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ON MAXWELL'S, KLEIN'S AND WIEGHARDT'S STRESS FUNCTIONS FOR DISCONTINUOUS MEDIA

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When working on reciprocal diagrams, later named after Cremona, Maxwell noticed that the statics of plane frames in the absence of external forces can be represented by a piecewise linear, discontinuous stress function of Airy type. Klein and Wieghardt visualized this by the "stress surface", portraying Airy's function degenerating into a piecewise plane "facet surface". In 1920 Funk proposed its practical application in connection with Castigliano's principle. Following lines of analogy proposed by Kron, Kardestuncer and others, the static quantities defining each "facet" prove to be analogous to the circular currents of electrical networks in the same sense as displacement and rotation of a joint in a frame are analogous to the potential of an electric network node. Generalizations into space are discussed.

1. INTRODUCTION

This paper does not deal with anything new in particular apart from some sketchy extensions in the concluding paragraph. The essence of the present work is due to MAXWELL [20]; it was later reformulated and exposed in a very clear and perceptible way by KLEIN and WIEGHARDT [21] and applied to civil engineering by FUNK [24]. The strange thing is that seemingly nobody used it, not even Funk himself who later published a standard work on variational methods [28]. It is only recently that he died in Vienna. On the other hand, the numerical advantages are apparent for frames that have fewer loops (meshes) the joints (nodes), and simultaneous use of deformation and stress function methods allows bounding for approximations of hypercircle type for large systems [6–12].

Maxwell had started from Airy's stress function and its boundary conditions [25]. Here we start from the theory of oriented graphs [1–5, 34], well known and extensively used for the calculation of currents and voltages in electrical networks (see, e.g., [13]). So we can establish analogies between the scalars of Ohm resistor networks and beams with a thin-walled cellular cross-section on the one hand, and the motors [29, 30] of elastic frames on the other hand. This allows to transfer hypercircle methods from elastic structures to electrical networks, and Prandtl's stress function for thin-walled cross-sections arises from a direct analogy with circular electric currents.

Reference should be made here to previous work on electro-mechanical analogies and introduction of topological concepts into the theory of elastic structures, such as in [1–5, 32] and the papers cited there, especially of G. Kron.

2. Branch, node and mesh methods in electricity

The topological concept usual in network analysis is sketched in its very simplest form in Fig. 1; we consider the branch z (heavy line), mark a direction on it (arrow) and note that, on the one hand, it connects two nodes k^- (z) and k^+ (z) and, on the other hand, it separates two meshes oriented by usually counterclockwise (circular arrows) m^- (z) amd m^+ (z), the $+$ and $-$ indices denoting coincidence or anticoincidence with the orientation of the branch. The formulation indicates duality betwen nodes and meshes with respect to the branches, and this duality indeed shows itself very clearly in the different methods of solution.

This duality is only obscured slightly by the use of the total number K of nodes, but only the number M of independent meshes which, after Euler's polyhedron formula, for simply connected planar nets add up to the number Z of branches plus one:

$$K+M=Z+1. \tag{2.1}$$

By suitable conventions this relation can also be preserved for non-planar nets of the Kuratowski type [33].

Some authors therefore consider only the number $N=M-1$ of independent nodes (one node is considered as an "outer node") [32]; another way of restoring of meshes to $L=M+1$ by adding an "outer mesh" oriented clockwise after Fig. 2

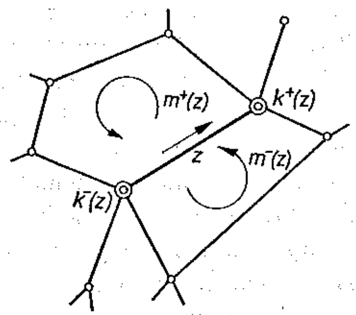

Fig. 1. Branches, nodes and meshes.

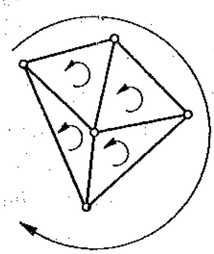

Fig. 2. The outer mesh.

in the simplest case, and containing, in general, all branches that have been passed only once in going around the independent meshes, in the opposite direction.

The unknowns are the branch currents or, connected to them by Ohm's law, the branch voltages. They are to be computed from prescribed inflows at the nodes and circular electromotive forces (EMFs) around the meshes (Fig. 3). Kirchhoff's laws yield K node conditions for currents and M mesh conditions for voltages in the "branch method" where the branch currents themselves serve as unknowns. From Eq. (2.1) we see that the number of equations exceeds the number of unknowns by one; it can be shown that one node equation is superfluous and will only be non-contradictory to the others if the prescribed inflows sum up to zero, corresponding to the conservation of electric charge. Therefore, only $K-1$ of the K node conditions are independent.

A similar solvability condition would arise if we included the Kirchhoff voltage condition for the outer mesh. In Fig. 4 it is shown how a circular EMF can be

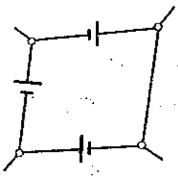

FIG. 3. Circular EMF.

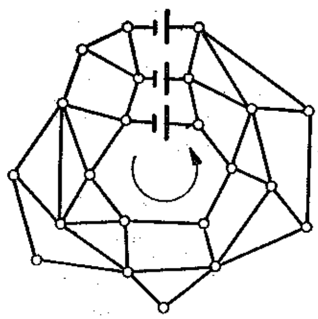

FIG. 4. A single circular EMF in an electrical network.

introduced into a single mesh of a complete net. We have to make some cut going through from the mesh in question to the outer mesh and to introduce identical EMFs into every branch passed on the way. For the rest the cut is arbitrary as long as it connects the two meshes; it is even possible to distribute the circular EMF on several cuts of the same sort. Now it is easily seen that the sum of circular EMFs of a complete system of meshes will cancel, as in passing through all circuits, every EMF is passed twice in opposite directions. And this is exactly the condition that the Kirchhoff equation for outer mesh will not contradict the other mesh equations.

It is clear, after Eq. (2.1), that either K or M or both of them are smaller than Z. It would therefore save a lot of computational labour if the number of unknowns could be reduced to $K-1$ resp. M or to an even smaller number if the problem posed allows it.

