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ON THE OPTIMIZATION OF ANNULAR PLATES RESTING ON INCOMPRESSIBLE LIQUID (*)

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The problem of minimization of weight is considered in the case of annular sandwich plates, statically indeterminate with respect to the support reactions and resting on an incompressible liquid; the plates are subjected to axi-symmetric loads, uniformly distributed over the annular region. Under various boundary conditions the fields of deflection rates and limit moments are calculated which correspond to the opitmum design; moreover, the reactions along the interior contour and relations between the loads and the pressure exerted by the liquid on the plate are derived. Numerical results are illustrated by graphs.

1. Introduction

The principles of the general theory of optimization of plastic structures have been formulated in the papers [1, 2] containing the conditions of absolute minimum of weight of sandwich plates and the relative minimum weight conditions of solid structures. The conditions were then used to obtain the solutions to a number of problems of optimum design of plastic structures subject to undirectional loads (cf., e.g. [3]). In the papers [4, 5] circular plates were considered, loaded by two systems of loads acting in opposite directions. Paper [6] presents an analysis of circular plates resting on incompressible liquids and acted upon by a centrally applied, uniformly distributed load.

In this paper we shall consider the problem of optimization of annular sandwich plates, statically indeterminate with respect to the support reactions, resting upon an incompressible liquid and subject to axi-symmetric load uniformly distributed within an annular region, under various boundary conditions.

In the case of analogous plates acted upon by uniformly distributed load over the entire surface, the corresponding problems of minimum of weight has been discussed in [7–10], and the problem of load carrying capacity — in [11].

2. Formulation of the problem and principal relations

Let us consider an annular plate subject to various boundary conditions and resting on an incompressible liquid (Figs. 1a–d). In the cylindrical system of co-ordinates r, φ, z the z-axis is directed downwards. Let A and B denote the respective

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inside and outside radii of the plate, p–the transversal load acting in the direction of z and uniformly distributed over the angular region $A \leqslant r \leqslant C$, q–the unknown pressure exerted by the liquid on the plate (in the case of perfectly plastic plates assumed constant 12) to be determined from the solution.

Let us consider the model of a sandwich plate consisting of two equal girder thin sheets of variable thickness h separated by a core made of a soft material of the thickness $H \gg h$, capable of transmitting shearing stresses only.

The material of the girder sheets is assumed to be homogeneous, perfectly plastic and obeys both the Tresca yield law and the associated flow law.

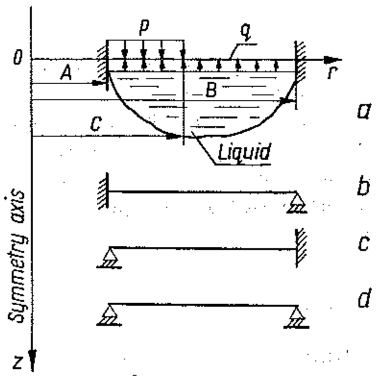

FIG. 1.

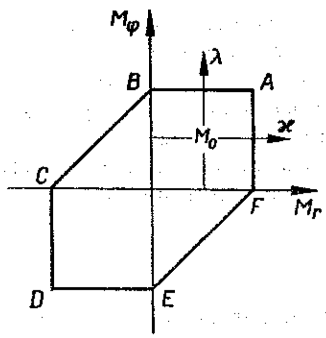

FIG. 2.

The problem of optimization of the plate is now reduced to the determination of the variable thickness h corresponding to the minimum weight condition, i.e. satisfying the condition

$$\int_A^B hr\,dr = \min. \tag{2.1}$$

Denoting by M_r and M_φ the radial and tangential bending moments respectively, the equation of equilibrium of the plate assumes the form

$$2[(rM_r) - M_\varphi] = \begin{cases} 2QA-(p-q)(r^2-A^2), & A \leqslant r \leqslant C, \\ 2QA-p(C^2-A^2)+q(r^2-A^2), & C \leqslant r \leqslant B. \end{cases} \tag{2.2}$$

Here Q denote the unknown interior support reaction forces referred to a unit length, to be determined later. Primes denote differentiation with respect to r.

The Tresca yield conditions is represented in the plane of M_r, M_φ by the hexagon $ABCDEF$ (Fig. 2), where $M_0 = \sigma_0 Hh$ — the total plastic bending moment, and σ_0 — yield limit at pure tension or compression.

The Drucker-Shield optimization criterion has the form [2]

$$\frac{D}{h} = \frac{M_r \varkappa + M_\varphi \lambda}{h} = \text{const}. \tag{2.3}$$

Here D is the rate of dissipation referred to a unit area of the middle surface of the plate, κ and λ — the respective radial and tangential curvature rates which may be expressed in terms of the rate of deflection W as follows:

$$\kappa = -W, \quad \lambda = -W'/r. \tag{2.4}$$

Due to incompressibility of the liquid, the additional condition

$$\int_A^B Wr\,dr = 0, \tag{2.5}$$

is satisfied in the process of bending of the plate. Thus, in order to determine h satisfying Eqs. (2.1), the equilibrium conditions (2.2) and Eqs. (2.3), (2.4) and (2.5) must be solved, the Tresca yield condition and the associated flow law being fulfilled.

