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SOME COMMENTS ON THE RESISTANCE EXPERIENCED BY A POROUS SPHERE MOVING IN A VISCOUS INCOMPRESSIBLE FLUID AT LOW REYNOLDS NUMBERS

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The formulae for viscous drag force experienced by a porous sphere falling in a viscous fluid at low Reynolds numbers which have been given by several authors are compared with one another and with the Stokes formula. The problem of these formulae, justified to apply in the cases of real porous media is discussed. It is shown that in many cases of "standard" porous material (for example sand, felt) the differences between the viscous drag force due to the Stokes formula and others are non-essential, independently of the forms of boundary conditions and filtration equation by the use of which these formulae were obtained. A fibrous medium with very large porosity (for example cotton) is given as an example of the porous medium in the case of which some of these formulae may have a practical meaning. A new proposition of an experimental method of determining both the constant appearing in the Beavers-Joseph condition and the effective viscosity in the Brinkman equation is given.

1. Introduction

The problem of the resistance experinced by an impermeable sphere moving slowly with a constant velocity U in a viscous incompressible fluid is one of the most classic and most elementary problems of fluid mechanics. This problem was solved by Stokes in 1851. His investigation was based on the assumption that the Reynolds number is much less than unity, i.e. $\mathrm{Re}=\frac{\rho RU}{\mu}\ll 1$, where R denotes the radius of the sphere, ρ and μ are the density and the dynamic viscosity of the fluid, respectively. In this way, Stokes found the drag force of the impermeable sphere to be

$$\text{Impermeable sphere drag force}=6\pi\mu RU. \tag{1.1}$$

The purpose of this paper is to find the changes of the Stokes formula (1.1) as a result of replacing the impermeable sphere by a porous one with non-negligible permeability. Several authors have found the formulae for porous sphere drag force assuming different forms of boundary conditions and the filtration equation. In order to find the above mentioned changes and to confirm the applicability of different porous medium models, we compare these formulae with one other and with one given by Stokes (1.1).

In determining porous sphere drag force at low Reynolds number one usually describes a flow within the sphere and outside the sphere using the Darcy equation {[1], pp. 61}:

$$\mathbf{q} = -\frac{k}{\mu} \nabla P \tag{1.2}$$

and the Stokes equations {[2], p. 60}:

$$\mu \nabla^2 \mathbf{v} = \nabla p, \tag{1.3}$$

respectively. Here $\mathbf{q}$ denotes the filtration velocity, k is the permeability of the sphere material and P is the pressure within the porous sphere; $\mathbf{v}$ and p denote the fluid velocity and the pressure outside the porous sphere (free flow region), respectively.

The above flow equations are considered together with the available boundary conditions which must be fulfilled on the porous sphere interface. Recently, the Beavers-Joseph boundary condition [3] has often been applied. This boundary condition can be written as follows:

$$\frac{dv_s}{dn} = \frac{\alpha}{\sqrt{k}} (v_s - q_s), \tag{1.4}$$

where v_s and q_s are the components of the velocities $\mathbf{v}$ and $\mathbf{q}$, respectively, tangent to the porous sphere interface, α is a dimensionless experimental constant, dv_s/dn denotes differentiation in the direction normal to the porous sphere interface.

Some authors suggest applying the Brinkman filtration equation [4]

$$\bar{\mu} \nabla^2 \mathbf{q} - \frac{\mu}{k} \mathbf{q} = \nabla P \tag{1.5}$$

instead of the traditional Darcy equation and the condition (requirement) of velocity and stress continuity across the porous sphere interface [5–7]. Here $\bar{\mu}$ denotes the effective viscosity.

To give a new experimental method of determining $\bar{\mu}$ and α is one of the aims of this paper.

2. Review of the viscous drag problem solutions for a porous sphere

In this section a review of several formulae available for the determination of the drag force exerted by the viscous fluid on a porous sphere is presented. In the following, T denotes the dimensionless drag force defined by the relation

$$T = \frac{\text{porous sphere drag force}}{\text{impermeable sphere drag force}}. \tag{2.1}$$

Gheorghitza [8–10] was the first to solve the problem under consideration in the present section. His idea is based on the following three assumptions.

