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STABILITY CONSTRAINTS IN OPTIMIZATION OF CRACKED COLUMNS SUBJECTED TO COMPRESSIVE FOLLOWER LOAD

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Maximization of the critical force of cracked columns, subjected to generalized follower force is discussed in the paper. The crack is assumed to be formed according to the opening and sliding modes and is modeled by a localized loss of stiffness. Influence of the crack stiffness and its localization on the value of the critical force is analyzed. The optimization process is based on the multimodal approach. The localization of crack with the critical force of the system equal to 137.17 EJ/L^2 is found.

1. Introduction

The structure considered in the paper is loaded by a follower force, i.e. the force that moves with the body on which it acts and that preserves the same attitude to this body during the loading process. In the present model, the force inclination and its eccentricity are proportional to the actual structural displacements. The contact force between the rail and the wheel, the reaction force of the bridge span support, the force of the vapour pressure acting in the Laval rotor or friction force of fluid against the pipes in fluid conveying pipes, are examples of the considered loading in mechanical systems. Some of them are discussed in literature, see references in [1] and [2]. Stability and optimization of systems subjected to follower forces have been intensively developed for the last three decades, whereas new results introduce an improvement in the stability criteria and new research in optimization.

In literature, the optimization of nonconservative systems is often developed on example of a column subjected to a concentrated compressive follower load, tangential to the deflection curve. This simple structural model leads however to some difficulties in structural and numerical modeling. It is due to the fact that the system has no potential and is governed by non-selfadjoint differential equations. For example, differences of about 20% in the value of the critical force are obtained by the authors of [3] and [4], by using the same optimality criteria but different methods of solution: the variational one and FEM.

It is observed that the result of optimization of nonconservative systems depends strongly on the starting point in the space of design variables, i.e. initial dimensions of the column. The proper choice of initial parameters can be essential for analysis. For example, a single crack located in the column span can increase the value of critical force of the column or beam subjected to follower compression [5, 14]. The optimal shape of a stepped column, shown in Fig. 1a, was found by Bogacz *et al.* [6]. Probably the first result in the class of continuous variation of the column cross-sections was obtained by Życzkowski and Gajewski [7]. As far as the authors know, the best present result of the column shape optimization, shown in Fig. 1b, is reported by Tada *et al.* [8].

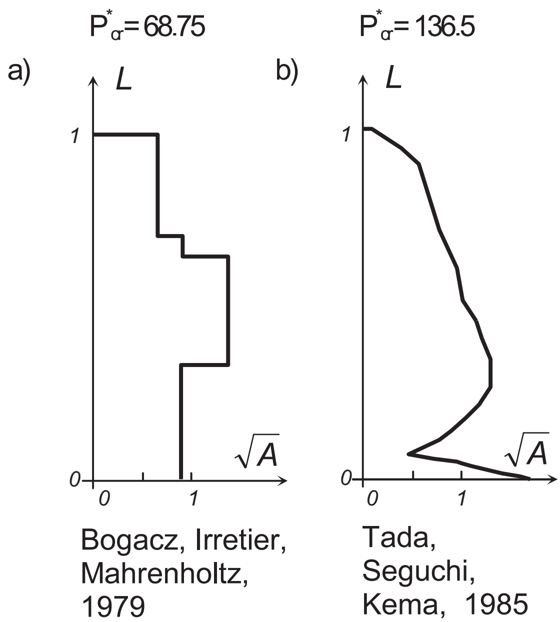

Fig. 1. Selected results of shape optimization of column subjected to follower compression.

In the present consideration the design variables, crack localization and stiffness are searched through to find the maximum value of the critical force. Crack is modeled by a localized loss of stiffness as an elastic joint with possible rotational and shear deformation. The critical force is determined from the configuration of characteristic curves on the force-frequency plane, which are obtained from the characteristic equation of the problem. The flutter force, the local maximum of characteristic curve graph, occurs at nonzero eigenfrequency, whereas buckling occurs at the eigenfrequency equal to zero. In the multimodal analysis 4–6 first eigenfrequencies are considered. More details of the optimization process are reported in Sec. 3.