The optimal design [13] is known to yield the plastic regimes A, C, D and F. Taking this into account, from the equations of equilibrium (2.2) we obtain for the states A and D $M_r = M_\varphi = \pm M_0$, the upper plus sign referring to A, and minus — to D

$$2rM_0' = \pm \begin{cases} F_1 - (p-q)r^2, & A \leqslant r \leqslant C, \\ F_2 + qr^2, & C \leqslant r \leqslant B. \end{cases} \tag{2.6}$$

For the plastic states F and C ($M_\varphi = 0$, $M_r = \pm M_0$, the plus sign referring to F) we obtain

$$2(rM_0)' = \pm \begin{cases} F_1 - (p-q)r^2, & A \leqslant r \leqslant C, \\ F_2 + qr^2, & C \leqslant r \leqslant B, \end{cases} \tag{2.7}$$

where

$$F_1 = [2Q + (p-q)A]A, \quad F_2 = F_1 - pC^2. \tag{2.8}$$

From the optimization condition (2.3) and Eq. (2.4) it follows that

$$W'' + W'/r = \pm\alpha \tag{2.9}$$

for the states A and D (the minus sign to be referred to A), and

$$W'' = \pm\alpha \tag{2.10}$$

for states F and C (the minus sign to be referred to F), α denoting an additional constant.

3. Examples

3.1. *Annular plates clamped at the contours.*

Under the loading and supporting conditions assumed, in the region adjacent to the interior contour, the plate will be deflected downwards, and in the remaining region — upwards (due to incompressibility of the fluid). The axial symmetry conditions will be preserved. It is easily seen that in the case of a clamped plate (Fig. 1a) there exist the regions $A \leqslant r \leqslant R$, $R_1 \leqslant r \leqslant R_2$, $R_2 \leqslant r \leqslant R_3$, $R_3 \leqslant r \leqslant R_4$, $R_4 \leqslant r \leqslant R_5$ and $R_5 \leqslant r \leqslant B$ characterized by the respective plastic states D, F, A, C, D and F. Such a plastic state is shown schematically in Fig. 3a.

Hence, in order to determine the deflection rates, after solving the system of six differential equations of the type (2.9) and (2.10), we must determine 12 integration constants and five unknown radii of the plastic regions. To this end the following boundary and continuity conditions for W and W' are used

$$\begin{aligned} W=W'&=0 \quad \text{at} \quad r=A, \\ W=W'&=0 \quad \text{at} \quad r=B, \\ [W]=[W']&=0 \quad \text{at} \quad r=R_i, \\ &(i=1, 2, \dots, 5). \end{aligned} \tag{3.1}$$

Here $[W]$ and $[W']$ denote the jumps of W and W'. Since $W'=0$ at $r=R_2$ and $r=R_4$, Eqs. (3.1) yield 16 conditions, and for the determination of the 17th unknown the additional condition (2.5) may be used.

In order to determine the thickness of the plate, seven differential equations of the type (2.6) and (2.7) must be solved; seven integration constants will then appear and, in addition, the liquid pressure and the support reaction at the internal contour will also be unknown. To determine the unknowns, let us apply the following continuity conditions for M_r:

$$[M_r]=0 \quad \text{at} \quad r=R_i \quad \text{and} \quad r=C \quad (i=1, 2, \dots, 5). \tag{3.2}$$

Since the radial bending moment changes its sign in passing through $r=R_1, r=R_3$ and $r=R_5$, we obtain

$$M_r=0 \quad \text{at} \quad r=R_1, \quad r=R_3 \quad \text{and} \quad r=R_5, \tag{3.3}$$

and Eqs. (3.2) yield the 9 necessary conditions.

Before determining W and $M_0=\sigma_0 Hh$ let us introduce the dimensionless quantities

$$\begin{gathered} m_r=4\frac{M_r}{pB^2}, \quad m_\varphi=4\frac{M_\varphi}{pB^2}, \quad m_0=4\frac{M_0}{pB^2}, \\ w=4\frac{W}{\alpha B^2}, \quad \bar{F}_1=\frac{F_1}{pB^2}, \quad \bar{F}_2=\frac{F_2}{pB^2}, \quad \bar{Q}=\frac{Q}{pB}, \\ \xi=\frac{r}{B}, \quad a=\frac{A}{B}, \quad c=\frac{C}{B}, \quad \rho_i=\frac{R_i}{B}, \quad k=\frac{q}{p} \\ (i=1, 2, \dots, 5). \end{gathered} \tag{3.4}$$

From Eqs. (2.9), (2.10) and (3.1), (3.4) we obtain

$$
(3.5)\quad w=\begin{cases}
\xi^2-a^2\left(1+2\ln\dfrac{\xi}{a}\right), & a\leqslant\xi\leqslant\rho_1,\\[2mm]
2\left[\xi(2\rho_2-\xi)-2\rho_1\rho_2\right]-a^2\left(1+2\ln\dfrac{\rho_1}{a}\right)+3\rho_1^2, & \rho_1\leqslant\xi\leqslant\rho_2,\\[2mm]
-\xi^2+2\left(\rho_2^2\ln\dfrac{\xi}{\rho_2}+a^2\ln\dfrac{a}{\rho_1}\right)-a^2+3(\rho_1^2+\rho_2^2)-4\rho_1\rho_2, & \rho_2\leqslant\xi\leqslant\rho_3,\\[2mm]
2\left[\xi(\xi-2\rho_4)+\rho_4^2\ln\dfrac{\rho_5}{\rho_4}-1\right]+3\rho_4^2+\rho_5(4-3\rho_5), & \rho_3\leqslant\xi\leqslant\rho_4,\\[2mm]
\xi^2+2\left[\rho_4^2\ln\dfrac{\rho_5}{\xi}-1\right]+\rho_5(4-3\rho_5), & \rho_4\leqslant\xi\leqslant\rho_5,\\[2mm]
-2(\xi-1)^2, & \rho_5\leqslant\xi\leqslant1.
\end{cases}
$$