The first assumption requires that a flow within and outside the porous sphere be governed by the Darcy equation (1.2) and Stokes equations (1.3) (for $\mathrm{Re} \ll 1$), respectively. The second assumption is the requirement of pressure and normal velocity component continuity across the porous sphere interface. The third assumption is that the tangent velocity components on both sides of the sphere interface are equal to zero (the slip absence condition) Making use of these assumptions, Gheorghitza found that the viscous drag force T has the form

$$T=\frac{1}{1+\frac{k}{2R^2}}, \tag{2.2}$$

k being the sphere permeability.

A few years later JOSEPH and TAO [11] examining the considered problem under the same assumptions as Gheorghitza obtained a similar result. They probably did not know Gheorghitza's works.

SUTHERLAND and TAN [12] have agreed with all Gheorghitza's assumptions with the exception of the tangent velocity component being equal to zero on the exterior side of the porous sphere interface. Taking this condition to be not natural, they tried another approach: instead of the slip absence condition, they assumed that the tangent velocity components are continuous across the porous sphere interface. In this way, they found the viscous drag force T to be

$$T=\frac{1}{1+\frac{3k}{2R^2}}. \tag{2.3}$$

The formula (2.3) differs slightly from the formula (2.2).

Replacing the slip absence condition by the Beavers-Joseph's condition (1.4), NEALE, EPSTEIN and NEADER [7] have arrived at the following formula for

$$T=\frac{1+\alpha\frac{\sqrt{k}}{R}}{1+\frac{k}{2R^2}+2\alpha\frac{\sqrt{k}}{R}+\frac{3}{2}\alpha\left(\frac{\sqrt{k}}{R}\right)^3}, \tag{2.4}$$

where α is the same dimensionless quantity which appears in the Beavers-Joseph condition (1.4). By comparing the formula (2.4) with the formula (2.2) one can find the change in the form of the Gheorghitza's formula (2.2) due to replacing the slip absence condition by that of Beavers-Joseph (1.4).

It should be stressed that the formulae (2.2)–(2.4) were obtained under the assumption that the flow within the porous sphere is governed by the Darcy equation (1.2). Even a rough examination of the formulae (2.2)–(2.4) shows that these formulae may have a practical meaning only in the case when the ratio k/R^2 of the permeability k of the porous sphere to its squared radius R^2 is of order greater than 10^{-2} or equal to 10^{-2} (the quantity α is of order unity). In our opinion, such a large value of α may be achieved only in the case of a porous medium with large porosity. As it is known, to calculate the viscous drag force T in the case of large porosity, some authors have proposed replasing the Darcy equation (1.2) by Brinkman's one (1.5). Considering the discussed problem, YAMAMOTO [5] made use of this idea and found the viscous drag force T to be

$$(2.5) \qquad T=\frac{I_{\frac{3}{2}}\left(\frac{R}{\sqrt{k}}\right)}{\left(\frac{3k}{2R^2}+1\right)I_{\frac{3}{2}}\left(\frac{R}{\sqrt{k}}\right)+\frac{\sqrt{k}}{R}I_{\frac{1}{2}}\left(\frac{R}{\sqrt{k}}\right)},$$

where $I_{3/2}$ and $I_{1/2}$ are the modified Bessel functions of order 3/2 and 1/2. The formula (2.5) has been obtained on the basis of the following assumptions:

a) flows within and outside the porous sphere are governed by the Brinkman equation (1.5) and the Stokes equations (1.3), respectively;

b) the ratio of the effective viscosity appearing in the Brinkman equation (1.5) to the dynamic viscosity of the fluid in the free flow region is equal to unity;

c) the fluid velocity vector and the fluid stress tensor are continuous across the porous sphere interface.

3. Some comments

Figure 1 presents the values of the drag force T exerted by the fluid on a porous sphere falling with the constant velocity U. These values are calculated on the basis of the formulae (2.1)–(2.4). In order to make the discussion easier, we give in Table 1 the values of permeability k and porosity ε of some "standard" porous materials.