2. Formulation of the problem

2.1. Model of the column

The considered model of column consists of segments of length l_i connected by elastic joints. The column is subjected to a generalized, follower compressive force P. The equation of motion for the i-th column segment, for small harmonic vibrations, has the form

$$\frac{\partial^2}{\partial x^2}\left(EI_i\frac{\partial^2 y_i}{\partial x^2}\right)+P\frac{\partial^2 y_i}{\partial x^2}+\rho A_i\frac{\partial^2 y_i}{\partial t^2}=0, \tag{2.1}$$

where ρA_i denotes the mass per unit length and EI_i – stiffness of the i-th segment. In the further analysis the cross-sections, masses and stiffnesses of the segments are assumed to be the same in the whole column. The separation of variables is considered in the form

$$y_i(x,t)=w_i(x_i)\exp(i\omega t), \tag{2.2}$$

where ω is the angular frequency. The exact solution for the segment of uniform mass and stiffness distribution is given by

$$w_i(x)=A_1\sinh\lambda_1 x_i+A_2\cosh\lambda_1 x_i+A_3\sin\lambda_2 x_i+A_4\cos\lambda_2 x_i, \tag{2.3}$$

where

$$\lambda_{1/2}=\sqrt{\pm\frac{P}{2EI_i}+\sqrt{\left(\frac{P}{2EI_i}\right)^2+\frac{\rho A_i\omega^2}{EI_i}}}. \tag{2.4}$$

2.2. Boundary conditions

The considered model of loading, the so-called generalized follower force, includes the variation of both the force horizontal displacement e and the force inclination at an angle of χ, as shown in Fig. 2a, [9]. These two parameters depend on both the actual generalized displacements, f and α, of the structure at the point of the force application. The bending moment $M=Pe$ and the transversal force component $H=P(\alpha-\chi)$ are finally included in the boundary conditions

$$w(0,t)=0, \qquad \frac{\partial}{\partial x}w(x_1,t)_{/x_1=0}=0, \tag{2.5}$$

$$\begin{gathered}\frac{\partial^2}{\partial x^2}w(x_n,t)_{/x_n=L}+\lambda\left[\delta\frac{\partial}{\partial x}w(x_n,t)+\vartheta w(x_n,t)\right]_{x_n=L}=0,\\ \frac{\partial^3}{\partial x^3}w(x_n,t)_{/x_n=L}+\lambda\left[\mu\frac{\partial}{\partial x}w(x_n,t)+\gamma w(x_n,t)\right]_{x_n=L}=0,\end{gathered} \tag{2.6}$$

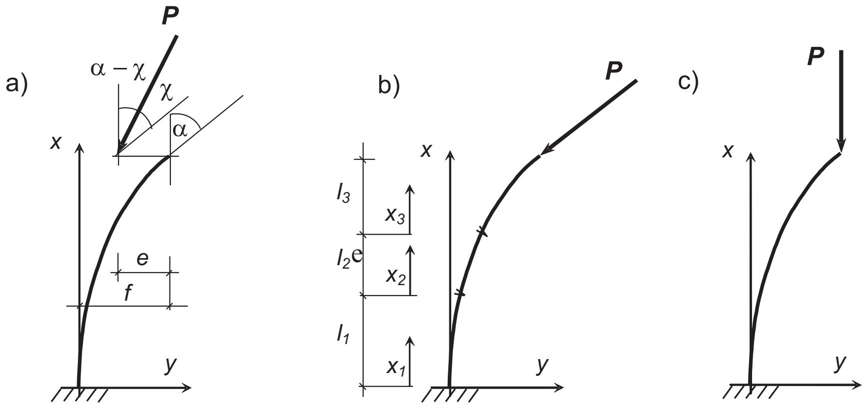

FIG. 2. a) Model of column subjected to generalized follower load, b) Segmentation of the structure on example of Beck column, c) Euler column.

where δ, ϑ, μ, γ are the non-dimensional parameters and L is the length of the column. In this way, values of the parameters δ, ϑ, μ, γ determine the boundary conditions.