The values of $\rho_1, \rho_2, \ldots, \rho_5$ are determined from the relations

$$a^2=\rho_1(3\rho_1-2\rho_2),\qquad \rho_2^2=\rho_3(3\rho_3-2\rho_4),\qquad \rho_4^2=\rho_5(3\rho_5-2),$$

$$
(3.6)\quad 2\left[a^2\ln\frac{\rho_1}{a}-\rho_2^2\ln\frac{\rho_3}{\rho_2}+\rho_4^2\ln\frac{\rho_5}{\rho_4}-1\right]=
=3(\rho_1^2+\rho_2^2-\rho_3^2-\rho_4^2+\rho_5^2)+4(-\rho_1\rho_2+\rho_3\rho_4-\rho_5)-a^2,
$$

and from Eq. (2,5) we obtain the additional condition

$$
(3.6')\quad 27(\rho_1^4-\rho_3^4+\rho_5^4)+\rho_2^4-\rho_4^4-16(\rho_1^3\rho_2-\rho_3^3\rho_4+\rho_5^3)=
=6(a^2\rho_1^2-\rho_2^2\rho_3^2+\rho_4^2\rho_5^2)+3a^4+2.
$$

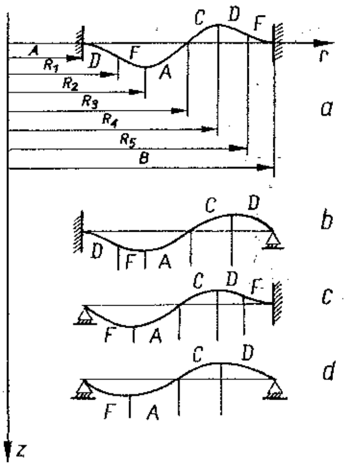

FIG. 3.

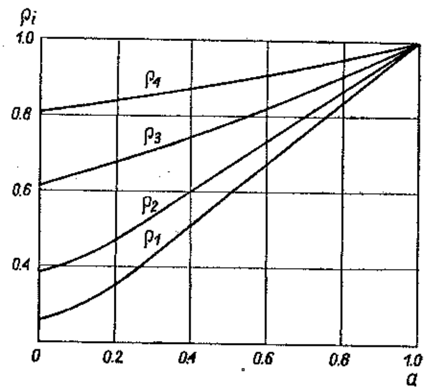

FIG. 4.

The dependence of ρ_i $(i=1, 2, \ldots, 5)$ on a is shown in Fig. 4. From the associated flow law it follows that for the regime A $\kappa\geqslant0$, $\lambda\geqslant0$ $(w''\leqslant0,\ w'\leqslant0)$ and for the regime D $\kappa\leqslant0$, $\lambda\leqslant0$ $(w''\geqslant0,\ w'\geqslant0)$.

It is seen from Eq. (3.5) that these conditions are satisfied. From the regime F the condition $0 \leqslant -\lambda \leqslant \kappa$, and for C the condition $0 \leqslant \lambda \leqslant -\kappa$ is also satisfied what is evident from Eq. (3.5) if $\rho_1 \geqslant \rho_2/2$, $\rho_3 \geqslant \rho_4/2$ and $\rho_5 \geqslant 1/2$. On the basis of Eq. (3.6) these inequalities are also found to be satisfied. In this manner the kinematically possible deflection rate field (3.5) satisfies the associated flow law. In determining the thickness h, the solution will be found to depend on the value of the radius C. Let us consider the particular cases.

1) Assume that $a \leqslant c \leqslant \rho_1$. Then

$$(3.7) \qquad m_0 = \begin{cases} (1-k)\xi^2 + 2\left(\bar{F}_1 \ln \dfrac{\rho_1}{\xi} + c^2 \ln \dfrac{c}{\rho_1}\right) - c^2 + k\rho_1^2, & a \leqslant \xi \leqslant c, \\ k(\rho_1^2 - \xi^2) + 2\bar{F}_2 \ln \dfrac{\rho_1}{\xi}, & c \leqslant \xi \leqslant \rho_1, \\ 2(3\xi)^{-1}[k(\xi^3 - \rho_1^3) + 3\bar{F}_2(\xi - \rho_1)], & \rho_1 \leqslant \xi \leqslant \rho_2, \\ k(\xi^2 - \rho_3^2) + 2\bar{F}_2 \ln \dfrac{\xi}{\rho_3}, & \rho_2 \leqslant \xi \leqslant \rho_3, \\ 2(3\xi)^{-1}[k(\rho_3^3 - \xi^3) + 3\bar{F}_2(\rho_3 - \xi)], & \rho_3 \leqslant \xi \leqslant \rho_4, \\ k(\rho_5^2 - \xi^2) + 2\bar{F}_2 \ln \dfrac{\rho_5}{\xi}, & \rho_4 \leqslant \xi \leqslant \rho_5, \\ 2(3\xi)^{-1}[k(\xi^3 - \rho_5^3) + 3\bar{F}_2(\xi - \rho_5)], & \rho_5 \leqslant \xi \leqslant 1. \end{cases}$$

