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MAXIMUM STIFFNESS BEAM-COLUMNS

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The problem considered is that of maximising the flexural stiffness of a simply supported elastic bar of given length and volume which is acted upon simultaneously by an axial compression and a transverse uniformly distributed force. The cross-section of the bar is of either sandwich or solid construction. Simple formulas are presented to calculate the minimum midspan deflection for prescribed axial compression and transverse force. It is shown that by maximising the flexural stiffness of the bar its lateral deflection can be substantially decreased.

1. Introduction

Over the past two decades considerable attention has been paid to the optimal design of structural members which had to act as a beam, column or plate (ref. [1–7]). Optimal design meant both the design which used the least amount of material to fulfil certain prescribed requirements such as restrictions on frequencies of natural vibrations, static deflection, buckling load and so forth and the design which, for a given volume of material, had the best mechanical properties (the maximum buckling load, the least static deflection, etc.). More recently, optimal design of structural members which have to perform several functions at different times during their design life has been considered (ref. [8–13]). Thus, for example, the member was expected to act as a beam for a part of its design life and as a column for the rest, but was not expected to perform both these functions simultaneously.

However, a very common situation in the design of structures is when a member is simultaneously acted upon by axial and lateral forces. Such a member is commonly referred to as a beam-column. In this paper we present the design of a simply supported, elastic beam-column which, for a given volume of material, has the maximum flexural stiffness distribution along the length. The beam-column is under the combined influence of axial compression and lateral uniformly distributed load. It is assumed that the flexural stiffness of the member is related to its mass by a simple power law. The need to maximise the flexural stiffness of a beam-column can arise because the designer wishes to either maximise the allowable axial compression when the lateral load and the maximum permissible deflection are prescribed or minimise the maximum transverse deflection when the axial and lateral forces are prescribed. The optimal beam-column design presented here is also applicable to other homogeneous boundary conditions, provided the length of the member is properly interpreted. Simple formulas are suggested relating the midspan deflection to the applied axial compression and lateral uniformly distributed load.

2. Mathematical formulation and optimality condition

Consider an elastic, simply supported beam-column of length L and volume V under the simultaneous action of an axial compressive force P^* and a transverse uniformly distributed load of intensity $2q$ per unit length. If shear and torsional deformations are neglected, the lateral deflection $u^*(\zeta)$ of the beam-column will satisfy the following differential equation and boundary conditions:

$$EI(\zeta)\, u^*_{\zeta\zeta} + P^* u^* + q\zeta (L-\zeta) = 0, \qquad 0 \leqslant \zeta \leqslant L/2\,;$$

$$u^*(0) = u^*_\zeta (L/2) = 0,$$

where $I(\zeta)$ is the second moment of area of the cross-section and the subscript ζ denotes differentiation with respect to this longitudinal coordinate. It is assumed that the lateral deflection function $u^*(\zeta)$ is symmetric about the midspan, as are the second moment of area $I(\zeta)$ and the cross-sectional area $A(\zeta)$. It is further assumed that the latter two are related through $I = cA^n$, where the constants c and n are defined by the cross-sectional shape. In particular, $n=1$ corresponds to a cross--section of sandwich construction or to a universal rolled section, $n=2$—to a geometrically similar solid cross-section (say, circular) and $n=3$—to a solid rectangular section with a constant width and variable depth along the length of the member.

It is convenient to work with nondimensional quantities. Accordingly, let $x = \zeta/L$, $u = u^*/L$ and $\alpha = AL/V$. The differential equation and boundary conditions therefore take the form

$$\alpha^n u_{xx} + Pu + Qx(1-x) = 0, \qquad 0 \leqslant x \leqslant 1/2\,;$$

(2.1)

$$u(0) = u_x(1/2) = 0,$$

where $P = (P^* L^{n+2}/EcV^n)$ and $Q = (qL^{n+3}/EcV^n)$ is one-half the total lateral force on the beam-column, E being Young's modulus.

