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A CRACK IN AN ANISOTROPIC LAYERED MATERIAL

D. L. CLEMENTS (KENTUCKY) (*)

The problem of determining the stress and displacement fields round a crack in an anisotropic layered material is considered. The problem is reduced to a system of three simultaneous Fredholm integral equations which are solved numerically in certain cases. In particular, the case when the material containing the crack is "strongly anisotropic" is examined in some detail.

1. Introduction

In a recent paper (Clements [1]) the author considered the problem of determining the stress field round a crack in an anisotropic elastic slab. The problem was reduced to three simultaneous Fredholm integral equations which were solved numerically in order to determine the crack energy for some particular anisotropic materials.

In the present paper we use a similar analysis to consider the problem of a crack in an anisotropic layer sandwiched between anisotropic half-spaces (Fig. 1). The

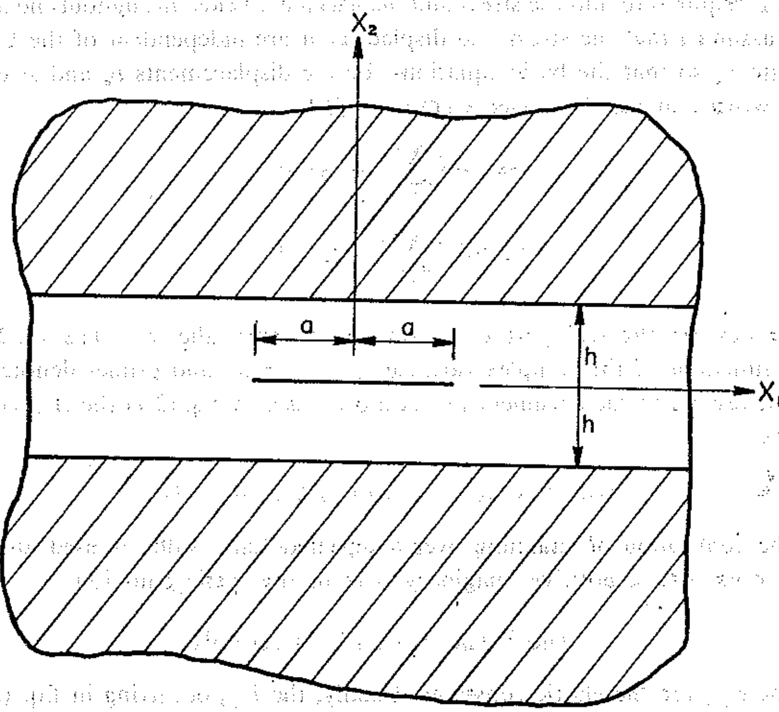

Fig. 1. A crack in a layer between two half-spaces.

(*) On leave from the University of Adelaide, Australia.

important case when the central layer of the composite is "strongly anisotropic" (or, more particularly, almost inextensible in a certain direction) is examined in some detail. Numerical values for the crack energy are obtained for several different values of the layer width and also for a number of combinations of different anisotropic materials. The results reveal that under normal loading the stability of the crack in the layer is not greatly affected by the nature of the material in the half-spaces, provided the layer width is approximately five times greater than the crack length. However, if the layer width is less than three times the crack length, then the effect of the surrounding material is considerable. Also, the results show that if the layer width is al least five times the crack length, then, even if the material containing the crack is "strongly anisotropic", the difference between the energies of a crack in an infinite homogeneous material and the corresponding crack in a layer of the same material sandwiched between two half-spaces is generally small.

2. Statement of problem and basic equations

Take the Cartesian coordinates x_1, x_2, x_3 in a layered anisotropic material. The regions $x_2<-h$, $-h<x_2<h$ and $x_2>h$ are occupied by different anisotropic materials. In the plane $x_2=0$ there exists a crack in the region $|x_1|<a$, $-\infty<x_3<\infty$ (Fig. 1). The crack is opened by equal and opposite tractions on each side of the crack. It is required to find the stress and displacement fields throughout the material.

