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ANALYSIS OF MEMBRANES STRETCHED OVER A UNILATERAL SUPPORT

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The main objective of the present paper is to find the equilibrium configuration of membranes of various shapes, stretched over a plane frame, in the vicinity of a rigid, two-dimensional profile which supplies unilateral constraints to its deflections. The problem is considered under the following conditions: (i) the membrane is subjected to assigned, constant tension forces and to transversal loads; (ii) boundary conditions of the membrane are given by zero displacements along the contour of the frame; (iii) the deformations are small; (iv) the contact is frictionless and reactions are assumed to be vertical.

1. Introduction

The solution of contact problems, that is, the analysis of structures subjected to unilateral constraints, is currently attracting growing attention in civil and mechanical engineering.

Most of the recent studies on the subject can be regarded as the further development of the mathematical theory of contact problems in linear elasticity, formulated in the early 30's; the same problems have been the object of theoretical approaches based on classical or modern functional analysis [1, 2, 3].

More recently, in the case of discrete (finite element) structural models, similarities and equivalence were pointed out between elastic analysis in the presence of unilateral constraints and holonomic anasis of elastic-plastic and elastic locking systems, [4, 5], with piecewise linear constitutive laws; a systematic recourse to mathematical programming methods was made in developing the latter approach [5, 6, 7].

In a previous paper [8] the authors presented some theoretical results and computational solution procedures in search o equilibrium configurations of cables (e.g. submarine energy—transmission lines) subjected to given transversal loads and given tension force in the vicinity of frictionless rigid ground of known profile under a small deformation hypothesis.

The problem dealt with in the present paper concerns the search of the equilibrium configuration of membranes of various shapes, stretched over a plane frame, in the vicinity of a rigid, two-dimensional profile which supplies unilateral constraints to its deflections. For the preliminary determination of the overall stress state in the membrane the following hypotheses can be assumed in practical engineering situations:

a) the membrane is subjected to assigned, constant tension forces and transversal loads;

b) boundary conditions of the membrane are given by zero displacements along the contour of the frame;

c) the deformations are small, in the sense that displacements referred to the original straight configuration are regarded as vertical and intervene linearly in the equilibrium equations ("second-order" geometrical effects);

d) the contact is frictionless and reactions are assumed to be vertical.

A discretization of the system is assumed in the sense that all active and reactive forces are considered to be lumped in joints chosen at suitably small intervals on the surface of the undeformed membrane (orthogonal to the plane of the contour). As a discretization the partial (linear second order) differential equation of equilibrium is replaced by finite difference equations. The analysis problem becomes a "linear complementarity problem" associated to a symmetric, positive-definite, sparse pentadiagonal matrix, the entries offdiagonal of which turn out to be all non-positive. In analogy to the case of the cable studied in [8], these characteristics of the finite difference formulation allow to use, for the numerical solution, some recently developed algorithms of mathematical programming.

The iterative overrelaxation method devised by CRYER [9] turns out to be particularly efficient for these large size problems.

A monotonicity property of the local displacements under proportional loading is pointed out on the basis of the nature of the matrix involved in the linear complementarity problem, and the extremum properties of the solution are established and interpreted in mechanical terms.

2. FORMULATION OF THE PROBLEM AND DISCRETIZATION

Consider a membrane stretched over a plane frame, as shown in Fig. 1. Let T (force per unit length) denote the tension which defines the (isotropic, constant) membrane stress state; $p(x, y)$, $\rho(x, y)$ indicate the active and reactive forces per unit surface (positive downwards and upwards respectively), the latter being supplied by an underlying smooth, rigid and frictionless surface. This is defined by its ordinates $f(x, y)$ (positive downwards) referred to the plane of the original, undeformed membrane, and bounds from above the deflections $w(x, y)$.