Taking into account that the loading is an inseparable part of the whole system, the type of the boundary condition implies its conservativeness or nonconservativeness. Let us consider an operator F_x of the Eq. (2.1)

$$F_x = \frac{\partial^2}{\partial x^2}\left(EI_i \frac{\partial^2 y_i}{\partial x^2}\right) + P\frac{\partial^2 y_i}{\partial x^2}. \tag{2.7}$$

The system is conservative if the operator (2.7) with respect to boundary conditions (2.5) and (2.6) is selfadjoint. E.g. for $e = 0$ and $\chi = \alpha$ ($\delta = \vartheta = \gamma = 0$, $\mu = 1$), the condition describes a conservative Euler column, shown in Fig. 2c, subjected to a force applied to the free end, the direction of which does not change during the loading process. For $e = 0$, $\chi = 0$ ($\delta = \vartheta = \gamma = \mu = 0$) the condition describes a nonconservative Beck column, shown in Fig. 2b, loaded by a force tangent to the column free end. This model will be considered in the following section.

2.3. *Model of the crack*

We assume that a crack was formed according to the opening mode and the sliding mode of development of the crack, so that in the mechanical model presence of the crack is expressed by a discontinuity in displacement and slope. Due to the fact that the column is loaded by a follower force, at location of the crack, $x = x_C$, the change of slope and shear depends on the rotary stiffness

χ_R as written in formula (2.8a), whereas the change of deflection and bending depends on the shear stiffness χ_S, as given in formula (2.8b):

$$\begin{aligned} w^I_{i+1}(x_C) - w^I_i(x_C) &= -\gamma_R w^{II}_{i+1}(x_C), \\ w^{III}_{i+1}(x_C) - w^{III}_i(x_C) &= -\frac{P}{EJ}\gamma_R w^{II}_{i+1}(x_C), \end{aligned} \tag{2.8a}$$

$$\begin{aligned} w_{i+1}(x_C) - w_i(x_C) &= \gamma_S w^{III}_{i+1}(x_C), \\ w^{II}_{i+1}(x_C) - w^{II}_i(x_C) &= \frac{P}{EJ}\gamma_S w^{III}_{i+1}(x_C). \end{aligned} \tag{2.8b}$$

where $(.)^I = d(.)/dx$ and $\gamma_R = 1/\chi_R$, $\gamma_S = 1/\chi_S$ represent additional flexibilities of the column due to the crack, which can be calculated on the basis of fracture mechanics. A model which is valid for a beam with a transverse open crack is discussed in [10, 11].

2.4. Segmentation of the structure and design variables

In what follows, the dimensionless quantities are used:

$$\lambda^* = PL^2/EI, \qquad \omega^{*2} = \omega^2 \rho A L^4/EI, \qquad u_i = w_i/L, \qquad \varepsilon_i = x_i/L. \tag{2.9}$$

The problem can now be rewritten in the following final form:

$$[u^{II}_i(\varepsilon_i)]^{II} + \lambda^* u^{II}_i(\varepsilon_i) - \omega^{*2} u_i(\varepsilon_i) = 0, \qquad 0 \le \varepsilon_i \le 1,$$

$$u_1(0) = u^I_1(0) = u^{II}_n(1) = [u^{II}_n(1)]^I = 0,$$

$$\begin{aligned} u^I_{i+1}(\varepsilon_C) - u^I_i(\varepsilon_C) &= -\gamma^*_R u^{II}_{i+1}(\varepsilon_C), \\ u^{III}_{i+1}(\varepsilon_C) - u^{III}_i(\varepsilon_C) &= -\frac{P}{EJ}\gamma^*_R u^{II}_{i+1}(\varepsilon_C), \\ u_{i+1}(\varepsilon_C) - u_i(\varepsilon_C) &= \gamma^*_S u^{III}_{i+1}(\varepsilon_C), \\ u^{II}_{i+1}(\varepsilon_C) - u^{II}_i(\varepsilon_C) &= \frac{P}{EJ}\gamma^*_S u^{III}_{i+1}(\varepsilon_C), \end{aligned} \tag{2.10}$$

where $\gamma^*_i = \gamma_i EI/L$, $\chi^*_i = 1/\gamma^*_i$ are the dimensionless parameters of the joint flexibility and stiffness, respectively.