The continuity conditions of m_r at $\xi = \rho_2$ and $\xi = \rho_4$ yield the following relations to be used for determining k and Q:

$$(3.8) \qquad \begin{cases} 12a\bar{Q} - A_1 k = 6(c^2 - a^2), \\ 12a\bar{Q} - B_1 k = 6(c^2 - a^2), \end{cases}$$

Here

$$(3.9) \qquad A_1 = 6a^2 + \frac{\rho_2(3\rho_3^2 - \rho_2^2) - 2\rho_1^3}{\rho_1 - \rho_2\left((1 + \ln \dfrac{\rho_3}{\rho_2}\right)}, \qquad B_1 = 6a^2 + \frac{\rho_4(3\rho_5^2 - \rho_4^2) - 2\rho_3^3}{\rho_3 - \rho_4\left(1 + \ln \dfrac{\rho_5}{\rho_4}\right)}.$$

From Eqs. (3.8) it is seen that

$$(3.10) \qquad k = 0, \quad \bar{Q} = \frac{c^2 - a^2}{2a}.$$

and hence we finally obtain

$$(3.11) \qquad m_0 = \begin{cases} \xi^2 + c^2\left(2 \ln \dfrac{c}{\xi} - 1\right), & a \leqslant \xi \leqslant c, \\ 0 & c \leqslant \xi \leqslant 1. \end{cases}$$

2) If we assume that $\rho_1 \leqslant c \leqslant \rho_2$, then

$$(3.12)\quad m_0=\begin{cases}(1-k)(\xi^2-\rho_1^2)+2\bar{F}_1 \ln\dfrac{\rho_1}{\xi}, & a\leqslant\xi\leqslant\rho_1,\\ 2(3\xi)^{-1}[(1-k)(\rho_1^3-\xi^3)+3\bar{F}_1(\xi-\rho_1)], & \rho_1\leqslant\xi\leqslant c,\\ 2(3\xi)^{-1}[k\xi^3+3\bar{F}_1(\xi-\rho_1)+c^2(2c-3\xi)+(1-k)\rho_1^3], & c\leqslant\xi\leqslant\rho_2.\end{cases}$$

In the region $\rho_2 \leqslant \xi \leqslant 1$ the corresponding expressions of Eqs. (3.7) are valid. To determine k and $\bar{Q}$ we have now the equation

$$(3.13)\quad \begin{cases}12a\bar{Q}-A_1 k=C_1,\\ 12a\bar{Q}-B_1 k=6(c^2-a^2).\end{cases}$$

Here

$$(3.14)\quad C_1=6(c^2-a^2)+\frac{2(2c^3+\rho_1^3-3c^2\rho_1)}{\rho_1-\rho_2\left(1+\ln\dfrac{\rho_3}{\rho_2}\right)}.$$

In particular, when $c=\rho_1$ all the formulae of Sections 1 and 2 coincide.

3) If $\rho_2 \leqslant c \leqslant \rho_3$, then

$$(3.15)\quad m_0=(k-1)\xi^2+2\left(\bar{F}_1 \ln\frac{\xi}{\rho_3}+c^2\ln\frac{\rho_3}{c}\right)-k\rho_3^2+c^2, \quad \rho_2\leqslant\xi\leqslant c.$$

Regarding m_0 in the regions $a \leqslant \xi \leqslant \rho_2$ and $c \leqslant \xi \leqslant 1$, the values of m_0 will be the same as those calculated for the regions $a \leqslant \xi \leqslant c$ and $\rho_2 \leqslant \xi \leqslant 1$ in the preceding section.

For the evaluation of k and $\bar{Q}$ we use the conditions

$$(3.16)\quad \begin{cases}12a\bar{Q}-A_1 k=C_2,\\ 12a\bar{Q}-B_1 k=6(c^2-a^2).\end{cases}$$

Here

$$(3.17)\quad C_2=-6a^2+\frac{\rho_2(\rho_2^2-3c^2)+2\rho_1^3-6c^2\rho_2\ln\dfrac{\rho_3}{c}}{\rho_1-\rho_2\left(1+\ln\dfrac{\rho_3}{\rho_2}\right)}.$$

At $c=\rho_2$ all the formulae derived in Sections 2 and 3 coincide.

4) Let us assume $\rho_3 \leqslant c \leqslant \rho_4$. Then

$$(3.18)\quad m_0=\begin{cases}(1-k)(\rho_3^2-\xi^2)+2\bar{F}_1 \ln\dfrac{\xi}{\rho_3}, & \rho_2\leqslant\xi\leqslant\rho_3,\\ 2(3\xi)^{-1}[(1-k)(\xi^3-\rho_3^3)+3\bar{F}_1(\rho_3-\xi)], & \rho_3\leqslant\xi\leqslant c,\\ 2(3\xi)^{-1}[-k\xi^3+3\bar{F}_1(\rho_3-\xi)+c^2(3\xi-2c)-(1-k)\rho_3^3], & c\leqslant\xi\leqslant\rho_4.\end{cases}$$

In the regions $a \leqslant \xi \leqslant \rho_2$ and $\rho_4 \leqslant \xi \leqslant 1$ the corresponding formulae derived in the preceding section will hold true. In order to determine k and $\bar{Q}$ we obtain

$$\begin{cases} 12a\bar{Q}+(1-k)\,A_1=0, \\ 12a\bar{Q}-B_1\,k=C_3\,. \end{cases} \tag{3.19}$$

Here

$$C_3=6\,(c^2-a^2)+\frac{2\left[\rho_3\,(\rho_3^2-3c^2)+2c^3\right]}{\rho_3-\rho_4\left(1+\ln\dfrac{\rho_5}{\rho_4}\right)}\,. \tag{3.20}$$

If $c=\rho_3$, all the formulae of Sections 3 and 4 coincide.