Under a small deformation hypothesis, by which the slopes dw/dx, dw/dy and, as a consequence, df/dx, df/dy can be dealt with as infinitesimal, the equilibrium of the present structure is expressed (see Fig. 2) by the classical equation

$$T\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+p-\rho=0. \tag{2.1}$$

Let introduce the auxiliary variable $\omega(x, y)$

$$\omega \stackrel{\Delta}{=} f-w, \tag{2.2}$$

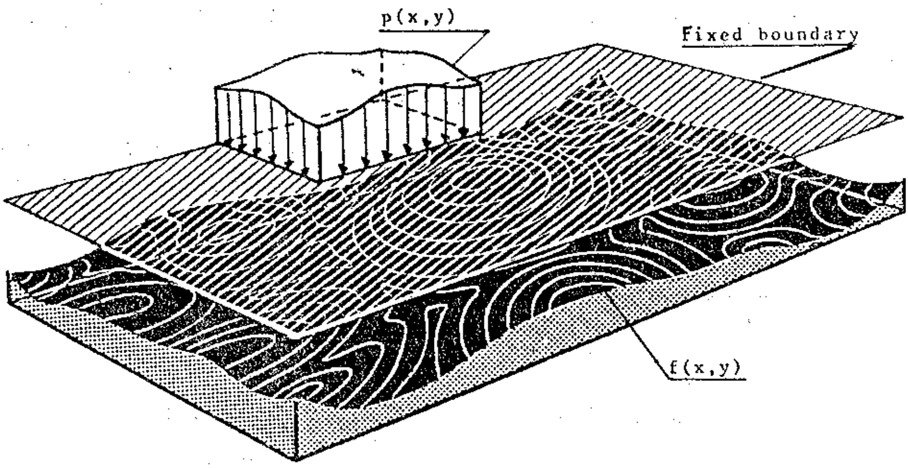


FIG. 1.

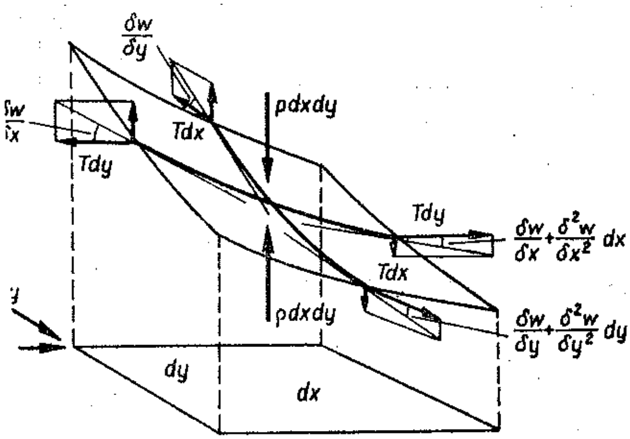


FIG. 2.

which represent the vertical distance of the deformed membrane from the obstacle defined by $f(x, y)$.

Thus the original formulation becomes

$$T\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right)+\rho=p+T\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right), \tag{2.3}$$

where the nonnegative unknowns $\omega(x, y)$ and $\rho(x, y)$ are related by the complementarity relation

$$\rho\omega=0, \tag{2.4}$$

the nonlinearity of which is due to the unilateral nature of the supports provided by the rigid profile. The association of Eqs. (2.3) and (2.4) with the nonnegativity conditions

$$\rho \geq 0, \quad \omega \geq 0 \tag{2.5}$$

referred to the whole area of the membrane and with the boundary condition $w=0$ along its supporting frame, provides the set of relations governing the structural response of the membrane to loads.