Indeed, the "node method" allows the number of unknowns and equations to be reduced to the number K of "free" nodes, for which this potential is not prescribed, by ascribing to each node a potential $-u_k$, which means $N=K-1$ equations at the most as, if for no node the potential is ascribed, it must be fixed arbitrarily for one of them. Here voltage U_z and current I_z in branch z take the form

$$U_z = u_{k^+(z)} - u_{k^-(z)}, \quad I_z = \frac{1}{R_z} U_z. \tag{2.2}$$

This form of solution guarantees the vanishing of all circular EMFs; and the unknown potentials have now to be determined so as to comply with Kirchhoff's node conditions. Using the node-incidence matrix $(\mathscr{E}ZK)$, with element -1 for the origin node and $+1$ for the end node of each branch and zeros for the rest, we obtain for the inflows J_k at the nodes k, collecting them into a column matrix

$$(\mathscr{K})_{KK}(u_k)_K = (J_k)_K, \tag{2.3}$$

$$(\mathscr{K})_{KK} = (\mathscr{E}ZK)^T (1/R_z)_{ZZ} (\mathscr{E}ZK), \tag{2.4}$$

where $(1/R_z)_{ZZ}$ is the diagonal matrix of the conductivities $1/R_z$. The symmetry of the matrix (2.4) is clearly apparent. In the sequel prescribed quantities will be marked by a hook. If $J_k=\check{J}_k$ is prescribed on every node, then the sum must comply with the above condition of the disappearing sum, if the system (2.3) is to have a solution, whilst, on the other hand, one u_k may be prescribed at will. This means $K-1$ equations for $K-1$ unknown potentials. If, on the other hand, the inflows $\check{J}_{kJ}$ are only prescribed on the K_J nodes with numbers k_J, and the potentials $-\check{u}_{ku}$ on the K_u remaining ones numbered k_u, the system (2.3) reduces, after appropriate reordering, to the general form

$$(2.5)\qquad (\mathscr{K})_{K_J K_J}(u_{kJ})_{K_J}=(\check{J}_{kJ})_{K_J}-(\mathscr{K})_{K_J K_u}(\check{u})_{K_u},$$

where $(\mathscr{K})_{K_J K_J}$ and $(\mathscr{K})_{K_J K_u}$ are the corresponding quadratic respective rectangular sections of the original matrix (2.4).

For non-disappearing circular EMFs, some distribution of currents complying with Kirchhoff's mesh conditions is constructed without regard to the node conditions, be it by trial and error or systematically along a complete tree of a dual net, which arises from the original net by interchanging nodes and meshes. Then, after solving an equation of the type (2.5), another distribution of currents of the form (2.2) is added to give the corect inflows and potentials at the nodes.

The "mesh method" on the other hand as the dual counterpart of the node method reduces the number of unknowns and equations to the number M of independent meshes by ascribing to each independent mesh m a circular current i_m. Here I_z and U_z take the form

$$(2.6)\qquad I_z=i_{m^+(z)}-i_{m^-(z)},\qquad U_z=R_z I_z.$$

This form of solution guarantees the vanishing of all inflows at the nodes, and the unknown circular currents have now to be determined so as to comply with Kirchhoff's mesh conditions with prescribed circular EMFs (Fig. 3)

$$(2.7)\qquad \check{W}_m=\sum_{\zeta=1}^{\langle Zm\rangle}\pm\check{V}_{z^\pm(m\zeta)}+\check{W}_{\text{ind}},$$

where $\langle Zm\rangle$ is the number of branches of the mesh m, whilst $z^\pm(m\zeta)$ are the successive branch indices when going around the mesh, and the marks $+$ respectively $-$ on z denote whether orientation of branch and mesh do or do not coincide. $\check{V}_z$ is the prescribed EMF in branch z, referred to the branch orientation, and $\check{W}_{\text{ind}}$ allows for the inductive contribution of a time-dependent magnetic field. If the circular current i_{M+1} of the outer mesh is assumed to be zero, we obtain, after introducing the mesh-incidence matrix $(\mathscr{E}ZM)$, the matrix equation for the unknowns i_m

$$(2.8)\qquad (\mathscr{M})_{MM}(i_m)_M=-(W_m)_M,$$

$$(2.9)\qquad (\mathscr{M})_{MM}=(\mathscr{E}ZM)^T(R_z)_{ZZ}(\mathscr{E}ZM),$$

where $(R_z)_{ZZ}$ is the diagonal matrix of the Ohm resistances R_z.

The case of prescribed circular currents in meshes other than the outer one may be omitted for lack of practical relevance, but non-disappearing inflow must be considered. If non-zero inflows J_{kJ} are prescribed in some nodes k_J, we construct, by trial and error or using a complete tree, some distribution of currents complying with Kirchhoff's node conditions and then add, after solving an equation of the type (2.8), currents of the type (2.6) to arrive at the correct circular EMFs. The nodes k_u, where the potentials $-u_u$ are prescribed, should be connected by zero-resistance "blind branches" with EMFs to a formal outer node; then, in the newly-formed meshes circular EMFs arise that can be included in the above treatment. If it is necessary to fulfil the sum condition on inflows, a non-zero inflow must attributed to the outer node.

3. Electrical networks and beaws with cellular cross-section

There is a one-to-one correspondence between branches of electrical networks and the traces in the transverse plane of the single web plates of a beam with multiply connected, thin-walled ("cellular") cross-section (Figs. 5–8). Flow quantities of the

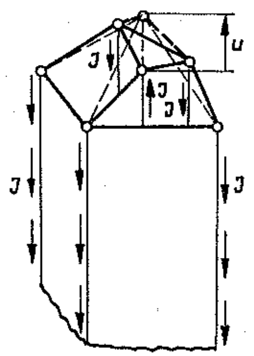

Fig. 5. Forces and displacements of a beam with cellular cross-section.

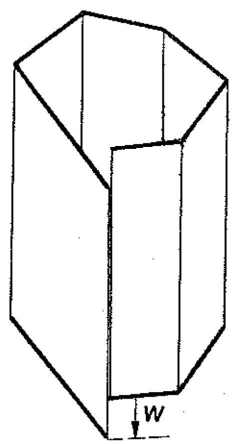

Fig. 6. Screw dislocation in a mesh of the beam's cross-section. The cell corresponding to the mesh is cut along its axis, rewelded in dislocated position and allowed to relax. Then W is the relative displacement of the edges of the cut.

electric net correspond to statical quantities of the elastic beam, and voltages resp. potentials to geometrical quantities.

So, if I_z denotes, in the usual terminology of elasticity and civil engineering, the shear flow in the web plate z, and, after appropriate sign convention, J_k a uniform distribution of forces along the (infinite straight) nodal line k (Fig. 5), then Kirchhoff's node condition

$$\sum_{\zeta=1}^{\langle Zk \rangle} \pm I_z \pm_{(k\zeta)} + J_k = 0 \tag{3.1}$$

describes exactly the equilibrium condition for the nodal line k.

On the other hand, if U_z^E denotes the elastic relative displacement of the nodal line $k^+(z)$ with respect to $k^-(z)$ along the beam's axis, then, by appropriate sign convention, Kirchhoff's mesh condition

$$\sum_{\zeta=1}^{\langle Zm\rangle} \pm U^E_{z\pm(m\zeta)} + W_m = 0 \tag{3.2}$$

describes exactly the compatibility condition for the mesh m where

$$W_m = \sum_{\zeta=1}^{\langle Zm\rangle} \pm V_{z\pm(m\zeta)} \tag{3.3}$$

is the negative Burgers vector of a screw dislocation [15] in this mesh (Fig. 6), and V_z is the non-elastic (e.g. plastic) relative displacement of the nodal lines bounding the web plate z. The analogy to Eq. (2.7) is obvious.

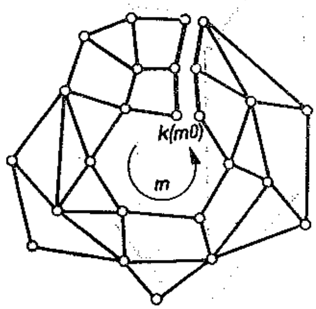

FIG. 7. Introducing a dislocation into an elastic structure (cross-section of a thin walled beam or plane framework).