5) If it is assumed that $\rho_4 \leqslant c \leqslant \rho_5$, then

$$m_0=(1-k)\,\xi^2+2\left(\bar{F}_1\,\ln\frac{\rho_5}{\xi}+c^2\ln\frac{c}{\rho_5}\right)+k\rho_5^2-c^2\,, \qquad \rho_4\leqslant\xi\leqslant c\,. \tag{3.21}$$

In the regions $a \leqslant \xi \leqslant \rho_4$ and $c \leqslant \xi \leqslant 1$ the formulae derived in the preceding section for the regions $a \leqslant \xi \leqslant c$ and $\rho_4 \leqslant \xi \leqslant 1$ hold true.

The formulae needed for the determination of k and $\bar{Q}$ have the form

$$\begin{cases} 12a\bar{Q}+(1-k)\,A_1=0, \\ 12a\bar{Q}-B_1\,k=C_4\,, \end{cases} \tag{3.22}$$

with

$$C_4=-6a^2+\frac{\rho_4\,(\rho_4^2-3c^2)+2\rho_3^3-6c^2\,\rho_4\,\ln\dfrac{\rho_5}{c}}{\rho_3-\rho_4\left(1+\ln\dfrac{\rho_5}{\rho_4}\right)}\,. \tag{3.23}$$

In particular, for $c=\rho_4$ all the formulae of Sections 4 and 5 coincide.

6) Finally, if we assume that $\rho_5 \leqslant \xi \leqslant 1$, then

$$m_0=\begin{cases} (1-k)\,(\xi^2-\rho_5^2)+2\bar{F}_1\,\ln\dfrac{\rho_5}{\xi}\,, & \rho_4\geqslant\xi\geqslant\rho_5\,, \\ 2\,(3\xi)^{-1}\left[(1-k)\,(\rho_5^3-\xi^3)+3\bar{F}_1\,(\xi-\rho_5)\right], & \rho_5\leqslant\xi\leqslant c\,, \\ 2\,(3\xi)^{-1}\left[k\xi^3+3\bar{F}_1\,(\xi-\rho_5)+c^2\,(2c-3\xi)+(1-k)\,\rho_5^3\right], & c\leqslant\xi\leqslant 1\,. \end{cases} \tag{3.24}$$

Regarding m_0 in the region $a \leqslant \xi \leqslant \rho_4$, the expressions derived in the preceding section are valid.

For the determination of k and $\bar{Q}$ we use the equations

$$\begin{cases} 12a\bar{Q}+(1-k)\,A_1=0, \\ 12a\bar{Q}+(1-k)\,B_1=0, \end{cases} \tag{3.25}$$

whence

$$k=1, \quad \bar{Q}=0. \tag{3.26}$$

In this manner we finally obtain

$$m_0=\begin{cases}0, & a\leqslant\xi\leqslant c,\\ 2\,(3\xi)^{-1}\,[\xi^3-3c^2\,\xi+2c^3], & c\leqslant\xi\leqslant 1.\end{cases} \tag{3.27}$$

In a similar way the solution may be obtained is the case of the load p acting in the region $c\leqslant\xi\leqslant 1$. Then the equilibrium equation (2.2) is replaced with

$$2\,[(rM_r)^1-M_\varphi]=\begin{cases}2QA+q\,(r^2-A^2), & A\leqslant r\leqslant C,\\ 2QA-p\,(r^2-C^2)+q\,(r^2-A^2), & C\leqslant r\leqslant B.\end{cases}$$

The region adjacent to the interior contour will now be deflected upwards, and the remaining region — downwards, and it is easily seen that in the regions $a\leqslant\xi\leqslant\rho_1$, $\rho_1\leqslant\xi\leqslant\rho_2$, $\rho_2\leqslant\xi\leqslant\rho_3$, $\rho_3\leqslant\xi\leqslant\rho_4$, $\rho_4\leqslant\xi\leqslant\rho_5$, $\rho_5\leqslant\xi\leqslant 1$ of the plate, the plastic states A, C, D, F, A and C will appear.

In conclusion, the solution may be obtained corresponding to the case of the load p acting within an arbitrary annular region $c\leqslant\xi\leqslant d$, with $c\geqslant a$ and $d\leqslant 1$.

3.2. Annular plates clamped at the interior contour and simply supported at the exterior contour

For such plates the boundary conditions are

$$\begin{aligned} w=w'&=0 \quad \text{at} \quad \xi=a,\\ w&=0 \qquad\qquad \xi=1,\\ m_r&=0 \qquad\qquad \xi=1.\end{aligned} \tag{3.28}$$

It is easily demonstrated that the plastic states (shown schematically in Fig. 3b) differ from the case discussed in Section 1 by the absence of the plastic regime F corresponding to the region $\rho_5\leqslant\xi\leqslant 1$ adjacent to the exterior contour. Hence, in the annular regions $a\leqslant\xi\leqslant\rho_1$, $\rho_1\leqslant\xi\leqslant\rho_2$, $\rho_2\leqslant\xi\leqslant\rho_3$, $\rho_3\leqslant\xi\leqslant\rho_4$, $\rho_4\leqslant\xi\leqslant 1$ the plastic regimes D, F, A, C, D will be formed. Except for the third expression (3.6) obtained from the condition of continuity of w' at $\xi=\rho_5$, all the formulae derived in Section 3.1 remain valid in this case if $\xi=\rho_5$ is replaced with the dimensionless radius of the interior contour, i.e. $\xi=1$. The dependence of ρ_i $(i=1, ..., 4)$ is shown in Fig. 5.