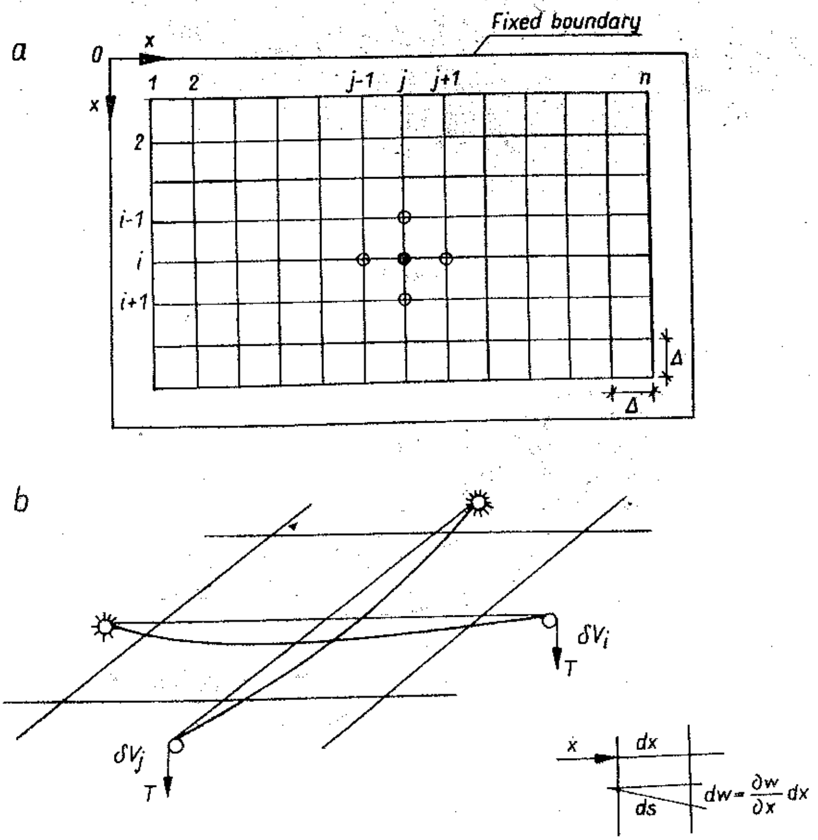


FIG. 3a. Finite difference discretization net. b. Cables „algebraically corresponding" to the i-th row and j-th column.

A central finite difference scheme applied to Eq. (2.3) with equal intervals $\Delta x = \Delta y = \Delta$ and the node numbering shown in Fig. 3a (where a rectangular membrane is represented for the sake of simplicity) leads to $m \cdot n$ algebraic linear equations. For each node belonging to the i-th row and to the j-th column (marked by a heavy circle in Fig. 3a) and distant not less than 2Δ from the edges, the finite difference equations reads

$$\rho_i \Delta^2 = p_i \Delta^2 + T(-\omega_{i,j-1} + 4\omega_{ij} - \omega_{i,j+1} - \omega_{i-1,j} - \omega_{i+1,j}) - -T(-f_{i,j-1} + 4f_{ij} - f_{i,j+1} - f_{i-1,j} - f_{i+1,j}). \tag{2.6}$$

When a node adjacent to the edge is considered, Eq. (2.6) has to be slightly modified in a self-evident way in view of the boundary condition.

By arranging all the $m \cdot n$ equations thus generated in a system according to Fig. 3a, the matrix collecting the coefficients of the $m \cdot n$ unknowns ω_{ij} (and of the data f_{ij}) turns out to exhibit the following three-block-diagonal structure:

$$GT\begin{bmatrix} \mathbf{M} & -\mathbf{I} & \mathbf{0} & \cdot \\ -\mathbf{I} & \mathbf{M} & -\mathbf{I} & \cdot \\ \mathbf{0} & -\mathbf{I} & \mathbf{M} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{2.7}$$