It is assumed that the stress and displacement are independent of the Cartesian coordinate x_3 so that the basic equations for the displacements u_k and stresses δ_{ij} may be written in the form (see Clements [2])

$$u_k=2\mathscr{R}\sum_{\alpha} A_{k\alpha}\chi_\alpha(z_\alpha), \tag{2.1}$$

$$\sigma_{ij}=2\mathscr{R}\sum_{\alpha} L_{ij\alpha}\chi'_\alpha(z_\alpha), \tag{2.2}$$

where $\mathscr{R}$ denotes the real part of a complex number, the $\chi_\alpha(z_\alpha)$ $(\alpha=1, 2, 3)$ are analytic functions of the complex variable $z_\alpha=x_1+\tau_\alpha x_2$ and primes denote derivatives with respect to the argument in question. Also, in Eq. (2.1) the $A_{k\alpha}$ satisfy the equations

$$(c_{i1k1}+\tau_\alpha c_{i1k2}+\tau_\alpha c_{i2k1}+\tau_\alpha^2 c_{i2k2})A_{k\alpha}=0, \tag{2.3}$$

where the convention of summing over a repeated Latin suffix is used and the τ_α are the roots with a positive imaginary part of the sextic equation

$$|c_{i1k1}+\tau c_{i1k2}+\tau c_{i2k1}+\tau^2 c_{i2k2}|=0, \tag{2.4}$$

where the c_{ijkl} are the elastic constants. Finally, the $L_{ij\alpha}$ occurring in Eq. (2.2) are related to the $A_{k\alpha}$ by the equation

$$L_{ij\alpha}=(c_{ijk1}+\tau_\alpha c_{ijk2})A_{k\alpha}. \tag{2.5}$$

In order to distinguish between the equations for the three regions $x_2<-h$, $-h<x_2<h$ and $x_2>h$, the superscripts L and R will be used to denote the regions $x_2>h$ and $x_2<-h$, respectively. Thus the basic equations for $x_2>h$ are

$$u_k^L=2\mathscr{R}\sum_\alpha A_{k\alpha}^L\chi_\alpha^L(z_\alpha^L),\qquad \sigma_{ij}^L=2\mathscr{R}\sum_\alpha L_{ij\alpha}^L\chi_\alpha^{L'}(z_\alpha^L)$$

with the same expressions with the superscript L replaced by R for the region $x_2<-h$.

3. Solution of the problem

In order to satisfy the boundary conditions on the crack and the continuity conditions at the material interfaces, it is convenient to choose representations for the displacement and stress as follows.

In the region $x_2>h$ we substitute

$$\chi_\alpha^L(z_\alpha)=\frac{1}{2\pi}\int_0^\infty D_\alpha^L(p)\exp(ipz_\alpha)\,dp \tag{3.1}$$

into Eqs. (2.1) and (2.2) to obtain

$$u_k^L=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha A_{k\alpha}^L D_\alpha^L(p)\exp(ipz_\alpha)\,dp\,, \tag{3.2}$$

$$\sigma_{ij}^L=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha L_{ij\alpha}^L D_\alpha^L(p)\,ip\exp(ipz_\alpha)\,dp\,, \tag{3.3}$$

where the functions $D_\alpha^L(p)$ will be determined by the continuity conditions on $x_2=h$. Similarly, in $x_2<-h$ the expressions for the displacement and stress are

$$u_k^R=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha A_{k\alpha}^R D_\alpha^R(p)\exp(-ipz_\alpha)\,dp\,, \tag{3.4}$$

$$\sigma_{ij}^R=-\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha L_{ij\alpha}^R D_\alpha^R(p)\,ip\exp(-ipz_\alpha)\,dp\,, \tag{3.5}$$

where the $D_\alpha^R(p)$ will be determined by the continuity conditions on $x_2=-h$.