Together with the above beam-column consider another beam-column under the simultaneous action of an axial compressive force P and a transverse midspan unit concentrated force and denote the transverse deflection by $v(x)$. In the same nondimensional variables the differential equation and boundary conditions are

$$\alpha^n v_{xx} + Pv + x/2 = 0, \qquad 0 \leqslant x \leqslant 1/2\,;$$

(2.2)

$$v(0) = v_x(1/2) = 0.$$

We are now in a position to use the well-known unit load theorem to derive an expression for the midspan transverse deflection $u(1/2)$. Denoting by $M(x) = -Pu - Qx(1-x)$ and $m(x) = -x/2$ the bending moments due to P and the uniformly distributed load Q and due to P and the midspan unit load, respectively, it follows from the unit load theorem that

$$u(1/2) = 2 \int_0^{1/2} M(x)\, m(x)\, dx/\alpha^n(x).$$

Substituting for $M(x)$ and $m(x)$ from Eqs. (2.1) and (2.2), respectively, we get

$$u(1/2)=2\int_0^{1/2}\alpha^n(x)\,u_{xx}\,v_{xx}\,dx.$$

The aim of the optimisation problem is to minimise the midspan beam-column deflection

(2.3) $$u(1/2)=2\int_0^{1/2}\alpha^n(x)\,u_{xx}\,v_{xx}\,dx\rightarrow\min,$$

subject to the isoperimetric condition that the beam-column use a prescribed volume of material V. In dimensionless variables the isoperimetric condition takes the form

(2.4) $$2\int_0^{1/2}\alpha(x)\,dx=1.$$

In order to derive the necessary optimality condition for the optimization problem (2.1)-(2.4), an auxiliary functional is established by including the constraint (2.4) through a Lagrange multiplier. By setting the first variation of the auxiliary functional with respect to α to zero and noting that the various functionals are stationary with respect to variations in u caused by variations in α, [14], the following optimality condition results:

(2.5) $$\alpha^{n-1}\,u_{xx}\,v_{xx}=\nu^2,$$

where the Lagrange multiplier ν is defined by the isoperimetric condition (2.4). Note the necessary optimality conditions (2.5) is a special case of a well-known general optimality condition, [15].

It is possible to relate the unknown Lagrange multiplier ν to the minimum midspan deflection $u_{min}(1/2)$. To this end, multiply both sides of Eq. (2.5) by $\alpha(x)$ and integrate from 0 to $1/2$. On making use of the isoperimetric condition (2.4) it follows that

(2.6) $$u_{min}(1/2)=\nu^2.$$

Before presenting the solution to the optimization problem it is expedient to investigate the behaviour of the area function, $\alpha(x)$, and the lateral deflection, $u(x)$, in the vicinity of the simply supported ends $x=0$ and 1. From symmetry consideration we need only to study the behaviour in the vicinity of one of the ends, say, $x=0$. Note that both the deflection function, $u(x)$, and the bending moment, $\alpha^n u_{xx}$, vanish at $x=0$. It follows, therefore, that $\alpha(0)=0$. The differential equation (2.1) is identically satisfied, but the curvature, u_{xx}, could well tend to approach infinity, as $x\rightarrow 0$. To investigate such a possibility, assume that in the neighbourhood of $x=0$ the deflection function can be expanded in a power series $u(x)=C_1 x+ + C_2 x^m+\ldots$, where m is the lowest noninteger power. Substituting for u, u_{xx} and α (from the optimality condition) into the differential equation (2.1) and equating the coefficients of like powers of x, it follows that $m=(n+3)/(n+1)$. Therefore, in the vicinity of $x=0$, u_{xx} varies as $x^{(1-n)/(1+n)}$ and $\alpha(x)$ as $x^{2/(n+1)}$. Note that the optimal beam and the optimal column area functions exhibit an identical behaviour, [1] and [3]. Note also that v_{xx} exhibits a similar singular behaviour near $x=0$.

3. Solution procedure

The optimisation problem (2.1)–(2.5) does not seem to have a closed form solution for any value of n. An iterative scheme was therefore designed to arrive at the solution. The scheme was based on the integration of regular functions. In view of the singular behaviour of u_{xx} and v_{xx} near $x=0$, the regular functions $g(x)$ and $f(x)$ were defined in the following manner:

$$g(x)=x^{-(1-n)/(1+n)}\,u_{xx},$$

$$f(x)=x^{-(1-n)/(1+n)}\,v_{xx}.$$

For prescribed values of P_0 and Q the iterations proceeded in the following sequence:

Assume $g_i(x)=f_i(x)\equiv -1$ in the first iteration ($i=1$)

$$(u_x)_i=-\int_x^{1/2} y^{(1-n)/(1+n)}\,g_i(y)\,dy \qquad (u_x(1/2)=0),$$

$$u_i=\int_0^x (u_y)_i\,dy \qquad (u(0)=0),$$

(3.1) $$(v_x)_i=-\int_x^{1/2} y^{(1-n)/(1+n)}\,f_i(y)\,dy \qquad (v_x(1/2)=0),$$

$$v_i=\int_0^x (v_y)_i\,dy \qquad (v(0)=0),$$

$$\alpha_i(x)=\left\{\frac{(P_0 v_i+x/2)(P_0 u_i+Qx(1-x))}{u_i(1/2)}\right\}^{1/(n+1)}$$

The above expression follows from Eqs. (2.1), (2.2), (2.5) and (2.6).