In the matrix (2.7) there are $m \cdot n$ blocks specified as follows: $\mathbf{I}$=identity matrix of order n; $\mathbf{0}$=null matrix of order n, $\mathbf{M}$=tridiagonal positive definite matrix of order n with the entries specified below:

$$\mathbf{M} \approx \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 4 & -1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 4 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 4 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & -1 & 4 \end{bmatrix}. \tag{2.8}$$

let us form the vectors:

$$\mathbf{R} \approx \{\mathbf{R}_1 \ldots \mathbf{R}_i \ldots \mathbf{R}_m\}^T, \qquad \mathbf{P} \approx \{\mathbf{P}_1 \ldots \mathbf{P}_i \ldots \mathbf{P}_m\}^T,$$

$$\boldsymbol{\omega} \approx \{\boldsymbol{\omega}_1 \ldots \boldsymbol{\omega}_i \ldots \boldsymbol{\omega}_m\}^T, \qquad \mathbf{f} \approx \{\mathbf{f}_1 \ldots \mathbf{f}_i \ldots \mathbf{f}_m\}^T,$$

where

$$\mathbf{R}_i \approx \{\rho_{i1} \ldots \rho_{in}\}\, \Delta^2, \qquad \mathbf{P}_i \approx \{p_{i1} \ldots p_{in}\}\, \Delta^2,$$

$$\boldsymbol{\omega}_i \approx \{\omega_{i1} \ldots \omega_{in}\}, \qquad \mathbf{f}_i \approx \{f_{i1} \ldots f_{in}\}.$$

With these symbols Eqs. (2.6) can be written in the following matrix form:

$$\mathbf{R} = \mathbf{P} + \mathbf{G}\boldsymbol{\omega} - \mathbf{G}\mathbf{f}. \tag{2.9}$$

The algebraized problem is completed by the nonnegativity conditions

$$\mathbf{R} \geqslant \mathbf{0}, \quad \boldsymbol{\omega} \geqslant \mathbf{0} \tag{2.10}$$

and by the complementarity-condition which now can be equivalently expressed as an orthogonality requirement:

$$\boldsymbol{\omega}^T \mathbf{R} = 0. \tag{2.11}$$

These relations can be interpreted as governing a discrete mechanical model of the membrane, namely a net of equally spaced and equally tensioned orthogonal cables, with loads and possible reactions of unilateral supports acting on the (internal) nodes alone. In fact, the points of the finite difference can be regarded as the nodes of a weightless cable net: $P_{ij} = p_{ij}\, \Delta^2$ and $R_{ij} = \rho_{ij}\, \Delta^2$ can be though of as the resultant of the active and the reactive force respectively, distributed over the surface Δ^2 around the node ij.

The matrix **G** is pentadiagonal, symmetric, sparse and positive definite. This sign definition can be shown by means of algebraic considerations based on the associated quadratic form or by discussing the energy variations $\delta\varepsilon$ for geometric "disturbances" with respect to the undeformed membrane configuration. Assuming the latter (mechanical) standpoint, let us consider the discretized structure as a cable plane net stretched by the tension force T: the total potential energy change is

$$\delta\varepsilon = \sum_{1}^{m}{}_{i}(-T\delta V_i) + \sum_{1}^{n}{}_{j}(-T\delta V_j), \tag{2.12}$$

where δV_i and δV_j are the lenght variation along the i-th and j-th orthogonal "cables", corresponding, from the algebraic point of view, to the i-th row and j-th column. I the membrane is conceived as a set o continuous strips around the single "cable" (Fig. 3b) then

$$\delta V_i = -\int_{1i} (ds - dx) = \int_{1i} (dx - ds),$$

$$\delta V_j = -\int_{1j} (ds - dy) = \int_{1j} (dy - ds),$$

$$ds = \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2}\, dx, \tag{2.13}$$

$$ds = \sqrt{1 + \left(\frac{\partial w}{\partial y}\right)^2}\, dy.$$

Substituting into Eq. $(2.13)_1$ the expression $(2.13)_2$ of ds, expanded up to the quadratic terms, we obtain

$$\delta\varepsilon = \frac{1}{2} T \sum_{1}^{m}{}_{i} \int_{1i} \left(\frac{\partial w_i}{\partial x}\right)^2 dx + \frac{1}{2} T \sum_{1}^{n}{}_{j} \int_{1j} \left(\frac{\partial w_j}{\partial y}\right)^2 dy. \tag{2.14}$$