For $-h<x_2<h$ we consider the regions $-h<x_2<0$ and $0<x_2<h$ separately. Guided by the analysis in [1] we obtain the following expressions for the displacement and stress in these two regions.

In $0<x_2<h$:

$$u_k=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha A_{k\alpha}\{[E_\alpha(p)+M_{\alpha l}\psi_l(p)]\exp(ipz_\alpha)+F_\alpha(p)\exp(-ipz_\alpha)\}\,dp\,, \tag{3.6}$$

$$\sigma_{ij}=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha L_{ij\alpha}\{[E_\alpha(p)+M_{\alpha l}\psi_l(p)]\exp(ipz_\alpha)-F_\alpha(p)\exp(-ipz_\alpha)\}\,ip\,dp\,. \tag{3.7}$$

In $-h<x_2<0$:

$$u_k=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha A_{k\alpha}\{E_\alpha(p)\exp(ipz_\alpha)+[M_{\alpha i}\bar{\psi}_i(p)+F_\alpha(p)]\exp(-ip\bar{z}_\alpha)\}\,dp\,, \tag{3.8}$$

$$\sigma_{ij}=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha L_{ij\alpha}\{E_\alpha(p)\exp(ipz_\alpha)-[M_{\alpha l}\bar{\psi}_l(p)+F_\alpha(p)]\exp(-ip\bar{z}_\alpha)\}\,ip\,dp\,, \tag{3.9}$$

where the $M_{\alpha j}$ are defined by

$$\sum_\alpha L_{i2\alpha}M_{\alpha j}=\delta_{ij}\,. \tag{3.10}$$

In Eqs. (3.6)–(3.9) the $E_\alpha(p)$ and $F_\alpha(p)$ will be determined by the continuity conditions on $x_2=\pm h$ while the $\psi_i(p)$ will be determined by the continuity and boundary conditions on $x_2=0$.

It is apparent from Eqs. (3.7) and (3.9) that the stress is continuous across $x_2=0$. The difference in the displacement across the plane $x_2=0$ is, from Eqs. (3.6) and (3.8),

$$\Delta u_k=\frac{1}{\pi}\mathscr{R}(B_{kj}-\bar{B}_{kj})\int_0^\infty\psi_j(p)\exp(ipx_1)\,dp\,, \tag{3.11}$$

where

$$B_{kj}=\sum_\alpha A_{k\alpha}M_{\alpha j}\,. \tag{3.12}$$

Now Δu_k must be zero outside the crack and this condition together with the stress boundary conditions on the crack face yield, using Eqs. (3.11), (3.7) and (3.10),

$$\mathscr{R}(B_{kj}-\bar{B}_{kj})\int_0^\infty\psi_j(p)\exp(ipx_1)\,dp=0\quad\text{for}\quad|x_1|>a\,, \tag{3.13}$$

$$\frac{1}{\pi}\mathscr{R}\int_0^\infty\Big[\psi_j(p)+\sum_\alpha\{L_{j2\alpha}E_\alpha(p)+\bar{L}_{j2\alpha}\bar{F}_\alpha(p)\}\Big]ip\exp(ipx_1)\,dp=-P_j(x_1)\quad\text{for}\quad|x_1|<a\,, \tag{3.14}$$

where the $P_j(x_1)$ are the given surface tractions.