It is reasonable to regard a sphere being made of one of the above materials as a porous one if the radius R of the sphere is much greater than the size of an aver-

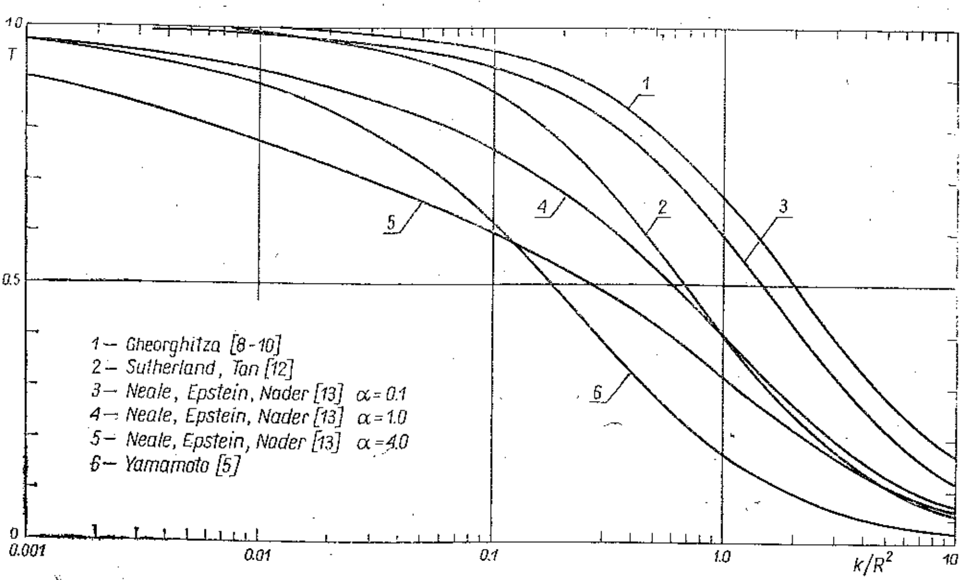

FIG. 1. Dimensionless viscous drag force of a porous sphere.

Table 1.

Porous material	Permeability k [m²]	Porosity ε
Quartz powder	$1.3\cdot10^{-14}$—$5.1\cdot10^{-14}$	0.37—0.49
Sand	$2.0\cdot10^{-11}$—$1.8\cdot10^{-10}$	0.36—0.50
Ground	$2.9\cdot10^{-13}$—$1.4\cdot10^{-11}$	0.43—0.54
Sandstone	$5.0\cdot10^{-16}$—$3.0\cdot10^{-12}$	0.08—0.38
Skin	$9.5\cdot10^{-14}$—$1.2\cdot10^{-13}$	0.56—0.59
Brick	$4.8\cdot10^{-15}$—$2.2\cdot10^{-13}$	0.12—0.34
Felt	$8.3\cdot10^{-10}$—$1.2\cdot10^{-9}$	0.63—0.74

age grain. In this case, however, the ratio k/R^2 is very small as compared with unity and the drag force T experienced by the porous sphere may be determined by using the Stokes formula (1.1). In other words, the formulae (2.1)–(2.4) have not any practical meaning in the case when the sphere is made of any material given in Table 1. It should be pointed out that these materials have small or moderate porosity. On the contrary, the problem of these formulae (2.1)–(2.4), justified to apply in the case of large porosity, is unsolved. In order to shed new light on this problem, we have determined the function $F(\varphi)$ presented in Fig. 2. φ has

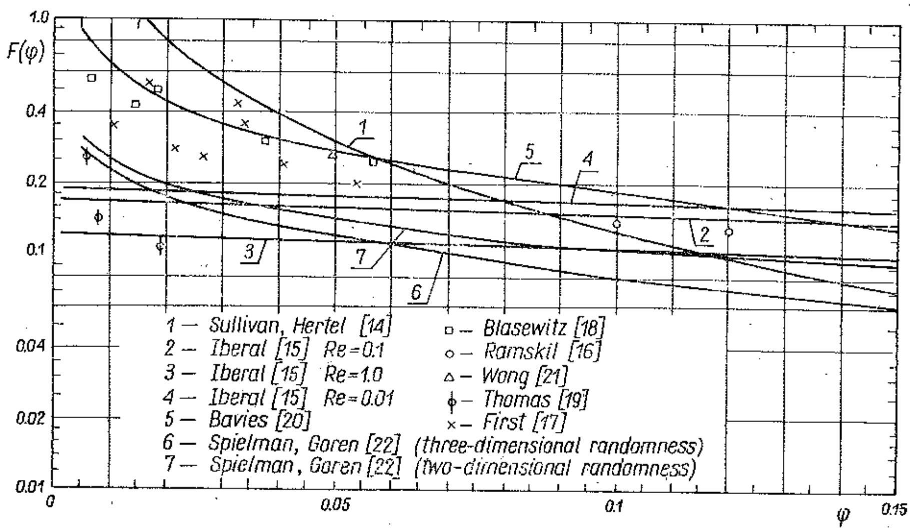

FIG. 2. The function $F(\varphi)$ for $\varphi \ll 1$.

been defined as $1-\varepsilon$. We have found the value of this function on the basis of the results of experimental and theoretical investigations of several authors [14–22]. The above function $F(\varphi)$ enables us to find the value of the permeability of a system consisting of fibres randomly distributed in a domain. We can do it by using the following formula:

$$k = \frac{d^2}{4\pi} F(\varphi), \tag{3.1}$$

where d is the fibre diameter.