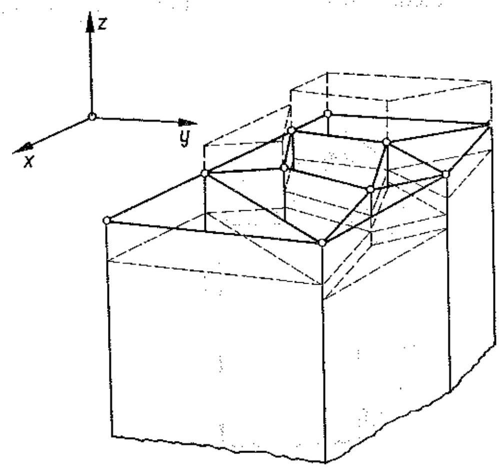

FIG. 8. Prandtl's stress function for a beam with cellular cross-section.

Figure 7 shows how to introduce such a screw dislocation into a single mesh of a complete cross-section. We have first to make a cut as in Fig. 4; for simplicity we now cut very close to a sequence of branches (web plates) connecting the mesh in question with the outer mesh (Fig. 7). Both ends of the cut are now displaced by an equal amount with respect to one another and perpendicularly to the cross--section plane as in Fig. 6, and refixed in the dislocated position. For a better understanding of the problem, one may stiffen the web plates on one edge of the cut to complete rigidity and insert new rigid web plates on the other edge before cutting and dislocating. In order to resoften them to their former elasticity, one must remove them after refixing in the dislocated position. The analogy to the circular EMF is evident from comparison of Figs. 4 and 7.

The role of Ohm's resistance of a branch is, in the analogy, being assumed by the elastic compliance of a web plate. If the web plate is plane, has the thickness h_z, the width l_z, and its material has the shear modulus G_z, then

$$R_z = \frac{l_z}{G_z h_z}. \tag{3.4}$$

It is easy to see now that the negative potential u_k of a node in the electrical network corresponds to the longitudinal displacement in the elastic analogy (Fig. 5). Equation (2.3), therefore, formulates the problem for a cellular beam without dislocations, or for the dislocation-free part of its elastic state after the "deformation method".

For the "equilibrium method" or "force method" we may, in analogy to circular currents, introduce circular shear flows [35, 36]. But it is easier to perceive the connection to the usual form of the thick-walled cross-section theory [37, 38] if i_m is considered following Fig. 8 as a function of position in the cross-section plane, constant in the area enclosed by each mesh and discontinuous on every stressed branch. It turns out then to be Prandtl's stress function in its general form.

The analogy, of course, may be used to model the shear flows in a cellular beam by the currents of an electric network. But it is to be noted that the analogy rests on the hypothesis that the web plates are deformed by shear only and remain plane. This will not, in general, be possible without drilling torques on the (far away) ends of the beam. So, for example, a screw dislocation in a one-cell cross-section will, within the limits of the linear theory, never produce stresses by itself as, in three--dimensional space, the thin walls can relax by bending (paper model).

The most important applications of this theory are neither real force distribution on nodal lines nor real screw dislocations in meshes, but the calculation of secondary stress systems to primary stresses in three dimensions which either would correspond to systems of screw dislocations if the nodal lines were kept straight and parallel and not bent to screw-shape (de St. Venant's torsion), or could not be maintained without forces on nodal lines (warping of cross-section by secondary stresses in bending with shear force).

Without going into details of derivation [26], we state that for de St. Venant's torsion the stresses are the same as if there were a screw dislocation of a negative Burgers vector in every mesh m enclosing the area F_m

$$W_m = -2F_m \Omega, \tag{3.5}$$

where Ω is the drilling angle per unit length. For a homogeneous beam material, Prandtl's stress function is often reduced, by division with $G\Omega$, to a function Ψ with purely geometrical meaning.

From Bredt's formulae it is well known that the drilling torque transmitted by a single circular shear flow is the product of the shear flow with the double cross--sectional area enclosed. In our interpretation this is proportional to the volume bounded by the step function of Fig. 8. Thus, the proportionality of the torque transmitted with the volume of Prandtl's stress hill, well-known from thick-walled or full cross-sections, does also apply for thin-walled ones. Only Prandtl's stress hill is no more the smooth surface of a soap film, but a landscape of plane plateaus separated by vertical cliffs.

We omit here the more complicated case of bending with shear force and refer the reader to [26]. But it should be remarked that there are two definitions of the "center of shear" in literature, both limited to the case of a homogeneous beam material or, at least, the constant Poisson's ratio over the whole cross-section. One definition starts from the disappearing of local torsion on the cross-section barycenter [38, 39] and the other one from the postulate of orthogonality, in a sense to be defined in the next section, of torsional and shear force stresses [40, 41]. We propose to call the first one, with [38, 39], "flexure center" and the second one "torsion center". In general, the two definitions do not coincide, except for symmetric and — within the order of wall thickness — open thin-walled cross-sections. They differ also for the closed thin-walled cross-sections treated here, such that in the principal axes system of the cross-section

$$Y_T - Y_F = \frac{1}{\mathcal{J}_y(m+1)} \int Y\Psi\, dS, \tag{3.6}$$

where $\mathcal{J}_y$ is the geometric moment of inertia with respect to the y-axis, m Poisson's number and the integral represents the moment of the reduced Prandtl stress hill with respect to the x-axis, parallel to which the resultant shear force acts.

Before generalizing this picture to the plane framework, we shall, in the next section, discuss the dual variational principles of elasticity which lend themselves very well to combination with simultaneous application of node and mesh methods.

4. On the hypercircle method

Already Maxwell [14], knew that for a network without circular EMFs "*the heat... generated when Ohm's law is fulfilled is mechanically equivalent... to the sum of the quantities of electricity supplied at the different external electrodes, each multiplied by the potential at which it is supplied*". Now, if there are inflows and circular currents at the same time, we have, by the linearity of Ohm's law, a linear superposition of a system of currents or "state" f' caused by the inflows alone, and another state f'' caused by the circular EMFs alone. The electric power, in symbolic form, may then be written

$$\{f'+f'', f'+f''\} = \{f', f'\} + \{f'', f''\} + 2\{f', f''\}. \tag{4.1}$$

The last term, the double "interaction power", may then be written as the sum of inflows of f'', each multiplied with the potential of f' at the same node. But as, by definition, each inflow of f'' disappears, the whole interaction power of a "curl-free" state f' and a "source-free" state f'' will disappear. In the language of functional analysis, f' and f'' are orthogonal.

For two arbitrary states $\hat{f}$ and $\hat{\hat{f}}$, the interaction power

$$\{\hat{f}, \hat{\hat{f}}\} = \sum_{z=1}^{Z} R_z \hat{I}_z \hat{\hat{I}}_z \tag{4.2}$$

exhibits all properties of a scalar product defining a Hilbert space $\mathscr{H}$. By the orthogonality of all states f' to all states f'', the Hilbert space $\mathscr{H}$ is divided into two orthogonal subspaces $\mathscr{H}'$ and $\mathscr{H}''$. This is the basis of Prager's and Synge's hypercircle method [5–10]. In elasticity, the states f' correspond to states without dislocations and prescribed non-zero displacements (pure load stress states), whilst the states f'' are pure self stress states, that is, states of stresses without external forces, except on points where displacements are prescribed. The scalar product is the elastic interaction energy per unit length of the beam.