The displacements u_k and stresses δ_{i2} must be continuous across $x_2=\pm h$. From Eqs. (3.2)–(3.10) it follows that this requirement will be satisfied if

$$\sum_\alpha[L_{i2\alpha}\{E_\alpha(p)+M_{\alpha j}\psi_j(p)\}\exp(ip\tau_\alpha h)+\bar{L}_{i2\alpha}\bar{F}_\alpha(p)\exp(ip\bar{\tau}_\alpha h)]=\sum_\alpha L^L_{i2\alpha}D^L_\alpha(p)\exp(ip\tau^L_\alpha h)\,, \tag{3.15}$$

$$\sum_\alpha[L_{i2\alpha}E_\alpha(p)\exp(-ip\tau_\alpha h)+\bar{L}_{i2\alpha}\{\bar{F}_\alpha(p)+\bar{M}_{\alpha j}\psi_j(p)\}\exp(-ip\bar{\tau}_\alpha h)]=\sum_\pi\bar{L}^R_{i2\alpha}\bar{D}^R_\alpha(p)\exp(ip\tau^R_\alpha h)\,, \tag{3.16}$$

$$(3.17)\qquad \sum_\alpha [A_{k\alpha}\{E_\alpha(p)+M_{\alpha j}\psi_j(p)\}\exp(ip\tau_\alpha h)+\bar{A}_{k\alpha}\bar{F}_\alpha(p)\exp(ip\bar{\tau}_\alpha h)]= \\ =\sum_\alpha A^L_{k\alpha}D^L_\alpha(p)\exp(ip\tau^L_\alpha h),$$

$$(3.18)\qquad \sum_\alpha [A_{k\alpha}E_\alpha(p)\exp(-ip\tau_\alpha h)+\bar{A}_{k\alpha}\{\bar{F}_\alpha(p)+\bar{M}_{\alpha j}\psi_j(p)\}\exp(-ip_\alpha h)]= \\ =\sum_\alpha \bar{A}^R_\alpha(p)\bar{D}^R_\alpha(p)\exp(ip\tau^R_\alpha h).$$

It is now convenient to put

$$(3.19)\qquad D^L_\alpha(p)=M^L_{\alpha j}\exp(-ip\tau^L_\alpha h)\varphi_j(p),$$

$$(3.20)\qquad D^R_\alpha(p)=M^R_{\alpha j}\exp(ip\tau^R_\alpha h)\omega_j(p),$$

where the φ_j and ω_j are to be determined and

$$(3.21)\qquad \sum_\alpha L^L_{i2\alpha}M^L_{\alpha j}=\delta_{ij},$$

$$(3.22)\qquad \sum_\alpha L^R_{i2\alpha}M^R_{\alpha j}=\delta_{ij}.$$

Use of Eqs. (3.19)–(3.22) in Eqs. (3.15)–(3.18) gives rise to the equations

$$(3.23)\qquad \mathbf{NE}+\overline{\mathbf{R}}\overline{\mathbf{F}}=-\mathbf{NM\Psi}+\mathbf{\Phi},$$

$$(3.24)\qquad \mathbf{RE}+\overline{\mathbf{N}}\overline{\mathbf{F}}=-\overline{\mathbf{N}}\overline{\mathbf{M}}\mathbf{\Psi}+\overline{\mathbf{\Omega}},$$

$$(3.25)\qquad \mathbf{UE}+\overline{\mathbf{V}}\overline{\mathbf{F}}=-\mathbf{UM\Psi}+\mathbf{B}^L\mathbf{\Phi},$$

$$(3.26)\qquad \mathbf{VE}+\overline{\mathbf{U}}\overline{\mathbf{F}}=-\overline{\mathbf{U}}\overline{\mathbf{M}}\mathbf{\Psi}+\overline{\mathbf{B}}^R\overline{\mathbf{\Omega}},$$

where

$$\mathbf{N}=[L_{i2\alpha}\exp(ip\tau_\alpha h)],\qquad \mathbf{R}=[L_{i2\alpha}\exp(-ip\tau_\alpha h)],$$

$$\mathbf{U}=[A_{k\alpha}\exp(ip\tau_\alpha h)],\qquad \mathbf{V}=[A_{k\alpha}\exp(-ip\tau_\alpha h)],$$

$$\mathbf{\Phi}=[\varphi_j],\qquad \mathbf{\Omega}=[\omega_j],\qquad \mathbf{\Psi}=[\psi_j],$$

$$\mathbf{E}=[E_j],\qquad \mathbf{F}=[F_j],\qquad \mathbf{M}=[M_{\alpha j}],$$

$$\mathbf{B}^L=[B^L_{kj}]=\Big[\sum_\alpha A^L_{k\alpha}M^L_{\alpha j}\Big],\qquad \mathbf{B}^R=[B^R_{kj}]=\Big[\sum_\alpha A^R_{k\alpha}M^R_{k\alpha}\Big].$$