Consider as an example the case of $d=0.2$ cm, $\varphi=0.02$, $R=5$ cm. Using Fig. 2 one gets $k/R^2 \approx 0.025$. As it follows from Fig. 2 the formula (2.4) given by Yamamoto may have in this case a practical meaning. The same goes for the formula (2.3) given by Neale, Epstein and Nader if α reaches sufficiently large values. However, the results obtained with the help of these formulae differ from each other. Beavers and Joseph formulating the condition (1.2) proposed an experimental method of determinating the constant α. The idea here is to perform two measurements of the efflux in the case of a Poisseuille flow through a rectangular channel with one permeable wall being replaced by an impermeable one before taking the second measurement. The ratio of the results of these two measurements allows us to determine the constant α.

The above review and discussion lead us to propose an experiment. The point of this proposition is that one can determine the constant α in the Beavers-Joseph condition (1.2) or the effective viscosity $\bar{\mu}$ in the Brinkman equation (1.5) by measuring the velocity of a falling porous sphere the permeability of which is known. It may be expected that this experiment will shed new light both on the problem of the Beavers-Joseph condition justified to apply in a case of non-Poisseuille flow, and on the relation between α and effective viscosity $\bar{\mu}$.

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Streszczenie

PEWNE UWAGI O OPORZE POROWATEJ KULI OPADAJĄCEJ ZE STAŁĄ PRĘDKOŚCIĄ W PŁYNIE LEPKIM NIEŚCIŚLIWYM PRZY MAŁYCH LICZBACH REYNOLDSA

Ze wzorem Stokesa porównuje się wzory otrzymane przez różnych autorów, rozważających problem opadania porowatej kuli w płynie lepkim przy małych liczbach Reynoldsa. Dyskutuje się sensowność wprowadzania tego typu wzorów w zastosowaniu dla rzeczywistych ośrodków porowatych. Wykazuje się, że dla wielu "standardowych" ośrodków porowatych (np. piasek, filc) poprawka wnoszona przez te wzory w porównaniu ze wzorem Stokesa jest nieistotna, niezależnie od założeń odnośnie do warunków brzegowych i równania filtracji, przy jakich wzory te zostały otrzymane. Ośrodek porowaty o budowie "włóknistej" i bardzo dużej porowatości (np. wata) podaje się jako przykład, dla którego niektóre z podanych wzorów mogą mieć praktyczne znaczenie.

Podaje się również nową propozycję eksperymentalnego wyznaczania stałej w warunku Beaversa-Josepha i lepkości efektywnej w równaniu Brinkmana.

Резюме

НЕКОТОРЫЕ ЗАМЕЧАНИЯ О СОПРОТИВЛЕНИИ ПОРИСТОГО ШАРА ПАДАЮЩЕГО С ПОСТОЯННОЙ СКОРОСТЬЮ В ВЯЗКОЙ НЕСЖИМАЕМОЙ ЖИДКОСТИ ПРИ МАЛЫХ ЧИСЛАХ РЕЙНОЛЬДСА

С формулой Стокса сравниваются формулы, полученные разными авторами, в которых рассматривается задача падения пористого шара в вязкой жидкости при малых числах Рейнольдса. Обсуждается смысл введения этого типа формул в применении для реальных пористых сред. Показывается, что для многих „стандартных" пористых сред (например песка, войлока) поправка, вносимая этими формулами по сравнению с формулой Стокса, несущественна независимо от предположений относительно граничных условий и уравнения фильтрации, при которых эти формулы были получены. Пористая среда с „волокнистым" строением и с очень большой пористостью (например вата) приводится как пример, для которого некоторые из приведенных формул могут иметь практическое значение.

Приводится тоже новое предположение экспериментального определения постоянной в условии Биверса-Джозефа и эффективной вязкости в уравнении Бринкмена.

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