Normalise $\alpha_i(x)$ such that

(3.2) $$2\int_0^{1/2} \alpha_i(x)\,dx=1;$$

3.3) $$g_{i+1}(x)=-x^{(n-1)/(n+1)}(P_0 u_i+Qx(1-x))/\alpha_i^n(x);$$

(3.4) $$f_{i+1}(x)=-x^{(n-1)/(n+1)}(P_0 v_i+x/2)/\alpha_i^n(x).$$

Note:

(3.5) $$g_{i+1}(0),\quad f_{i+1}(0)\neq 0.$$

Repeat steps (3.1), (3.2) and (3.3)

(3.6) $$|u_{i+1}(1/2)-u_i(1/2)|<10^{-5}.$$

Repeat steps (3.1)–(3.6) for various values of P_0 and Q.

Table 1. $\alpha(x)$ of a beam-column with geometrically similar cross-sections ($n=2$) for various values of axial compression P_0, including $P_0=13.16$ (optimal column). Note that $\alpha(x)$ is independent of the lateral force Q.

x	P_0					
	4.00	6.00	8.00	10.00	11.00	13.16
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.005	0.0947	0.0979	0.1009	0.1038	0.1052	0.1081
0.010	0.1495	0.1543	0.1588	0.1632	0.1653	0.1696
0.020	0.2353	0.2422	0.2489	0.2552	0.2583	0.2647
0.030	0.3059	0.3144	0.3225	0.3303	0.3341	0.3419
0.040	0.3679	0.3775	0.3808	0.3956	0.3999	0.4088
0.050	0.4239	0.4344	0.4445	0.4542	0.4588	0.4686
0.100	0.6516	0.6637	0.6755	0.6869	0.6924	0.7040
0.150	0.8278	0.8390	0.8499	0.8606	0.8658	0.8769
0.200	0.9725	0.9810	0.9894	0.9978	1.0019	1.0108
0.250	1.0937	1.0983	1.1031	1.1079	1.1103	1.1158
0.300	1.1959	1.1957	1.1957	1.1960	1.1963	1.1971
0.350	1.2816	1.2758	1.2703	1.2652	1.2628	1.2579
0.400	1.3525	1.3404	1.3287	1.3174	1.3118	1.3002
0.450	1.4094	1.3906	1.3721	1.3537	1.3446	1.3251
0.500	1.4529	1.4271	1.4012	1.3751	1.3620	1.3333

Table 2. $\alpha(x)$ of a beam-column of solid construction ($n=3$) for various values of the axial compression P_0, including $P_0=13.88$ (optimal column). Note that $\alpha(x)$ is independent of the lateral force Q.

x	P_0					
	4.00	6.00	8.00	10.00	11.00	13.88
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.005	0.1765	0.1819	0.1869	0.1915	0.1937	0.1997
0.010	0.2479	0.2549	0.2613	0.2674	0.2703	0.2781
0.020	0.3471	0.3560	0.3641	0.3718	0.3755	0.3856
0.030	0.4219	0.4317	0.4409	0.4496	0.4537	0.4651
0.040	0.4838	0.4943	0.5041	0.5134	0.5178	0.5300
0.050	0.5375	0.5484	0.5586	0.5683	0.5728	0.5855
0.100	0.7395	0.7501	0.7602	0.7699	0.7746	0.7875
0.150	0.8834	0.8920	0.9003	0.9084	0.9124	0.9235
0.200	0.9957	1.0014	1.0071	1.0127	1.0155	1.0236
0.250	1.0867	1.0890	1.0913	1.0939	1.0953	1.0993
0.300	1.1616	1.1600	1.1587	1.1577	1.1573	1.1566
0.350	1.2234	1.2176	1.2123	1.2073	1.2049	1.1987
0.400	1.2738	1.2637	1.2540	1.2447	1.2402	1.2276
0.450	1.3141	1.2995	1.2853	1.2713	1.2643	1.2445
0.500	1.3450	1.3257	1.3067	1.2876	1.2780	1.2500

Numerical results showed, that for all values of n, the cross-sectional variation along the length of the member was independent of the lateral load Q. The values of $\alpha(x)$ as a function of P_0 are tabulated in Tables 1 and 2 for $n=2$ and 3, respectively. The values for $n=1$ are not tabulated because it was found possible to derive them from the following simple analytical expression:

$$\alpha(x)=\frac{P_0\,x(1-x)}{2}+\frac{(12-P_0)\,x(1-x)^{\frac{1}{2}}}{2.44}, \qquad n=1. \tag{3.7}$$

The absolute maximum deviation from the above expression was less than 0.5%. It is worth pointing out that the expression for $\alpha(x)$ does not include the lateral force Q explicitly. Note also that the two terms on the right hand side of Eq. (3.7) are, respectively, the column and beam contributions to the beam-column design. In fact, the maximum buckling load of the optimal column of prescibed volume can attain the value 12.