Elimination of $\mathbf{\Phi}$ and $\mathbf{\Omega}$ from Eqs. (3.23)–(3.26) yields

$$(3.27)\qquad [\mathbf{N}-(\mathbf{B}^L)^{-1}\mathbf{U}]\mathbf{E}+[\overline{\mathbf{R}}-(\mathbf{B}^L)^{-1}\overline{\mathbf{V}}]\overline{\mathbf{F}}=[-\mathbf{NM}+(\mathbf{B}^L)^{-1}\mathbf{UM}]\mathbf{\Psi},$$

$$(3.28)\qquad [\mathbf{R}-(\overline{\mathbf{B}}^R)^{-1}\mathbf{V}]\mathbf{E}+[\overline{\mathbf{N}}-(\overline{\mathbf{B}}^R)^{-1}\overline{\mathbf{U}}]\overline{\mathbf{F}}=[-\overline{\mathbf{N}}\overline{\mathbf{M}}+(\overline{\mathbf{B}}^R)^{-1}\overline{\mathbf{U}}\overline{\mathbf{M}}]\mathbf{\Psi}.$$

Elimination of **F** now yields

(3.29) $$\mathbf{E}=\mathbf{Q}\mathbf{\Psi},$$

where

(3.30) $$\mathbf{Q}=\{[\overline{\mathbf{R}}-(\mathbf{B}^L)^{-1}\overline{\mathbf{V}}]^{-1}[\mathbf{N}-(\mathbf{B}^L)^{-1}\mathbf{U}]-[\overline{\mathbf{N}}-(\overline{\mathbf{B}}^R)^{-1}\overline{\mathbf{U}}]^{-1}[\mathbf{R}-(\overline{\mathbf{B}}^R)^{-1}\mathbf{V}]\}^{-1}\times$$
$$\times\{[\overline{\mathbf{R}}-(\mathbf{B}^L)^{-1}\overline{\mathbf{V}}]^{-1}[-\mathbf{N}\mathbf{M}+(\mathbf{B}^L)^{-1}\mathbf{U}\mathbf{M}]+\overline{\mathbf{M}}\}.$$

Similar expressions for **F**, **Φ** and **Ω** (in terms of **Ψ**) may now be obtained by back substitution into Eqs. (3.27), (3.23) and (3.24).

For simplicity we now restrict our attention to the case when the material in $x<-h$ is the same as the material in $x_2>h$ so that $\mathbf{B}^L=\mathbf{B}^R$. Also, if we require the applied tractions $P_j(x_i)$ to be even functions of x_i, then it is sufficient to take the functions $\psi_j(p)$ in the form

(3.31) $$\psi_j(p)=i\int_0^a r_j(t)J_0(pt)\,dt,$$

where the $r_j(t)$ are real functions to be determined and J_0 is the Bessel function of order zero. It is apparent that this choice of the $\psi_j(p)$ automatically satisfies the condition (3.13). Also, Eqs. (3.27) and (3.28) may now be readily used to show that