We look for the cracks localization ε_C and stiffnesses χ^*_R, χ^*_S which maximize the critical load under the following constraints: a constant total mass of the column, a constant column cross-section A and the total column length L. For n cracks this condition is written as

$$\sum_{i=1}^{n+1} l_i = 1, \tag{2.11}$$

where l_i is the length of the i-th column segment. For the localization of the crack at one of the column ends, $l_1 = 0$ or $l_{n+1} = 0$. The design variables of the problem are

$$\Delta\alpha \in \{\varepsilon_C, \chi_R^*, \chi_S^*\}. \tag{2.12}$$

3. Optimality criteria based on multimodal analysis

Consider the optimization of a nonconservative Beck column under the constraints of dynamic stability. The objective function $\lambda_{\text{cr}}^* = \lambda_{\text{cr}}^*(\Delta\alpha)$, where $\Delta\alpha$ stands for a set of design variables, is not defined explicitly. The optimization conditions, imposed in the frequency domain, are defined as limitations on variation of the shape of characteristic curves. Tada *et al.* [12] proposed a definition of the optimal point as the one that represents the state for which all pairs of eigenvalues become double roots with the same value of critical forces, as shown in Fig. 3. In the present research it means that we should determine localization and stiffnesses of the joint, for which the successive double roots have equal values:

$$\lambda_{12}^* = \lambda_{34}^* = \lambda_{56}^* = ..., \tag{3.1}$$

where λ_{ij}^* denotes the critical load corresponding to the i-th and j-th frequency branches in the force-frequency plane.

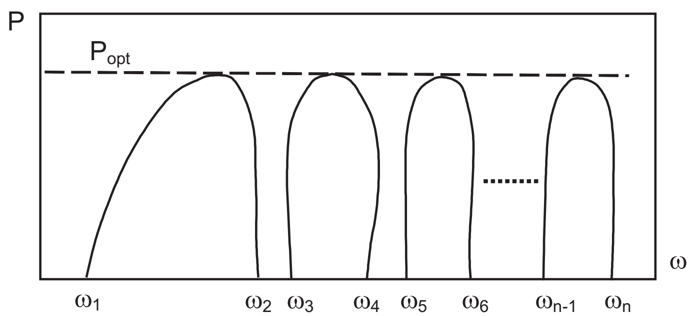

Fig. 3. Configuration of characteristic curves for the optimal shape.

The authors of [13] have noticed that for the developed cracks, flutter can occur with frequency tending to zero. Increasing of the joint stiffness produces increasing of the flutter frequency. The condition of optimal structure can now be rewritten in the following form

$$\lambda^*_{01} = \lambda^*_{23} = \lambda^*_{45} = \lambda^*_{67} = ... \tag{3.2}$$

The first term λ^*_{01} of formula (3.2) can be understood as flutter occurring with frequency tending to zero. Notice that this term can have a second meaning. With respect to divergent-flutter systems it can be understood as buckling for which the frequency is equal to zero.

In first optimizations the authors increased the value of λ^*_{12} like in [3] or $\lambda^*_{12} = \lambda^*_{34}$, similarly to [4]. In the analysis presented in [6] and [12] the first six eigenfrequencies were considered. Finally, we reduce the problem to increasing the value of λ^*_{opt} under the condition

$$\lambda^*_{\mathrm{opt}} \leq \lambda^*_{ij}, \tag{3.3}$$

where $\lambda^*_{\mathrm{opt}} = \lambda^*_{01}$, λ^*_{12} or λ^*_{23}.

Notice that the assumption (3.3) is very useful in calculations since the necessity of keeping the first two or three first values of the critical force within the range of acceptable accuracy can be eliminated.

The next constraint is introduced to preserve a high value of the critical force against the shape perturbation. Due to possible interactions between the successive characteristic curves, a discontinuous decrease in the critical force value can appear. The assumption of minimal distance between two successive characteristic curves prevents such interactions. The formula for a sufficient distance between two curves is as follows:

$$\omega^*_{i+1} - \omega^*_i \geq c, \tag{3.4}$$

where c is a positive number and i denotes the i-th frequency branch. Such formulation of the condition was introduced in [12]. It is seen that a switch-over of characteristic curves resulting in determination of a critical force higher than λ^*_{opt} of formula (3.3) is permitted.