Numerical results also showed that the minimised value of the midspan deflection was a linear function of the lateral load Q. The values of $Q/u_{min}(1/2)$ for various values of axial compression are tabulated in Table 3 for all values of n.

Table 3. Percentage reduction in the midspan deflection of the optimal beam-column ($u_{min}(1/2)$) compared to that of a prismatic beam-column ($u_{pr}(1/2)$) of the same volume for various values of P_0 ($<\pi^2$), Q and n.

P_0	$Q/u_{min}(1/2)$ for various n			Prismatic beam-column $Q/u_{pr}(1/2)$ for all n	$(1-u_{min}/u_{pr})\cdot 100$ for various n		
	1	2	3		1	2	3
0	48.00	54.00	58.10	38.40	20.00	28.90	34.00
4	32.00	37.77	41.52	22.75	28.91	39.63	45.07
5	28.00	33.62	37.26	18.88	32.57	43.75	49.26
6	24.00	29.47	33.04	15.02	37.42	49.02	54.53
7	20.00	25.33	28.81	11.14	44.33	56.33	61.35
8	16.00	21.19	24.60	7.25	54.70	65.78	70.52
9	12.00	17.06	20.39	3.37	71.92	80.25	83.47
10	8.00	12.95	16.20	—	—	—	—
11	4.00	8.84	12.01	—	—	—	—
12	—	4.74	7.85	—	—	—	—

From the results of numerical computations it was possible to deduce (to within an accuracy of ±0.5%) the following explicit relations for $u_{min}(1/2)$ as a function of the prescribed axial compression P_0 and one half the total uniformly distributed lateral load Q:

$$u_{min}(1/2)=\begin{cases}0.250\,Q/(12.00-P_0), & n=1,\\ 0.244\,Q/(13.16-P_0), & n=2,\\ 0.239\,Q/(13.88-P_0), & n=3.\end{cases} \tag{3.8}$$

In deriving the expressions (3.8), we considered the fact that the maximum value of P_0 attained by the optimal column design is equal to 12.00, 13.16 and 13.88 for $n=1$, 2, and 3, respectively.

An alternative interpretation of Eq. (3.8) is also possible. Given the permissible midspan deflection u_0 under an axial compression and a uniformly distributed lateral load of total magnitude $2Q$, the expressions (3.8) specify the maximum axial compression that can be applied to the optimal beam-column without exceeding u_0:

$$P_{max}=\begin{cases}12.00-0.250\,Q/u_0, & n=1,\\ 13.16-0.244\,Q/u_0, & n=2,\\ 13.88-0.239\,Q/u_0, & n=3.\end{cases} \tag{3.9}$$

In order to judge the advantage gained by optimising the beam-column design, the minimised midspan deflection of Eq. (3.8) was compared with that of a prismatic beam-column of the same volume and subjected to the same axial and lateral forces as the optimal beam-column. The cross-sectional area of a prismatic beam-column of a given volume (3.4) is $\alpha(x)=1$, $0\leqslant x\leqslant 1/2$ for all n and its midspan deflection $u_{pr}(1/2)$ under an axial compression P_0 and uniformly distributed lateral load of total magnitude $2Q$ is given by [17]

$$u_{pr}(1/2)=\frac{2Q}{P_0^2}\left(\sec(\sqrt{P_0}/2)-1\right)-\frac{Q}{4P_0}. \tag{3.10}$$

Clearly, the comparison is meaningful only when P_0 is less than the Euler buckling load of the prismatic column for all n ($P_0<\pi^2$).

The values of $u_{pr}(1/2)$ for various Q and P_0 are tabulated in Table 3 alongside the values of $u_{min}(1/2)$. From the last column of Table 3 it is quite obvious that the lateral deflection of the optimally designed beam-column is substantially lower than that of a comparable prismatic beam-column. Care should, however, be exercised in judging the gain achieved for large values of $u_{min}(1/2)$, i.e. small values of $Q/u_{min}(1/2)$, in view of the limitations of the linear elastic beam-column theory used in arriving at the results.