(3.32) $$\mathbf{E}=-\mathbf{F}.$$

Use of Eqs. (3.31) and (3.32) in Eq. (3.14) yields

(3.33) $$\int_0^\infty \cos(px_1)p\,dp\int_0^a r_j(t)J_0(pt)\,dt+\int_0^\infty T_{jk}(p)\cos(px_1)p\,dp\int_0^a r_k(t)J_0(pt)\,dt=$$
$$=\pi P_j(x_1)\quad\text{for}\quad 0<x_1<a,$$

where $T_{jk}(p)$ is the real matrix defined by

(3.34) $$T_{jk}(p)=\sum_\alpha L_{j2\alpha}Q_{\alpha k}(p)+\sum_\alpha \bar{L}_{j2\alpha}\bar{Q}_{\alpha k}(p),$$

with the matrix $[Q_{\alpha k}]$ given by Eq. (3.30). Proceeding as in [1] we may reduce Eq. (3.33) to the form

(3.35) $$r_j(t)+t\int_0^a K_{jk}(s,t)r_k(s)\,ds=2t\int_0^t\frac{P_j(u)\,du}{(t^2-u^2)^{1/2}}\quad\text{for}\quad 0<t<a,$$

where

$$K_{jk}(s,t)=\int_0^\infty T_{jk}(p)J_0(ps)J_0(pt)p\,dp.$$

Equations (3.35) constitute three simultaneous Fredholm integral equations for the $r_j(t)$, $j=1,2,3$.

4. The crack energy

The energy of the crack is given by the integral

$$U=\frac{1}{2}\int_{-a}^{a} P_k(x_1)\Delta u_k\,dx_1, \tag{4.1}$$

where the increment in relative displacement Δu_k between the two sides of the crack is given by Eq. (3.20). Using Eqs. (3.40), (3.20) in Eq. (4.1) it follows that

$$\Delta u_k=i\pi^{-1}(B_{kj}-\bar{B}_{kj})\int_0^\infty \cos(px_1)\,dp\int_0^a r_j(t)J_0(pt)\,dt= \tag{4.2}$$
$$=i\pi^{-1}(B_{kj}-\bar{B}_{kj})\int_0^a r_j(t)\,dt\int_0^\infty J_0(pt)\cos(px_1)\,dp=$$
$$=i\pi^{-1}(B_{kj}-\bar{B}_{kj})\int_{x_1}^a \frac{r_j(t)\,dt}{(t^2-x_1^2)^{1/2}} \quad \text{for} \quad 0<x_1<a.$$

Hence Eq. (4.1) yields

$$U=i\pi^{-1}(B_{kj}-\bar{B}_{kj})\int_0^a r_j(t)\,dt\int_0^t \frac{P_k(x_1)\,dx_1}{(t^2-x_1^2)^{1/2}}. \tag{4.3}$$

5. Numerical results

In the case of a constant applied normal traction P_2^c and zero shear tractions ($P_1=P_3=0$), Eq. (3.47) yields

$$[a^{-1}P_2^{-1}r_j(at')]+t'\int_0^1 K'_{jk}(s',t')\,[a^{-1}P_2^{-1}r_k(as')]\,ds'=\pi t'\delta_{j2}, \tag{5.1}$$

where $t=at'$, $s=as'$, $p=p'/a$ and

$$K'_{jk}(s',t')=\int_0^\infty T_{jk}(p'/a)J_0(p's')J_0(p't')p'\,dp'. \tag{5.2}$$

Also, in this case Eq. (4.3) reduces to

$$U=\frac{1}{2}i(B_{2j}-\bar{B}_{2j})P_2^2a^2\int_0^1 [a^{-1}P_2^{-1}r_j(at')]\,dt'. \tag{5.3}$$

In the remainder of this section the behaviour of Eq. (5.3) for various combinations of particular anisotropic materials is examined.