4. Numerical examples – maximization of the critical force of a cracked column

Let us consider the process of maximization of the critical force of a cracked column. The optimization process consists of two steps. In the first one, possible combinations of the design variables $\Delta\alpha_i$ are checked to find the values for which the critical force takes a higher value. In the second step, this result is taken as the initial guess for the gradient procedure. The procedure selects the design parameter $\Delta\alpha$ by analyzing the configuration of characteristic curves and satisfying the conditions (3.2), (3.3) and (3.4). Notice that the condition (3.3) guarantees only a non-negative value of the gradient of the objective function and not the largest one. Discontinuous changes in the critical force are prevented by the condition (3.4). The obtained maximum value of the critical force is equal to $\lambda^*_{\rm cr} = 137.17\ {\rm EI/L^2}$ for the crack localization of $\varepsilon_C = 0.382$ and stiffnesses $\chi^*_S = 0.01$ and $\chi^*_R = 0.0001$. The column is shown in Fig. 4a and the corresponding configuration of the characteristic curves is depicted in Fig. 4b. The value of the critical force is compared with the best result obtained for continuous mass distribution, reported by TADA *et al.* [8], shown in Fig. 1b.

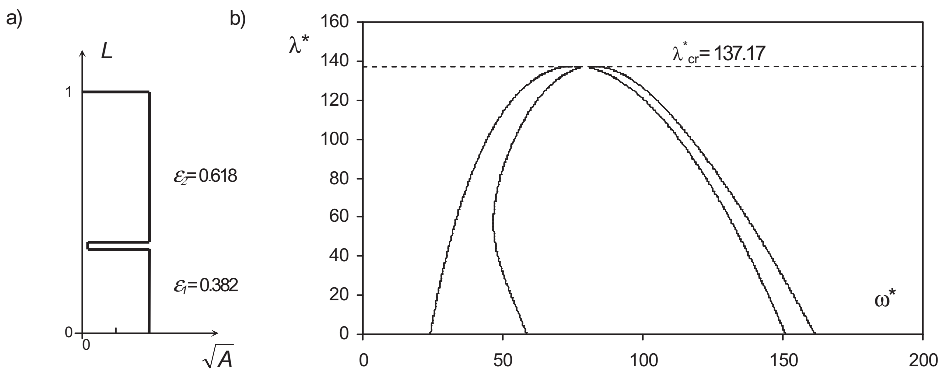

FIG. 4. Optimal localization of crack and respective shape of characteristic curves.

Notice that this result could be improved when higher frequencies would be taken into consideration. However, this requires greater computational efforts. On the other hand, due to a high sensitivity, such optimality is questionable from the viewpoint of the structure reliability and safety.

5. Discussion of stability of the cracked columns and conclusions

The authors of [13] analyzed the problem of stability of column with localized loss of stiffness described by a model similar to that given by expression (2.8). On the force-frequency plane they observed two kinds of characteristic curves. One of them can be attributed to a column with a joint located at the fixed end, whereas the second kind concerns the column with a joint located at the free end. Notice that the second localization of the joint does not influence the stability of the column, the configuration of characteristic curves is the same as that for a uniform Beck column. By changing the position of the joint, the characteristic curves of one kind approach the origin of coordinates whereas the curves of second kind move away from it. The same phenomenon is observed for the present model. In Fig. 5 the configurations of characteristic curves are shown on the example of a column with optimal localization of the crack, considered in the previous chapter. The curves of one kind are depicted by black line whereas the curves of the second kind – by a gray line.

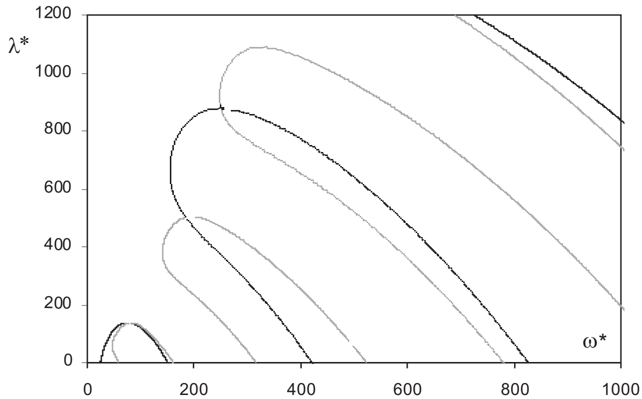

Fig. 5. Two kinds of eigencurves of column with optimal localization of crack.

An interesting observation concerns the stability of a column with crack which was formed according to the sliding mode, modeled by the loss of the shear stiffness, i.e. for $\chi_R^* \to \infty$ and arbitrary χ_S^*. Any localization of the joint and every value of the shear stiffness do not influence the stability of the system. For each joint location and its stiffness, the shape of characteristic curves and the value of critical force are the same as for a uniform column.

