описания смеси жидкость-взвесь, сводят влияние каждой частицы к силе Стокса действующей в её центру. Для решения исходных уравнений применен метод пертурбаций, требуя, чтобы безразмерное число $S=\tau_p/\tau_e$, где τ_p—характеристическое время для взвеси, а τ_e—характеристическое время речения, было малой величиной. Показано, что это условие совпадает с предположениями, но уравнения для смеси справедливы только внутри ограниченной области вблизи стенки. Результаты показывают, что частицы взаимодействуют с профилем скорости жидкости, если концентрация массы частиц при стенке по крайней мере порядка $1/\mathrm{Re}S$, где Re—число Рейнольдса потока жидкости. Для случая, когда это условие удовлетворено, вычислены профили скорости ,а решение представлено в замкнутом виде. Показано несколько типичных профилей скорости для равномерной инжекции частиц.

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