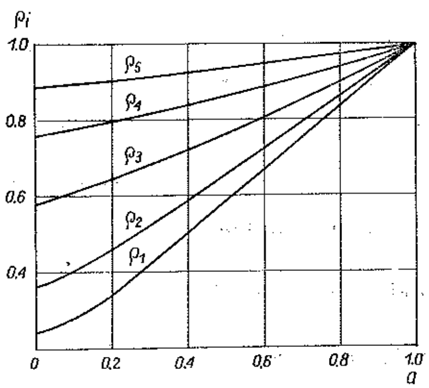

FIG. 5.

3.3. Annular plates simply supported at the interior contour and clamped at the exterior contour

For such plates the boundary conditions are

$$\begin{aligned} w&=0 \qquad\qquad \text{at} \quad \xi=a,\\ w=w'&=0 \qquad\qquad\quad\;\; \xi=1,\\ m_r&=0 \qquad\qquad\quad\;\; \xi=a.\end{aligned} \tag{3.29}$$

The plastic region adjacent to the inside contour and corresponding to the plastic regime D will now be absent from the plate. Such a state is schematically shown in Fig. 3c. Thus the plastic regimes F, A, C, D and F occur in the plate in the respective regions $a \leqslant \xi \leqslant \rho_1$, $\rho_1 \leqslant \xi \leqslant \rho_2$, $\rho_2 \leqslant \xi \leqslant \rho_3$, $\rho_3 \leqslant \xi \leqslant \rho_4$, and $\rho_4 \leqslant \xi \leqslant 1$. Except for the first expression (3.6) obtained from the condition of continuity of w' at $\xi = \rho_1$, all the formulae derived in Section 3.1 remain valid in the case considered, provided $\rho_1, \rho_2, \rho_3, \rho_4$ and ρ_5 are replaced with $a, \rho_1, \rho_2, \rho_3$, and ρ_4.

From the associated flow law for the regime F in the region $a \leqslant \xi \leqslant \rho_1$ it follows that $a \geqslant \rho_1/2$, and thus the solutions obtained will be valid for narrow annular plates.

If $a < \rho_1/2$, the plastic regime D occurs in the plate similarly to the situation encountered in plates clamped at interior contours near that edge. Let this region be $a \leqslant \xi \leqslant \rho_0$. In the regions $\rho_0 \leqslant \xi \leqslant \rho_1$, $\rho_1 \leqslant \xi \leqslant \rho_2$, $\rho_2 \leqslant \xi \leqslant \rho_3$, $\rho_3 \leqslant \xi \leqslant \rho_4$ and $\rho_4 \leqslant \xi \leqslant 1$ the previously mentioned regimes F, A, C, D, F will be preserved.

On using the boundary conditions (3.29) and the corresponding continuity conditions we obtain

$$(3.30)\quad w = \begin{cases} \xi^2 - a^2\left(1 - 2\ln\dfrac{\xi}{a}\right) & a \leqslant \xi \leqslant \rho_0, \\ 2\left[-\xi(\xi - 2\rho_1) + a^2 \ln\dfrac{\rho_0}{a}\right] + 3\rho_0^2 - a^2 - 4\rho_0\rho_1 & \rho_0 \leqslant \xi \leqslant \rho_1, \\ -\xi^2 + 2\left(\rho_1^2 \ln\dfrac{\xi}{\rho_1} + a^2 \ln\dfrac{\rho_0}{a}\right) + 3(\rho_0^2 + \rho_1^2) - a^2 - 4\rho_0\rho_1 & \rho_1 \leqslant \xi \leqslant \rho_2, \\ 2\left[\xi(\xi - 2\rho_3) + \rho_3^2 \ln\dfrac{\rho_4}{\rho_3} - 1\right] + 3\rho_3^2 + \rho_4(4 - 3\rho_4) & \rho_2 \leqslant \xi \leqslant \rho_3, \\ \xi^2 + 2\left(\rho_3^2 \ln\dfrac{\rho_4}{\xi} - 1\right) + \rho_4(4 - 3\rho_4) & \rho_3 \leqslant \xi \leqslant \rho_4, \\ -2(\xi - 1)^2 & \rho_4 \leqslant \xi \leqslant 1. \end{cases}$$

The values $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4$ are determined from the relations

$$a^2 = \rho_0(2\rho_1 - 3\rho_0), \quad \rho_1^2 = \rho_2(3\rho_2 - 2\rho_3), \quad \rho_3^2 = \rho_4(3\rho_4 - 2),$$

$$(3.31)\quad 2\left(a^2 \ln\frac{\rho_0}{a} + \rho_1^2 \ln\frac{\rho_2}{\rho_1} - \rho_3^2 \ln\frac{\rho_4}{\rho_3} + 1\right) - a^2 = \\ = 3(-\rho_0^2 - \rho_1^2 + \rho_2^2 + \rho_3^2 - \rho_4^2) + 4(\rho_0\rho_1 - \rho_2\rho_3 + \rho_4),$$

the first expression being derived from the flow law.