This shows that for $T>0$, $\delta\varepsilon>0 \forall \delta w \neq 0$, i.e. static stability is ensured. If follows that for the algebraic (discretized) model $1/2\delta\mathbf{P}^T\, \delta\mathbf{w} > 0 \forall \delta\mathbf{w} \neq \mathbf{0}$ and since $\delta\mathbf{P} = \mathbf{G}\delta\mathbf{w}$, we have $1/2\delta\mathbf{w}^T\, \mathbf{G}\delta\mathbf{w} > 0 \forall \delta\mathbf{w} \neq \mathbf{0}$, i.e. **G** is positive definite. The positive definiteness of the matrix **G** ensures the existence and uniqueness of the solution of the *LCP* (2.9)–(2.11) for any given vector, i.e. for any load distribution on the membrane. The evident analogy between the single cable and the membrane conceived as a cable net permits to extend to the present case some properties proved in [8] for the cable contact problem. In particular, the reference made so far for convenience to rectangular contours (Fig. 3a) does not mean loss of generality: different shapes would not alter the essential features of the mathematical model formulated and the mechanical conclusions derived through it; "in primis" extremum and monotonicity properties will still hold.

3. Computation remark and numerical example

The membrane contact analysis can be carried out by solving the linear complementarity problem (*LCP*) resulting from the finite difference discretization of Eqs. (2.3)–(2.5), or by solving the quadratic programming problem (*QP*) to which *LCP* is equivalent. Moreover, an alternative approach, due to Mangasarian [10], is possible; since a Minkowskian matrix belongs to the class of real square matrices with nonpositive offdiagonal elements, the *LCP* problem is solvable by a single linear program.

However, in this paper Cryer's method [9], a modified version of *SOR* (systematic overrelaxation), the brief outline of which is given in [8], is used; in fact, it is particularly efficient when, like in this case, the problem is characterized by a large size, "finite difference" matrix and nonnegative variables.

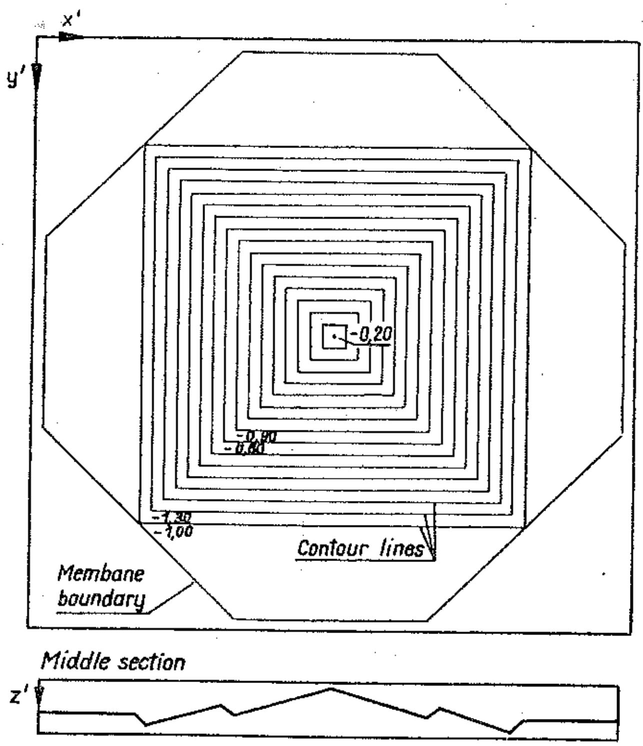


Fig. 4. Map and middle section of rigid surface underlying membrane ($x', y', z' = x/\Delta, y/\Delta, z/\Delta$).

The algorithm efficiency is tested by analyzing the hexagonal membrane drawn in Fig. 4. The numerical program is conceived to permit rather a general membrane boundary shape and rigid surface over which the membrane is stretched. There are no special reasons for the example chosen except for the fact that it is easier to check the numerical results, particularly the symmetry of the structural response.