The essence of the method may be visualized by the symbolic representation of Fig. 9 which depicts the states of an electric network or an elastic structure into

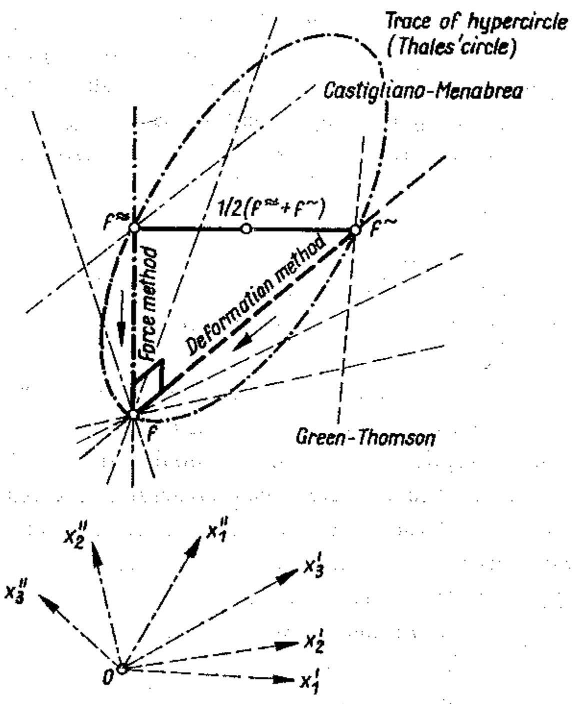

FIG. 9. Symbolic representation of the hypercircle method.

a multi- or infinite-dimensional Euclidean space and projects it onto a two-dimensional plane. The axes x_1', x_2', ... are all orthogonal to each other, and the hatched lines are parallel to some line in the subspace $\mathscr{H}'$ spanned by them. The corresponding assertion applies to the axes x_1'', x_2'', ..., spanning the subspace $\mathscr{H}''$, and the dot-hatched lines. Hence, any hatched line is perpendicular to any dot-hatched line.

Now, let f be the unknown state, complying — in the elastic (electric) case — with all geometric and static (voltage and inflow) conditions. Then we construct (see Sect. 2) two states $f^{\sim}$ and $f^{\approx}$, the first of which fulfills all geometric (voltage and potential) conditions, that is for networks, all mesh conditions, and the second one all static (inflow) conditions, that is for networks, all node conditions. Consequently,

$$(4.3) \qquad f^{\sim}-f\in\mathscr{H}', \quad f^{\approx}-f\in\mathscr{H}''.$$

The displacement or deformation method (node method of Sec. 2) then depicts itself on the fat hatched line; by solving Eq. (2.3), the unknown state f sought for is reached in one step. Correspondingly, the stress function or force method (mesh method of 2; solution of Eq. (2.8)) depicts itself on the fat dot-hatched line. And the hypercircle method makes use of the fact that the points depicting $f, f^{\sim}$ and $f^{\approx}$ form a rectangular triangle with its vertex in f. This means that in approaching $f^{\approx}$ on any of the hatched lines through $f^{\sim}$, that is, in adding an arbitrary load stress state g' with a multiplier γ' such that the distance $|||f^{\sim}+\gamma' g'-f^{\approx}|||$ becomes a minimum, one gets nearer to f, too. The same applies for approaching $f^{\sim}$ on one of the dot-hatched lines through $f^{\approx}$, or minimizing the distance $|||f^{\approx}+\gamma'' g''-f^{\sim}|||$ with respect to γ''. Replacing the single load stress state g' (self stress state g'') by a system of load stress states (self stress states) we arrive at the Ritz method, applied on the Green-Thomson principle of virtual work (Castigliano-Menabrea principle). The node method (the use of displacements resp. potentials) eases the construction of states g', whilst the mesh method (the use of circular currents resp. Prandtl's stress function) helps in constructing states g''.

Finally, $|||f^{\sim}-f^{\approx}|||$ gives an upper bound to $|||f^{\sim}-f|||$ and $|||f^{\approx}-f|||$, and half of it is the exact value of $|||\frac{1}{2}(f^{\sim}+f^{\approx})-f|||$. This last result is visualized by the Thales' circle over $f^{\sim}f^{\approx}$. Of course, the union of all these Thales' circules is a multi- or infinite-dimensional structure, called hypercircle and gives its name to the method. For its application to obtain global and pointwise bounds in the potential theory, elasticity and discrete elastic structures, the reader is referred to [7–12, 42]; for electrical networks we cite again Maxwell [14]: "*In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current*". This is exactly analogous to the Castigliano-Menabrea principle applied to a beam with cellular cross-section.

5. Plane elastic frames

"*The layman is struck by the similarity of an elastic structure, such as a bridge-network or the girders of a skyscraper, to an electrical network. ...Any layman would assume that the concepts of combinatorial topology are surely being applied equally and in a parallel manner to the study of both electrical and mechanical structures. How wrong the layman is*!" These lines of G. Kron [3] where written before the papers [1, 4, 5, 32] had appeared. But some of the difficulties he mentions still exist and block the way for wide-spread practical applications of the existing analogies.

First, Kron rightly points out that in going from electrical networks to elastic frameworks the scalars have to be replaced by motors, but for a monography on motor calculus he only cites Study's original book of 1903. Really, if the author is to cite recent literature on this subject, he can only mention [29] and [30], the first of which is mainly interested in differential forms describing a Cosserat continnum, and the second, whilst giving an elementary exposition from the standpoint of mechanics, offers language difficulties to any reader who does not understand Russian. Here we profit of the still comparatively simple structure of motors in the plane case.

Another difficulty arises in finding the analogue of the electric current. Section 3 suggests that it sould be a static quantity, probably of motor character. "*An attempt to define the admittance matrix ... for an isolated beam with two end-points, brings into sharp focus the utter lack of a topological viewpoints in the theory of elastic structures. The author was unable to find an expression in the literature for a beam accessible from both of its and-points* (Kron l.c.)". Kron then solves this difficulty by grounding his electric beam model in the middle, obtaining a 12×12 matrix characterizing the beam.

The present author prefers another way which allows him to dispose with grounding or anything like that, and to describe static and geometric quantities by motors related to the beam independent of its support. As a motor in the plane has only three components, the elastic properties of one beam are described by a 3×3 matrix; in space a 6×6 matrix would have to be expected. It is shown that by this representation the development of a mesh method, not yet common in structural mechanics, becomes obvious and, finally, we are led back to the idea of an Airy stress function for discontinuous systems already developed by MAXWELL [20–24].

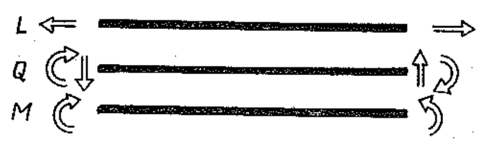

FIG. 10. Elementary equilibrium systems on a straight beam.

We consider a straight beam loaded only at the ends by an equilibrium system of forces and torques. Any such system can be composed of three elementary component equilibrium subsystems, one representing the longitudinal force L, one the shear force Q and one the mean bending moment M (Fig. 10). It is to be noted that Q also includes the linearly variable bending moment associated with the shear force and necessary for pointwise equilibrium.