The work reported here has clearly demonstrated that optimization of the design of a beam-column of a given volume can not only lead to a substantial reduction of its lateral deflection, but can also provide a design in those situations when the values of P_0 and Q preclude the use of a prismatic beam-column. The solution of the optimization problem also showed that the optimal beam-column design does not differ much from the corresponding optimal column design. In view of this observation, the optimal column design may be considered as a quasi-optimal beam-column design. Finally, it should be observed that the design presented here is applicable to beam-column with other homogeneous boundary conditions, provided the forces P_0, Q and the span are properly scaled.

References

1. J. B. Keller, *The shape of the strongest column*, Arch. Ratl. Mech. Anal., 5, 275, 1960.
2. F. I. Niordson, *On the optimal design of a vibrating beam*, Quart. Appl. Math., 23, 47, 1965.
3. N. C. Huang, *Optimal design of elastic beams for minimum-maximum deflection*, J. Appl. Mech., ASME, 38, 1078, 1971.

4. B. L. Karihaloo nd F. I. Niordson, *Optimum design of vibrating cantilevers*, J. Optimization Theory Appl., **11**, 638, 1973.
5. N. Olhoff, *Optimal design of vibrating circular plates*, Int. J. Solids Struct., **6**, 139, 1970.
6. N. Olhoff, *Maximising higher order eigenfrequencies of beams with constraints on the design geometry*, J. Struct. Mech., **5**, 107, 1977.
7. N. Olhoff, J. E. Taylor, *Designing continuous structures for minimum total cost of material and interior supports*, J. Struct. Mech., **6**, 367, 1978.
8. W. Prager, R. T. Shield, *Optimal design of multi-purpose structures*, Int. J. Solids Struct., Struct., **4**, 469, 1968.
9. N. V. Banichuk, B. L. Karihaloo, *Minimum-weight design of multipurpose cylindrical bars*, Int. J. Solids Struct., **12**, 267, 1976.
10. R. D. Parbery, B. L. Karihaloo, *Minimum-weight design of hollow cylinders for given lower bounds on torsional and flexural rigidities*, Int. J. Solids Struct., **13**, 1271, 1977.
11. B. L. Karihaloo, *Optimal design of tie-column of solid construction*, Int. J. Solids Struct., **15**, 104, 1979.
12. B. L. Karihaloo, G. L. Wood, *Optimal design of multi-purpose sandwich tie-column*, J. Engr. Mech. Div., ASCE, **105**, 465, 1979.
13. B. L. Karihaloo, R. D. Parbery, *Optimal design of multi-purpose beam columns*, J. Optimization Theory Appl., **27**, 439, 1979.
14. W. S. Hemp, *Optimal structures*, p. 49, Clarendom Press, Oxford 1973.
15. E. F. Masur, *Optimum stiffness and strength of elastic structures*, J. Engr. Mech. Div., ASCE, **96**, 621, 1970.
16. B. L. Karihaloo, R. D. Parbery, *The optimal design of beam columns*, Int. J. Solids Struct., **15**, 855, 1979.
17. G. J. Simitses, *An introduction to the elastic stability of structures*, Prentice–Hall, London 1976

Streszczenie

PRĘTY O MAKSYMALNEJ SZTYWNOŚCI PRZY RÓWNOCZESNYM ZGINANIU I ŚCISKANIU

Przedstawione zagadnienie dotyczy maksymalizacji sztywności giętnej swobodnie podpartego na końcach pręta o określonej długości i objętości, poddanego jednoczesnemu działaniu ściskającej siły osiowej i równomiernie rozłożonego obciążenia poprzecznego. Przekrój poprzeczny pręta może być sandwiczowy lub lity. Wyprowadzono proste wzory umożliwiające obliczenie minimalnej strzałki ugięcia pręta przy danych wartościach sił ściskających i obciążeń poprzecznych. Pokazano, że maksymalizacja sztywności giętnej pręta prowadzi do istotnego obniżenia jego ugięcia poprzecznego.

Резюме

МАКСИМАЛЬНО ЖЕСТКИЕ СТЕРЖНИ ПРИ ОДНОВРЕМЕННОМ ИЗГИБЕ И СЖАТИИ

Представленный вопрос касается максимализации изгибной жесткости свободно опирающегося на концах стержния определенной длины и объема, подверженного одновременно действующей сжимающей осевой силы и равномерно распределенной поперечной нагрузки. Поперечное сечение стержня может быть сплошное или типа сандвича. Вводятся простые формулы позволяющие расчитать минимальный прогиб стержня при данных значениях сжимающих сил и поперечных нагрузок. Показано, что максимизация изгибной жесткости стержня ведет к существенному снижению его поперечного прогиба.

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