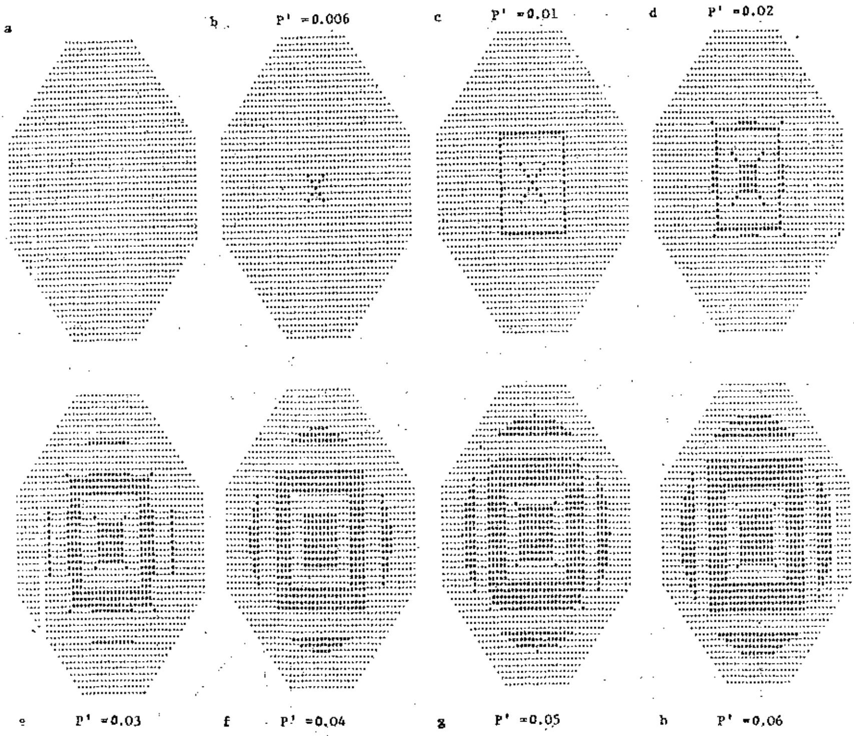


Fig. 5. "Contact maps".