Of the deformation of the beam, only the relative displacements and rotation of both ends are of interest. Following Kron's remark, we have to find a representation which does not depend on fixing one end or any other point of the beam. This is no problem for the relative rotation Φ (Fig. 11). It is also no problem for the longitudinal relative displacement U_x in the beam's coordinete system $x(z), y(z)$;

only for the transverse relative displacement do we have to find a quantity into which the absolute rotation of either end does not enter. This condition is fulfilled by the directed length $\tilde{U}_y$ depicted in Fig. 10 and illustrated by two rigid levers attached tangentially to the ends of the beam. This mechanism also exhibits a palpable way of applying the load corresponding to the shear force Q in Fig. 10 (*).

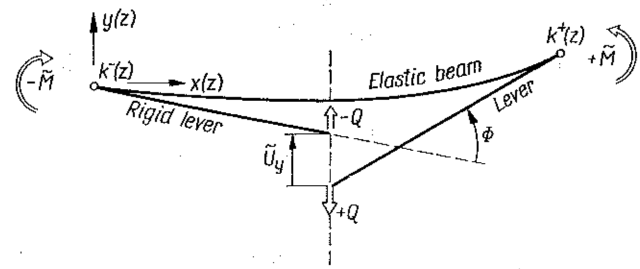

FIG. 11. Deformation of a beam ($Q<0$).

Hooke's law for a straight beam, connecting the "cinemate" $\mathscr{U}$ with the "dyname" $\mathscr{P}$, takes the form

$$\mathscr{P}=\mathscr{C}\mathscr{U}, \quad \mathscr{U}=\mathscr{S}\mathscr{P}, \quad \mathscr{C}=\mathscr{S}^{-1},$$

(5.1) $$\mathscr{P}=\begin{pmatrix} L \\ Q \\ M \end{pmatrix}, \quad \mathscr{U}=\begin{pmatrix} U_x \\ U_y \\ \Phi \end{pmatrix}, \quad \mathscr{S}=\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ & & S_{33} \end{pmatrix},$$

where the coefficients S_{ij} of the elastic compliance matrix $\mathscr{S}$ can be computed by the well-known methods of elementary structural mechanics. Generally, $\mathscr{S}$ and $\mathscr{C}$ are symmetric and positive matrices except that in special cases, as for hinged frameworks, the elastic stiffness matrix $\mathscr{C}$ may be positive semi-definite, so that certain components of the dyname $\mathscr{P}$ must be excluded. Also, an inextensible or otherwise rigid beam may have a semi-definite $\mathscr{S}$. Therefore, if appropriate precautions are taken for these special cases, the elastic interaction energy of the states f and $\hat{f}$

(5.2) $$\{f,\hat{f}\}=\sum_{z=1}^{Z}\hat{\mathscr{P}}_z\mathscr{S}_z\hat{\hat{\mathscr{P}}}_z$$

has, like Eq. (4.2), all properties of a scalar product for a physically realistic frame. For straight beams, by symmetry

(5.3) $$S_{12}=S_{13}=0.$$

But otherwise Eq. (5.1) applies also to curved beams. The only difference is that L and Q are no more the real longitudinal and shear forces all along the beam, but only fictive quantities along a straight line connecting the ends, and M is no longer the arithmetic mean of the bending moment along the beam, but just the mean of its value at both ends. So $\mathscr{P}$ has a real meaning only for the ends of the beam, and the same applies to $\mathscr{U}$. But this is enough for framework analysis.

(*) For generalizations see DAŠEK 1930 [60–62].

If forces are distributed along the beam, we calculate the dynames (forces and torques) that would act on the ends if it were rigidly fixed there, so that the relative cinemates cancel. If the beam is then inserted into the framework, it deforms as if the opposite dynames would act on both end nodes of it.

We can thus limit ourselves now to elastic frameworks, where external forces act only on the nodes. It these external forces are the only case of the stresses, we can speak, as in Sections 3, 4, of load stress states. Self-stress states, arising without action of external forces, are caused by Volterra distortions (dislocations and disclinations) or node cinemates prescribed by rigid supports. The latter can be reduced formally to Volterra distortions after connecting all supports by rigid beams joined together in a formal outer node. And the analogue to Kirchhoff's mesh condition will be obtained by generalization of Eq. (3.2).

To describe a Volterra distortion, we can make recourse to Fig. 7. We have only to replace the word "web plate" in the text by "beam", and to generalize the relative displacement of the edges of the cut to a general rigid movement, also to be described by a cinemate, that is, mathematically, a motor. Being limited to the plane, the cinemate reduces to one relative rotation (wedge disclination) and two relative translations (edge dislocations). An alternative way to introduce Volterra distortions is to replace the cuts by plastic or other non-elastic relative displacements analogously to Eq. (3.3). The only difference is that we now have to replace the simple addition of scalars by the superposition of cinemates according to the rules of the cinematics of rigid bodies. It will be described later in more detail.

To sum up, out basic problem is again, as in Sect. 3, the calculation of stress resultants in an elastic network under the action of dynames at the nodes and of Volterra distortions in the meshes. The only difference is that addition has to be replaced by superposition after the rules of statics and cinematics or, what means the same, to be understood in the sense of motor calculus. In analogy to Sect. 2, we establish now the appropriate node and mesh methods.

As forces are assumed to act on the nodes, the node method is here clearly identical with the displacement or deformation method which is firmly established in the practical computation of structures, so that we have only to describe some differences in formulation. We write the dyname acting on the node k and the cinemate describing its movement in the form of block matrices, marked by square brackets in contrast to ordinary matrices with scalar elements (round brackets).

$$\begin{bmatrix} \mathbf{K}_k \\ \mathscr{M}_k \end{bmatrix}, \quad \begin{bmatrix} \mathbf{u}_k \\ \varphi_k \end{bmatrix}. \tag{5.4}$$

Here the vectors $\mathbf{K}_k$ and $\mathbf{u}_k$ mean the force resp. the absolute displacement of the node; for practical calculation, they have to be represented by 2×1 column matrices with their Cartesian components, referred to some coordinate system associated to the node. The scalars $\mathscr{M}_k$ and φ_k are the torque resp. the absolute rotation of the node.