Another condition is found from Eq. (2.5)

$$(3.32)\quad 9(-a^4 + \rho_0^4 - \rho_2^4 - \rho_4^4) - \rho_1^4 + \rho_3^4 + 2 = 8(\rho_1^3\rho_1 - \rho_2^3\rho_3) + 6(a^2\rho_0 + \rho_1^2\rho_2^2).$$

Figure 6 shown the relations between ρ_i $(i = 0, \ldots, 4)$ and a defined by Eqs. (3.30) and (3.31).

It is easily seen that while in the case of plates clamped at interior contours the condition $w'=0$ is satisfied at $\xi=a$ for any a, the same condition in plates simply supported at that contour is satisfied only if $a=0$.

In the particular case of $a=\rho_1/2$ the solutions concerning narrow and wide plates are identical.

The value of m_0 is calculated by means of the formulae derived in Sect. 3.1. It is easily verified, however, that $m_r=0$ does not hold true at the interior contour. To fulfill this condition it is assumed [9] that the annular zone of an infinitesimal width adjacent to the interior contour is characterized by the plastic regime ED of the yield hexagon, and that in the limit of that width approaching zero the regime is transformed to D. The infinitesimal width of the annular zone is denoted by (ρ^*-a). If the assumption is made that

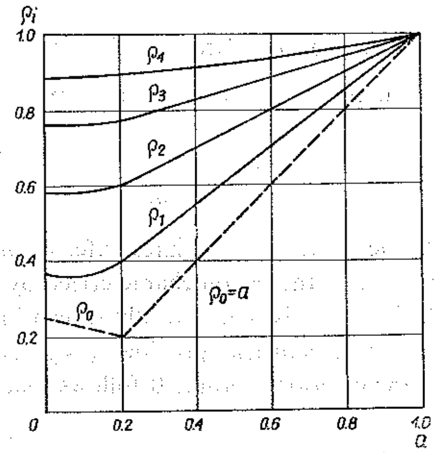

FIG. 6.

a) $\rho^*\leqslant c\leqslant\rho_0$, then the equilibrium equations yield

$$m_r=2\,(3\xi)^{-1}\,[3\,(\bar{F}_1-2\bar{m}_0)\,(\xi-a)-(1-k)\,(\xi^3-a^3)\qquad a\leqslant\xi\leqslant\rho^*,$$

$$(3.33)\qquad m_0=\begin{cases}(1-k)\,\xi^{+2}\left(\bar{F}_1\ln\dfrac{\rho_0}{\xi}+c^2\ln\dfrac{c}{\rho_0}\right)-c^2+k\rho_0^2, & \rho^*\leqslant\xi\leqslant c,\\[2ex] k\,(\rho_0^2-\xi^2)+2\bar{F}_2\ln\dfrac{\rho_0}{\xi}, & c\leqslant\xi\leqslant\rho_0.\end{cases}$$

Here

$$(3.34)\qquad \bar{m}_0=\frac{\sigma_0 Hh}{pB^2}=[12\,(\rho^*-a)]^{-1}\left\{6\bar{F}_1\left[\rho^*\left(1-\ln\frac{\rho^*}{\rho_0}\right)-a\right]-\right.$$

$$\left.-6\rho^*c^2\ln\frac{\rho_0}{c}+(1-k)\,(2a^3+\rho^{*3})+3\rho^*\,(c^2-k\rho_0^2)\right\},$$

and h is the constant width of the annular zone.

b) If $\rho_0\leqslant c\leqslant\rho_1$, then

$$m_r=2\,(3\xi)^{-1}[3\,(\bar{F}_1-2\bar{m}_0)\,(\xi-a)-(1-k)\,(\xi^3-a^3),\qquad a\leqslant\xi\leqslant\rho^*,$$

$$(3.35)\qquad m_0=(1-k)\,(\xi^2-\rho_0^2)+2\bar{F}_1\ln\frac{\rho_0}{\xi},\qquad \rho^*\leqslant\xi\leqslant\rho_0.$$

Here

$$(3.36)\qquad \bar{m}_0=[12(\rho^*-a)]^{-1}\left\{6\bar{F}_1\left[\rho^*\left(1-\ln\frac{\rho^*}{\rho_0}\right)-a\right]+\right.$$

$$\left.+(1-k)\,(2a^3+\rho^{*3}-3\rho^*\,\rho_0^2)\right\}.$$

In the remaining region $\rho_0 \leqslant \xi \leqslant 1$ the value of m_0 is determined on the basis of the formulae of Section 3.1 in which ρ_1 should be replaced with ρ_0. Similar changes should be done in determining k and $\bar{Q}$.

3.4. Annular plates simply supported at the contours

In this case the boundary conditions are

$$w=0 \quad \text{at} \quad \xi=a \quad \text{and} \quad \xi=1, \tag{3.37}$$
$$m_r=0 \qquad \xi=a \qquad \xi=1.$$

Plastic states of such plates differ from those defined in the preceding section by the absence of the region characterized by the regime F adjacent to the interior contour. Such a state is schematically shown in Fig. 3d. Thus the formulae of the preceding section remain true provided $\xi=\rho_4$ is replaced with the dimensionless radius $\xi=1$ of the exterior contour. It follows that in the case of wide plates, the plastic regimes

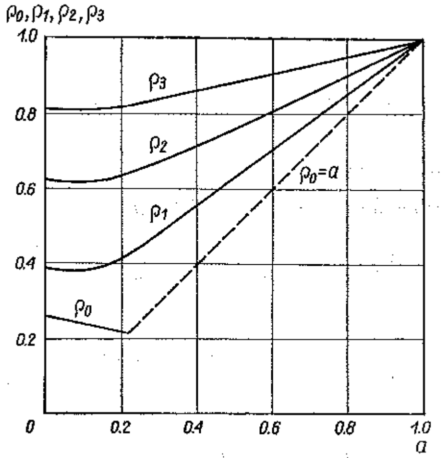

FIG. 7.