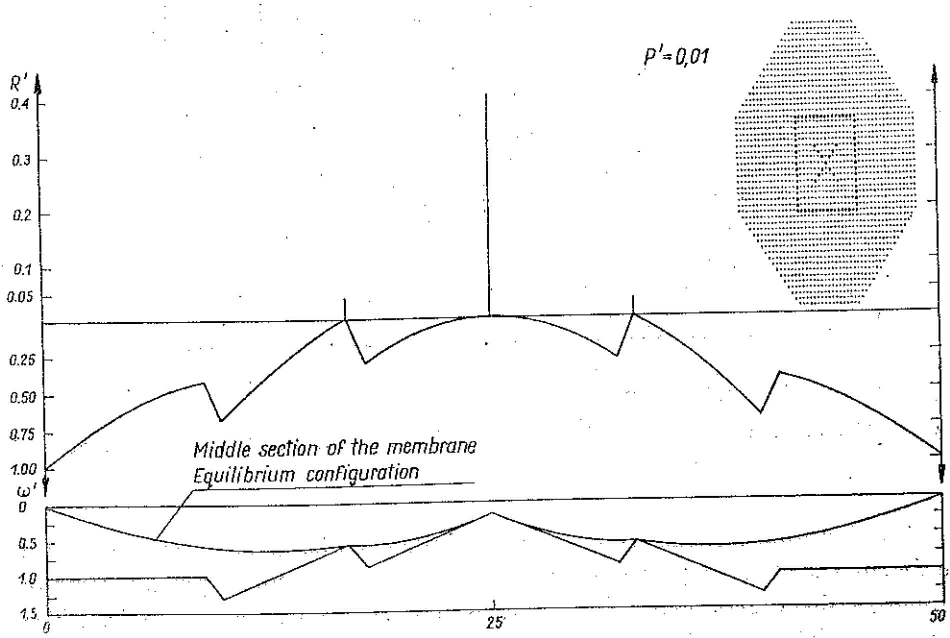
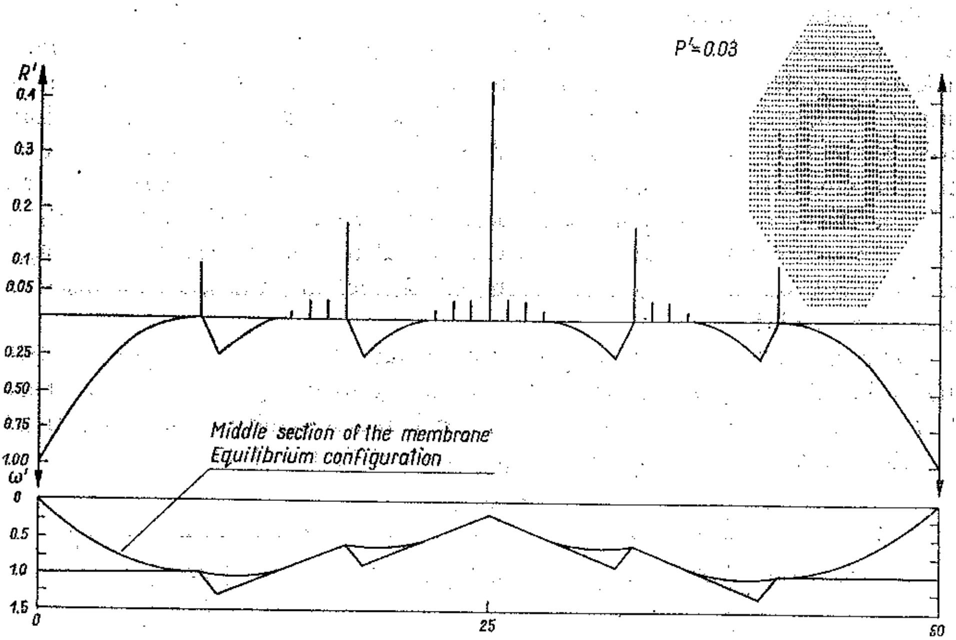
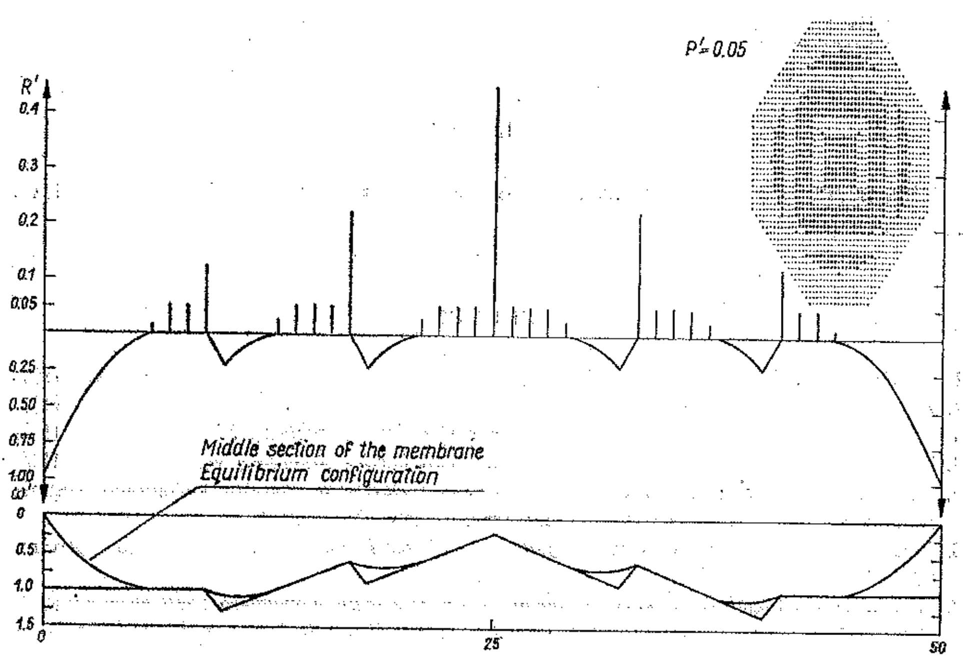


Fig. 6.