For calculation of the deformation cinemate $\mathscr{U}_z$ of the beam z in Eq. (5.1) this has to be transformed into the coordinate system of the beam by the transformation block matrix

$$[\mathscr{T}_{zk}]_{\pm} = \begin{bmatrix} +\mathbf{i}_z^T & 0 \\ +\mathbf{j}_z^T \mp \frac{1}{2}\,l_z \\ \mathbf{0}^T & 1 \end{bmatrix}. \tag{5.5}$$

Here $\mathbf{i}_z$ and $\mathbf{j}_z$ are the unit vectors of the beam coordinate system of Fig. 11; the symbol T ("transpose") means that, if writing them put in node coordinates, we have to use the form of a 1×2 row matrix (transpose of a column), so that matrix multiplication always gives the scalar product of the vectors. This also implies that in transposing the block matrix the sub-matrices must be transposed simultaneously. l_z is the length of beam z, and the use of $+$ or $-$ sign depends on whether $k=k^+(z)$ or $k=k^-(z)$. Now, Eq. (2.2) is replaced by

$$\mathbf{u}_z = [\mathscr{T}_{zk^+(z)}]_+ \begin{bmatrix} \mathbf{u}_{k^+(z)} \\ \boldsymbol{\varphi}_{k^+(z)} \end{bmatrix} - [\mathscr{T}_{zk^-(z)}]_- \begin{bmatrix} \mathbf{u}_{k^-(z)} \\ \boldsymbol{\varphi}_{k^-(z)} \end{bmatrix}. \tag{5.6}$$

To arrive at a system of equations analogous to Eq. (2.3), we have to replace in Eq. (2.4) the node-incidence matrix by a block matrix arising from the substitution

$$\pm 1 \Rightarrow \pm [\mathscr{T}_{zk}]_{\pm} \tag{5.7}$$

and the diagonal matrix of conductivities by a diagonal block matrix where the stiffness matrices of the beams take their place

$$\frac{1}{R_z} \Rightarrow \mathscr{C}_z. \tag{5.8}$$

The analogue to Kirchhoff's node condition is, apparently, the equilibrium condition for nodes, as in Section 3, and, therefore, on the right side of the equations we have to replace the inflows by the dynames acting on the nodes

$$J_k \Rightarrow \begin{bmatrix} \mathbf{K}_k \\ \mathscr{M}_k \end{bmatrix}. \tag{5.9}$$

And the solvability condition is here the condition of overall equilibrium.

Some complications may arise when establishing the analogue to Eq. (2.5) as there may be not only nodes with a completely rigid support, but also fixed hinges or supports with sidesway etc. That the stiffness matrices $\mathscr{C}$ also may be incomplete has already been remarked; extreme cases are inextensible beams, on the one hand, and the bars of a hinged framework, on the other hand. As the deformation method is a firmly established and widely used method, we can leave out details and refer to [26].

The same cannot be said of the force method. At present, it has the state of an art, depending on the skill of working with the hatchet on a statically indeterminate system until it becomes statically determinate, but not of an algorithm fitted to stand the "*never-ending race between the large-scale problems to be solved, and the* *size of the available computer*" (Kron l.c.). The mesh method for electrical networks allows for such algorithms, and we have to find now the analogue for the plane frame.

It seems natural to replace the scalar i_m in Eq. (2.6) by a motor describing a static quantity, that is, a dyname referring to the mesh m. Indeed, if we replace in Eq. (2.6) i_m by another scalar $f_m(x, y)$, and I_z by the bending moment $M_z(x_z)$ in the beam z, then the resulting torque on any node is zero. This follows from Eq. (2.6), for if we go around a node we find that every $f_m(x, y)$ occurs twice and with opposite sign on a closed circuit such that the sum cancels. The only difference is that we now have to contract the circuit on the node to get the value at the end, as after Fig. 10, the bending moment is in general variable along the beam, and the shear forces can also contribute to the torque on the node. But the connection between the bending moment and the shearing forces is correctly represented if $f_m(x, y)$ is taken to be linear and the shearing force as its derivative along the beam.

The longitudinal force, on the other hand, has to fulfil, together with the shear force, the equilibrium condition for forces. Therefore, it is obvious to replace the scalars in Eq. (2.6) by vectors representing on the left side the resultant of shear and longitudinal force, and on the right side forces referred to a mesh. To comply with the connection between the shearing force and the linear function

$$f_m(x, y) = \tilde{f}_m + \mathbf{f}_m \cdot (\mathbf{x} - \mathbf{x}_m), \tag{5.10}$$

we take this vector to be $-\mathbf{k} \times \mathbf{f}_m$, that is, the gradient of $f_m(x, y)$ rotated by right angle clockwise, as $\mathbf{k}$ means the unit vector of the z-direction perpendicular to the plane of the frame and right-handed with respect to x and y (Fig. 12). $\mathbf{x}$ is the position vector and $\mathbf{x}_m$ some fixed reference vector in the plane of the frame. If we specialize

$$\mathbf{x}_m = \mathbf{x}_{k(m0)}, \quad \tilde{f}_m = \overset{0}{\tilde{f}}_m, \quad \mathbf{x}_k = \overset{0}{\mathbf{x}}_k, \tag{5.11}$$

where $k(m, 0)$ is a fixed reference node of the mesh m, and introduce for transformation from the mesh system into the beam system the new transformation block matrix

$$[\overset{0}{\mathscr{R}}_{zm}] = \begin{bmatrix} \mathbf{i}_z^T & 0 \\ \mathbf{j}_z^T & 0 \\ -(\mathbf{k} \times \overset{0}{\tilde{\mathbf{x}}}_z)^T & 1 \end{bmatrix}, \tag{5.12}$$

where

$$\overset{0}{\tilde{\mathbf{x}}}_z = \tfrac{1}{2}(\overset{0}{\mathbf{x}}_{k(z^+)} + \overset{0}{\mathbf{x}}_{k(z^-)}) \tag{5.13}$$

is the position vector of the mid-point of the beam with respect to the reference node, then, using the notation (5.1), Eq. (2.6) may be replaced by

$$\mathscr{P}_z = [\overset{0}{\mathscr{R}}_{zm^+(z)}] \begin{bmatrix} -\mathbf{k} \times \mathbf{f}_{m^+(z)} \\ \overset{0}{\tilde{f}}_{m^+(z)} \end{bmatrix} - [\overset{0}{\mathscr{R}}_{zm^-(z)}] \begin{bmatrix} -\mathbf{k} \times \mathbf{f}_{m^-(z)} \\ \overset{0}{\tilde{f}}_{m^-(z)} \end{bmatrix}. \tag{5.14}$$

The elastic relative cinemate $\mathscr{U}_z^E$ of the beam then follows by the notation (5.1).

This is an elementary derivation, using only elements of structural analysis. Maxwell [20] obtained the same relations starting from the boundary conditions on Airy's stress function, summarized, for example, in [25]. Indeed, Eq. (5.10) is nothing else than the zero stress Airy stress function of a hole, producing no stresses. It is nicely illustrated by Klein's and Wieghartd's "facet surface" [21], of which Fig. 12 shows two facets. It is clearly a generalization of Fig. 8.

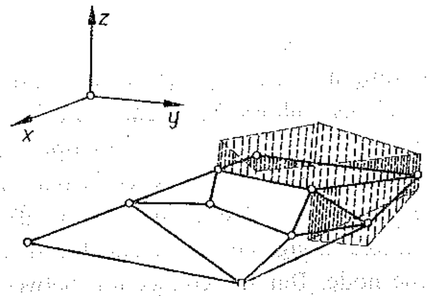

FIG. 12. Facet surface 16 of a plane frame in plane deformation.

The laws of statics make it easy to generalize the model to frameworks with curved beams.