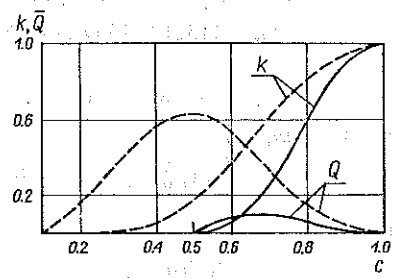

FIG. 8.

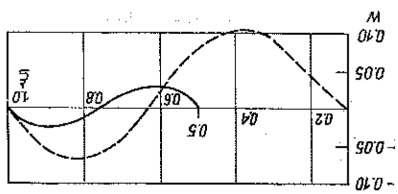

FIG. 9.

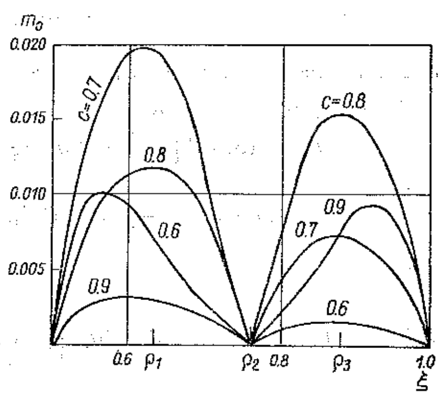

FIG. 10.

D, F, A, C, D occur within the regions $a \leqslant \xi \leqslant \rho_0$, $\rho_0 \leqslant \xi \leqslant \rho_1$, $\rho_1 \leqslant \xi \leqslant \rho_2$, $\rho_2 \leqslant \leqslant \xi \leqslant \rho_3$ and $\rho_3 \leqslant \xi \leqslant 1$. In the case of narrow plates the region $a \leqslant \xi \rho_0$ dissappears and in the corresponding formulae it should be assumed that $\rho_0 = a$.

A special program written in ALGOL was prepared to evaluate $\rho_0, \rho_1, \rho_2, \rho_3$ on a BESM–6 computer. The results of computations of $\rho_0 - \rho_3$ as functions of a are shown in Fig. 7. It is seen from the graphs that $a = 0.2121$ constitutes the limiting value between the narrow and wide plates.

Figure 8 illustrates the dependence of k and $\tilde{Q}$ on c for narrow plates $a = 0.5$ (continuous line) and for wide plates $a = 0.1$ (dashed line). The deflection rates w as functions of ξ in the two cases of plates are shown in Fig. 9, while the dependence of thickness upon ξ for various c in narrow and wide plates is illustrated by Figs. 10 and 11.

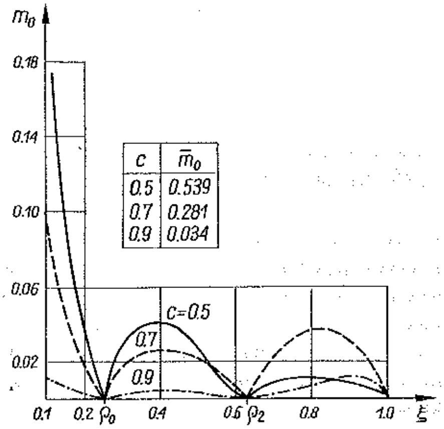

FIG. 11.

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STRESZCZENIE

O OPTYMALIZACJI PŁYT PIERŚCIENIOWYCH SPOCZYWAJĄCYCH NA CIECZY NIEŚCIŚLIWEJ

Rozpatrzono zagadnienie minimalizacji ciężaru uwarstwionych płyt pierścieniowych, statycznie niewyznaczalnych względem reakcji podporowych, spoczywających na cieczy nieściśliwej i poddanych działaniu osiowo-symetrycznego obciążenia równomiernie rozłożonego na obszarze pierścieniowym. Przy różnych warunkach brzegowych określa się pole prędkości ugięć oraz momenty graniczne odpowiadające projektowi optymalnemu, jak również reakcje na konturze wewnętrznym płyty oraz zależności między obciążeniem a ciśnieniem wywieranym na płytę przez ciecz. Wyniki obliczeń numerycznych przedstawiono w postaci wykresów.

Резюме

ОБ ОПТИМИЗАЦИИ КОЛЬЦЕВЫХ ПЛАСТИНОК, ЛЕЖАЩИХ НА НЕСЖИМАЕМОЙ ЖИДКОСТИ

Рассматривается задача минимизации веса статически неопределимых по отношению опорных реакций круглых кольцевых слоистых пластинок, лежащих на несжимаемой жидкости и находящихся под действием осесимметричной, равномерно распределенной в кольцевой области нагрузки. При различных граничных условиях определяются поля скоростей прогибов и предельных моментов, соответствующие оптимальному проекту, опорная реакция на внутреннем контуре, а также взаимодействие между приложенной нагрузкой и давлением жидкости на плстинку. Результаты численных расчетов приводятся в виде графиков.

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