P' = 0.03
R'
0.4
0.3
0.2
0.1
0.05
0.25
0.50
0.75
1.00
ω'
0
0.5
1.0
1.5
0
25
50
Middle section of the membrane
Equilibrium configuration

FIG. 7.



P' = 0.05
R'
0.4
0.3
0.2
0.1
0.05
0.25
0.50
0.75
1.00
ω'
0
0.5
1.0
1.5
0
25
50
Middle section of the membrane
Equilibrium configuration

FIG. 8.

The dots plotted in Fig. 5a represent the distribution of the 1857 variables corresponding to the finite difference discretization net. In Figs. 5b-h the stars replacing dots indicate the points where the membrane settles on the constraint surface; the sequence of those figures, which are a sort of contact maps at increasing loading levels, shows the monotonic behaviour of this type of structure. In Figs. 6, 7 and 8, for the dimensionless loading values $P'=P/T'=p\Delta^2/T\Delta=0.01, 0.03, 0.05$, the following dimensionless quantities are represented: the reactions $R'=R/T'=p\Delta^2/T\Delta$, the membrane-constraint surface distances $\omega'=\omega/\Delta$ and the displacements $w'=w/\Delta$ from the undeformed configuration at the middle section (in abscissa the distance have been divided by Δ).

Computational aspects of this example, analyzed with a tolerance $\varepsilon=0.001$ and an initial vector $\boldsymbol{\omega}=\mathbf{0}$, are summarized in the following table; for each load P', the optimal overrrelaxation factor β_0, the iteration number N and CPU time, in seconds, with a UNIVAC 1108 computer, are indicated (Table 1).

Table 1

P'	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
β_0	1.70	1.65	1.60	1.50	1.45	1.45	1.40	1.40	1.35	1.35
N iter	53	45	31	27	23	21	20	19	18	17
CPU time (sec)	17.04	14.74	13.13	8.97	7.65	7.02	6.59	6.39	6.08	5.80

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Streszczenie

ANALIZA MEMBRAN ROZCIĄGANYCH WZDŁUŻ JEDNOSTRONNEGO PODPARCIA

Głównym celem pracy jest znalezienie konfiguracji równowagi membran o różnych kształtach. rozciąganych wzdłuż płaskiej ramy w otoczeniu sztywnego dwuwymiarowego profilu, który stanowi jednostronne więzy dla ugięć. Zagadnienie jest rozważane przy następujących warunkach: (i) membrana jest oddana określonej, stałej sile rozciągającej i obciążeniom bocznym; (ii) warunki brzegowe membrany są określone żądaniem znikania przemieszczeń wzdłuż konturu ramy; (iii) deformacje są małe; (iv) kontakt jest beztarciowy i reakcje są przyjęte jako prostopadłe.

Резюме

АНАЛИЗ МЕМБРАН РАСТЯГИВАЕМЫХ ВДОЛЬ ОДНОСТОРОННЕЙ ПОДПОРЫ

Главной целью работы является нахождении конфигурации равновесия мембран разной формы, растягиваемых вдоль плоской рамы внутри жесткого двухразмерного профиля, который составляет одностороннюю связь для прогиба. Задача рассматривается в следующихся условиях: (I) мембрана подвергается определенной постоянной растягивающей силе и боковым нагрузкам. (II) краевые условия мембраны определены требованием исчезновения перемещений вдоль контура рамы, (III) деформации малы; (Iv) контакт без трения и реакции предполагаются перпендикулярными.

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