It remains to find the analogue to Eq. (2.7) resp. Eq. (3.3) and the mesh condition (3.2), that means, the computation of the negative Burgers vectors and rotation discontinuity of a Volterra distortion and its insertion into the compatibility conditio of a mesh. To this end we make a cut as in Fig. 7 and compute the contribution of elastic relative displacements of the single beams corresponding to electric (Ohm) voltages caused by currents in electric networks, starting from the reference node $k\,(m0)$. With the notation (5.1)–(4.12) we have for an arbitrary node $k\,(m\zeta)$ of the mesh m

$$(5.15)\qquad \begin{bmatrix} \mathbf{u}^E_{k\,(m\zeta)}-\mathbf{u}^E_{k\,(m0)} \\ \boldsymbol{\varphi}^E_{k\,(m\zeta)}-\boldsymbol{\varphi}^E_{k\,(m0)} \end{bmatrix} = \mathbf{k}\times \overset{0}{\mathbf{x}}_{k\,(m\zeta)} \sum_{\bar\zeta=1}^{\zeta} \pm \Phi^E_{z\pm\,(m\bar\zeta)} + \sum_{\bar\zeta=1}^{\zeta} \pm [\overset{0}{\mathscr{R}}_{zm}]^T\, \mathscr{U}^E_{z\pm\,(m\bar\zeta)}.$$

This can be proved easily for the contributions of U^E_{xz} and $\tilde U^E_{xz}$ with the help of Fig. 11, if all Φ^E_z, for simplicity, are kept zero. To prove it for the contribution of Φ^E_z, we assume, on the contrary, all U_{xz} and $\tilde U_{yz}$ to be zero. This means that the real mesh of elastic beams is replaced by a sequence of rigid levers with hinges in the mid-points of the original beams. Now the contribution of $\Phi^E_{z\,(m\bar\zeta)}$ to the displacement of the node $k\,(m\zeta)$ is apparently

$$(5.16)\qquad \Delta\,(\Phi^E_{z\pm\,(m\bar\zeta)})\,\mathbf{u}_{k\,(m\zeta)} = \pm\mathbf{k}\times(\overset{0}{\mathbf{x}}_{k\,(m\zeta)}-\overset{0}{\mathbf{x}}_{z\,(m\bar\zeta)})\,\Phi^E_{z\pm\,(m\bar\zeta)}$$

which, on summation, gives Eq. (5.15).

For a full circuit the first term disappears, and if the second term is not zero, it is the Burgers vector and the rotation discontinuity (or, as the author proposes to call it, the "Burgers cinemate") of a Volterra distortion

$$(5.17)\qquad \begin{bmatrix} -\mathbf{W}_m \\ -\boldsymbol{\Psi}_m \end{bmatrix} = \sum_{\zeta=1}^{\langle mZ\rangle} \pm [\overset{0}{\mathscr{R}}_{zm}]^T\, \mathscr{U}^E_{z\pm\,(m\zeta)},$$

where $\langle mZ\rangle$ is the number of branches forming the mesh. If the Burgers cinemate is prescribed, we mark it by a hook and find as the mesh condition for a frame the compatibility condition

$$(5.18)\qquad \sum_{\zeta=1}^{\langle mZ\rangle} \pm [\overset{0}{\mathscr{R}}_{zm}]^T\, \mathscr{U}^E_{z\pm\,(m\zeta)} + \begin{bmatrix} \check{\mathbf{W}}_m \\ \check{\boldsymbol{\Psi}}_m \end{bmatrix} = 0.$$

The negative Burgers cinemate can be composed from the non-elastic (e.g. plastic) deformation cinemates $\check{\mathscr{V}}_z$ of the single branches

$$\begin{bmatrix} \check{\mathbf{W}}_m \\ \mathbf{\Psi}_m \end{bmatrix} = \sum_{\zeta=1}^{\langle mZ \rangle} \pm [\overset{0}{\mathscr{R}}_{zm}]^T \check{\mathscr{V}}_z \pm_{(m\zeta)}, \tag{5.19}$$

which constitutes the analogy to Eq. (2.7) resp. Eq. (3.3).

To establish the mesh equations for the frame analogous to Eq. (2.8) we have first to replace the mesh-incidence matrix by a block matrix in substituting

$$\pm 1 \Rightarrow \pm [\overset{0}{\mathscr{R}}_{zm}] \tag{5.20}$$

and the diagonal matrix of Ohm resistances by a diagonal block matrix where the compliance matrices by the beams take their place

$$R_z \Rightarrow \mathscr{S}_z. \tag{5.21}$$

As unknowns we substitute for the circular currents the mesh dynames of Eq. (5.14)

$$i_m \Rightarrow \begin{bmatrix} -\mathbf{k} \times \mathbf{f}_m \\ \overset{0}{f}_m \end{bmatrix} \tag{5.22}$$

and on the right side the negative Burgers cinemates (5.19) show up instead of the circular voltages

$$\check{W}_m \Rightarrow \begin{bmatrix} \check{\mathbf{W}}_m \\ \mathbf{\Psi}_m \end{bmatrix}. \tag{5.23}$$

Henceforth we have a solution method completely analogous to the mesh method in electricity.

Of course, self-stress problems for frames are rather rare in applications. But as source or inflow problems in electricity could be solved by the mesh method with circular currents, so can load stress problems for frames be solved by the mesh method with mesh dynames. We have only to construct some stress state complying with the given loads without regard to the compatibility conditions.

This is exactly the same as the usual force method does in computing a preliminary solution for a statically determinate basic system, and if such a preliminary solution is at hand, it may be used also here. But we are no longer tied to constructing a statically determinate system first, as any statically admissible system of stress resultants will do. And for the second part of the calculation, it is no longer necessary to construct that usually rather involved set of substitute equilibrium systems replacing each structural member removed, since, by Eq. (5.14), these systems are generated automatically. So we have a force method which also for complicated and highly statically indeterminate systems is not more involved than the deformation method.

MAXWELL [20] and in his succession KLEIN and WIEGHARDT [21] laid special emphasis on the special case of the hinged framework. Here stiffness and compliance matrices (5.1) degenerate to a single scalar and, as the beams transmit only a longitudinal force and no shear force and bending moment any more, the facet surface becomes continuous with breaks (sharp bends) over the beams [21]. It is interesting to note that Maxwell developed from this model not a method of computation, but the graphical construction which later was made famous by Cremona.

The facet surface now depends only on its value over the nodes; there are its peaks and minimum corners, between which straight ridges span the inclined plane surfaces [21], and the mesh dyname of mesh m is determined by three of its nodes. From this intuitive graphic model, Klein and Wieghardt conclude that the number of statical indeterminacies, equal to the number of possible non-zero-stress Volterra distortions, is the same as the number of nodes of the facet surface that can be moved up and down independently of the neighbouring nodes. Applying this to a purely triangular framework, they find that it is as often statically indeterminate as the number of its inner points indicates. The "simple rosette", for example, with $K-1$ topologically radial beams ζ, circumferential beams $\zeta+\frac{1}{2}$, meshes ζ and boundary nodes ζ, and one inner node K has exactly one self-stress state, and, if we put the stress function value on the boundary to zero, this self-stress state is uniquely determined by the value f_K of this function in the inner node K (Fig. 13).