However, when the rotary stiffness of the crack is finite, the shear stiffness influences the shape of characteristic curves. An example is shown in Fig. 6,

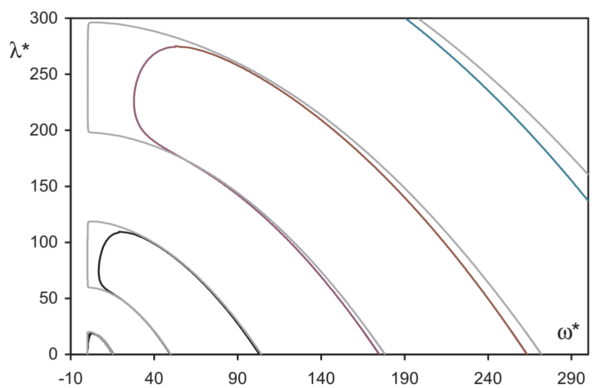

Fig. 6. Characteristic curves of column with crack placed at $\varepsilon_1 = 0.0$ for $\chi_R^* = 1e-4$, $\chi_S^* \to \infty$ (gray) and $\chi_R^* =$ 1e-4, $\chi_S^* =$ 1e4 (black).

where the shape of characteristic curves for a joint of rotary stiffness $\chi_R^* =$ 1e-4 placed at $\varepsilon_C = 0.0$ and two different values of shear stiffness $\chi_S^* =$ 1e10 and $\chi_S^* =$ 1e4 are shown. It is seen that the column with hinge-joint, $\chi_S^* \to \infty$, loses stability by divergence. The successive critical forces occur with frequencies approximately equal to zero. Release of the shear flexibility causes that successive critical frequencies arise.

Let us consider the column with a crack which was formed according to the opening mode, modeled by the localized loss of the rotary stiffness, i.e. for $\chi_S^* \to \infty$ and arbitrary χ_R^*. The critical load versus dimensionless joint location ε_C is shown in Fig. 7. Notice that the critical load decreases considerably for $\varepsilon_C > 0.3$. Discontinuous changes of the critical force occur when the crack of $\chi_R^* = 0.1$ and $\chi_R^* =$ 1e-4 is located near the free end of the column. The discontinuities on the chart result from the qualitative changes of the shape of characteristic curves. The phenomenon is described e.g. in [5, 6] or [13].

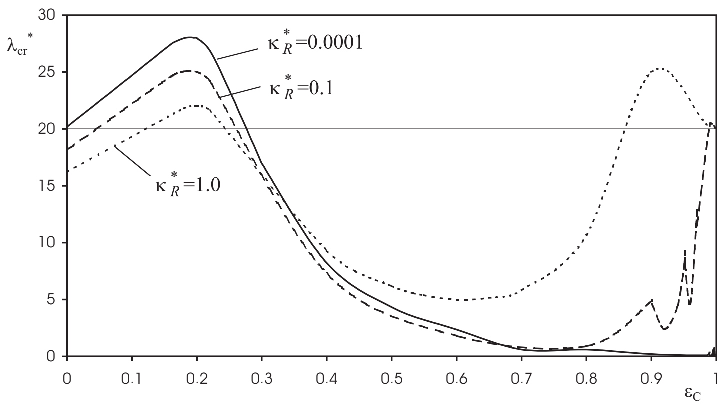

FIG. 7. Critical force versus hinge-joint localization for various joint stiffnesses.

Let us observe the shape of the curve plotted for the stiffness of $\chi_R^* = 1.0$. An analysis of this shape can give some information concerning optimal volume distribution of the column. The simplest way to increase the value of the critical force of the column of constant length and constant Young modulus, is to increase the moment of inertia, namely to enlarge the cross-sections of the column. We have concluded that the column cross-sections should be enlarged in these coordinates, for which the critical force is lower than that for a uniform column. On the other hand, the volume of the column can be taken away in these coordinates for which the critical force is higher than that for a uniform column. According to this, the optimal shape of the column relates directly to the shape of the curve plotted in Fig. 7.

In the modern mechanical systems the follower load can be caused e.g. by systems of active control. The present analysis shows that in the class of slender columns subjected to follower load, the existence of a crack not always destroyed the structure. There are localizations of the crack, for which the critical force is higher than that of a uniform column. Designer can initialize such localization of the crack for which the cracked element under the action of the follower load will not be destroyed.

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