In the following we shall consider anisotropic materials which are transversely isotropic. Such materials may be characterized by five elastic constants which we will denote by A, N, F, C and L. If the x_3-axis is normal to the transverse plane, then the non-zero c_{ijkl} are given in terms of the constant A, N, F, C and L by the relations

$$c_{1111}=c_{2222}=A\,,\quad c_{1122}=N\,,\quad c_{1133}=c_{2233}=F\,,$$

$$c_{1331}=c_{2332}=L\,,\quad c_{1212}=1/2\,(A-N)\,,\quad c_{3333}=C\,.$$

The general orientation of the Cartesian axes within the material may be conveniently described in terms of two angles α and θ as follows. Suppose that initially (that is, when $\alpha=\theta=0$) the x_3-axis is normal to the transverse plane so that the x_1-axis and x_2-axis lie in the transverse plane. The x_2-axis is kept fixed and the x_1-axis and x_3-axis are rotated through an angle α. The x_1-axis is then kept fixed in its new position at angle α to the transverse plane and the x_2-axis and x_3-axis are rotated through an angle θ. Both of these rotations are in the positive direction. Referred to the rotated frame the elastic constants (denoted by c'_{ijkl}) are given by

$$c'_{ijkl}=a_{ip}\,a_{jq}\,a_{kr}\,a_{is}\,c_{pqrs}$$

where

$$[A_{ij}]=\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ \sin\theta\cos\alpha & \cos\theta & \sin\theta\cos\alpha\\ \cos\theta\sin\alpha & -\sin\alpha & \cos\theta\cos\alpha\end{bmatrix}.$$

We consider a composite made up of transversely isotropic materials. For one of the materials (referred to subsequently as material I) we use the constants $A=5.96$, $N=2.57$, $F=2.14$, $C=6.14$ and $L=1.64$ while for the second material (material II) we use the constants $A=16.2$, $N=9.2$, $F=6.9$, $C=18.1$ and $L=4.67$. If each of these numerical values is multiplied by 10^{11}, then the units for the constants are dynes/cm^2. These are the values of the elastic constants for crystals of magnesium and titanium, respectively, although they are chosen here merely for illustrative purposes.

Using Eqs. (5.1)–(5.3) the variation in U/U_0 (where U_0 denotes the energy of the corresponding crack in an infinite homogeneous material with the same constants as the material in $-h<x_2<h$) was calculated for $\alpha=0$, $\theta=\pi/2$ (so that the crack lies in the transverse plane) and various values of h/a. The results obtained are shown in Table 1. Case I in the table refers to a composite with material I in the

Table 1. Variation of crack energy with layer width for $\alpha=0$ and $\theta=\pi/2$

h/a	1	2	3	4	5	10	20
U/U_0 (Case I)	0.74	0.89	0.94	0.96	0.98	0.99	1.00
U/U_0 (Case II)	1.38	1.13	1.06	1.03	1.02	1.01	1.00

region $-h<x_2<h$ and material II in the regions $x_2<-h$ and $x_2>h$. Case II refers to a composite with material II in the region $-h<x_2<h$ and material I in the regions $x_2<-h$ and $x_2>h$. The results in Table 1 show that, provided h/a is greater than five, the crack energy in the composite is practically the same as the energy of the corresponding crack in an infinite homogeneous material.

If h/a is less than three, then the difference between the materials in the half--spaces and the layer begins to influence appreciably the energy of the crack. When the material in the half-spaces is stronger than the material in the layer (Case I), then the energy of the crack in the composite decreases as h/a becomes small. Hence, according to the GRIFFITH [3] theory of fracture, the crack becomes more stable as h/a becomes small since less energy is available for the formation of the new surface which is inherent in any crack extension. Conversely, if the material in the half-spaces is weaker than the material in the layer (Case II), then the energy of the crack increases as h/a becomes small so that the crack becomes less stable.

Similar results to those shown in Table 1 were obtained for the angles $\alpha=\pi/6$, $\theta=\pi/4$ and $\alpha=\pi/3$, $\theta=\pi/4$. Hence it is reasonable to predict that this pattern would be repeated for all combinations of α and θ.

It is of interest to examine the case when the layer is "strongly anisotropic" since this case will be relevant to certain types of reinforced materials. In particular, it is of interest to consider a layer of material which is almost inextensible in a particular direction. Hence we suppose that the layer $-h<x_2<h$ consists of a material with the same values of A, N, F and L as for material I but with a larger value of C. As C becomes large, the inextensibility of the material in a direction normal to the transverse plane increases. The variation of U/U_0 with C for a layer of such a material sandwiched between two half-spaces of material II is shown in Tables 2, 3 and 4. Also, the variation of U/U_1 (where U_1 is the value of the energy of the

Table 2. Variation of crack energy with C for $\alpha=0$, $\theta=\pi/2$ and $h/a=5$

C	10	20	30	40	50	60	70	80	90	100
U/U_0	0.97	0.98	0.99	1.00	1.01	1.03	1.04	1.06	1.07	1.09
U/U_1	0.72	0.49	0.39	0.34	0.31	0.28	0.26	0.25	0.24	0.23

Table 3. Variation of crack energy with C for $\alpha=\pi/6$, $\theta=\pi/4$ and $h/a=5$

C	10	20	30	40	50	60	70	80	90	100
U/U_0	0.98	0.98	0.98	0.99	0.99	0.99	0.99	0.99	0.99	0.99
U/U_1	0.94	0.86	0.82	0.80	0.78	0.77	0.76	0.75	0.75	0.74

Table 4. Variation of crack energy with C for $\alpha=\pi/3$, $\theta=\pi/4$ and $h/a=5$

C	10	20	30	40	50	60	70	80	90	100
U/U_0	0.98	0.99	0.99	0.99	0.99	0.99	1.00	1.00	1.00	1.00
U/U_1	0.88	0.74	0.69	0.65	0.63	0.61	0.60	0.59	0.59	0.58

corresponding crack in an infinite region containing the homogeneous material I with $\alpha=0$ and $\theta=\pi/2$) is shown in these three tables. It should be noted that the angles above each table refer only to the material in the region $-h<x_2<h$. For the materials in $x_2<-h$ and $x_2>h$ the values of the angles were taken to be $\alpha=0$ and $\theta=\pi/2$.

For the range of values of C considered it is clear from the numerical results that there is little difference between the energies of the crack in the composite and the corresponding crack in an infinite homogeneous material with the same constants as the material in $-h<x_2<h$. Thus, the stability of a crack in an almost inextensible layer of a composite would seem to be almost identical to the stability of the corresponding crack in an infinite material with the same constants as those of the layer. This observation is, of course, dependent on the crack being sufficiently removed from the material interfaces of the composite. If the crack is close to an interface between the layer and the surrounding weaker material, then the results in Table 1 indicate that the crack would be less stable than the corresponding crack in an infinite material with the same constants as those of the layer.

The values of U/U_1 in Tables 2, 3 and 4 show that an increase in C causes a decrease in the crack energy. This decrease in energy is most marked when the crack in the transverse plane (so that $\alpha=0$ and $\theta=\pi/2$).

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Streszczenie

SZCZELINA W ANIZOTROPOWYM MATERIALE WARSTWOWYM

Rozpatrzono zagadnienie określenia pola naprężeń i przemieszczeń wokół szczeliny w anizotropowym materiale warstwowym. Zagadnienie sprowadzono do układu trzech równań całkowych Fredholma, które dla pewnych przypadków rozwiązano numerycznie. Szczególną uwagę poświęcono przypadkowi materiału «silnie anizotropowego».

Резюме

ЩЕЛЬ В АНИЗОТРОПНОМ СЛОИСТОМ МАТЕРИАЛЕ

Рассмотрена задача определения поля напряжений и перемещений вокруг щели в анизотропном слоистом материале. Задача сведена к системе трех интегральных уравнений Фредгольма, которые для некоторых случаев решены численно. Особое внимание посвящени случаю „сильно анизотропного" материала.

DEPARTMENT OF ENGINEERING MECHANICS
UNIVERSITY OF KENTUCKY

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