A detailed analysis shows [26], that of the Burgers vector in each mesh, only its component along the circumferential beam

$$\mathscr{W}_\zeta = -\mathbf{W}_\zeta \cdot \mathbf{i}_{\zeta+\frac{1}{2}} \tag{5.24}$$

enters into the self-stress state. The connection between the static quantity f_K and the geometric quantities K is established by the equation

$$f_k \sum_{\zeta=1}^{K-1} \frac{\mathscr{A}_\zeta}{h_\zeta} = \sum_{\zeta=1}^{K-1} \frac{\mathscr{W}_\zeta}{h_\zeta}, \tag{5.25}$$

where the coefficients $\mathscr{A}_\zeta$ are calculated from the length l_ζ of the radial beams ζ and the longitudinal elastic compliances S_ζ and $S_{\zeta+\frac{1}{2}}$ after

$$\mathscr{A}_\zeta = \frac{2\cos\beta_\zeta}{l_\zeta} S_\zeta - \frac{2\cos\gamma_\zeta}{l_{\zeta+1}} S_{\zeta+1} - \frac{1}{h_\zeta} S_{\zeta+\frac{1}{2}}. \tag{5.26}$$

Any self-stress state of a triangular hinged framework can be composed of the self-stress states of simple rosettes.

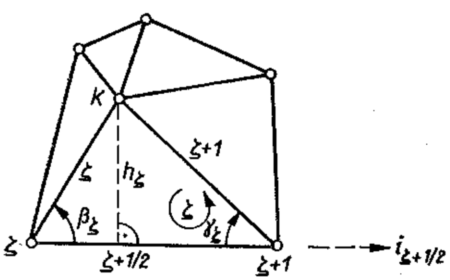

FIG. 13. The simple rosette hinged framework.

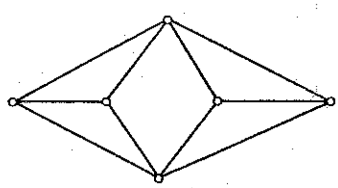
FIG. 14. Hinged framework with a quandrangle.

As far as not completely triangular hinged frameworks are concerned, we consider only the example of Fig. 14. It is evident that for fixed boundary nodes of the Klein-Wieghardt facet surface cannot move independently and therefore the framework has only one self-stress state.

6. Generalizations

We have considered in the preceding paragraph a plane frame deformed only by stretching and bending (displacement and rotation) of the beams in its plane. To encompass all possible deformations, we have to add to the plane case, in the terminology of Milne-Thomson [43], the antiplane case, that is, a plane frame with beams deformed by torsion and bending (displacement and rotation) perpendicular to its plane. This is the mode of deformation characteristic of a "slab" as opposed to the strictly plane "slice" [44].

Without going into detail, we only give a rough sketch of the appropriate stress function method, starting from the stress functions of a slab introduced by Mindlin [44], and to which Schaefer, by a simple change of indices, gave a very vivid visualization, completing Wieghardt's slice-slab-analogy [45] by its dual counterpart and aiding effectively the discussion of their properties in multiply-connected plates [46].

In Schaefer's representation, the vectorial two-component stress function of a stress-free region (zero stress function) and especially a hole in a plane slab is modelled geometrically by the movement of a rigid plane facet in its plane, described by two components of the plane movement of a translation point and the small angle of rotation. Indeed, if the component of the linear Schaefer type stress function vector normal to a beam adjacent to the mesh, where the stress function is defined, is interpreted as the contribution from this mesh to the bending moment in the beam's tangent plane perpendicular to the plane of the frame or, what means the same, about its normal in this plane, then its derivative in tangential direction gives correctly the appropriate shear force normal to the plane of the frame. There still remains to interpret the longitudinal component of the Schaefer stress function vector of the mesh as its contribution to the drilling moment of the beam. This completes our model of Fig. 12 to a general rigid movement of each facet in space [46].

The model of rigid movement of a model surface facet works even for spatial frameworks [59–61]. In [48–50] it was shown that in space the general zero stress function tensor of the Beltrami type can be written as the deformator of an arbitrary vector field in space. The components of this vector, in their turn, represent stress functions of a "crust shell" [50] replacing the boundary of the three-dimensional body in the same way as a closed beam can replace the boundary of a plane plate for static considerations. Indeed, three-component stress functions of this type are well known in the shell theory, underlying only certain restrictions by the symmetry of stresses in real shells [51, 53]. But these restrictions must not apply for the fictive crust shell which might be imagined to be a curved two-dimensional Cosserat continuum [48, 53], and the number of stress functions increases to five, including two normal derivatives of the vector field.

Here it is essential that the zero stress functions of the crust shell again are modelled by the rigid movement of a model surface congruent to the boundary of the body resp. the crust shell [48]. If, then, surfaces are spanned between the beams of a spatial framework, dividing in their turn the space between the beams into cells, then the cells are free of stresses if in each of them a vector A_c is defined generating for each cell c a zero stress function tensor Def A_c. The surface between two cells is also free of stresses if the discontinuity of the vector field **A** and its first normal derivatives correspond to a rigid movement of the surface. Thus stresses can only occur in the beams where the surfaces intersect, and the stress resultants in the beams follow, after [54], by summing all discontinuities in going round a beam once and taking the limit on contracting the circuit on the beam.

We shall not dwell on the topological questions arising for the beams of a three--dimensional framework but only mention that for the stress function field they play the same role as dislocations and disclinations do for the deformation field of an elastic body [15, 54] or vortex lines for a velocity field [57] or electric currents in a magnetic field [14]. Indeed, the lines of discontinuity of pseudo stress function fields treated in [54] may be considered as very special cases of beams whose stress resultants are derived from piecewise differentiable and one-valued zero stress functions in space of the same type as can be used for spatial frameworks.

The duality relations between static and geometric quantities in point structures connected by beams on the one hand and rigid-plated structures connected by springs on the other would be in interesting field of research; such dualities in special cases were already noted by SAUER [58]. Sauer also considered the passage to the limiting continuum for spatial structures, as WIEGHARDT [21] did for plane structures. Here, also, generalizations are still possible, for example, on the lines proposed by MINAGAWA and others [3].

In connection with Sect. 4, there might be also applications for the numerical treatment of big matrices if they may be regarded as generated from a real or fictive elastic network after, say, the node method. If one succeeds in constructing the corresponding matrix of the mesh method, it might be used for bounding in approximate calculations. Matrices derived after the Ritz method, e.g., in finite elements, are of this type even if the structures connecting the nodes might be more complicated than real beams. But it is to be kept in mind that this would be a bounding not for the real problem, as in Sect. 4, but bounds for a positive matrix within matrix calculus. Experience shows that this is also sometimes useful for big matrices treated by slowly convergent approximations.

7. ACKNOWLEDGEMENTS

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References

1. H. Kardestuncer, *Flexibility analysis of closed loops subject to temperature variations*, Wiss. Z. Hochschule Architektur und Bauwesen Weimar 19, 169–173, 1972.
2. H. Kardestuncer, *Matrix analysis of discrete elastic systems*, Int. Centre for Mech. Sciences (CISM), Courses and Le tures, Springer-Verlag, Wien 1974.
3. K. Kondo (ed.) *Raag memoirs of the unifying study of basic problems inengineer ing andphysical sciences by means of geometry*, vol. I, (1955), II (1958), III (1962), IV (1968).
4. K. Klöppel, D. Reuschling, *Zur Anwendung der Theorie der Graphen bei der Matrizenformulierung statischer Probleme*, Stahlbau, **35**/8, 236–245, 1966.
5. K.-H. Möller, C.-H. Wagemann, *Die Formulierung der Einheitsverformungs- und Einheitsbelastungszustände in Matrizenschreibeweise mit Hilfe der Theorie der Graphen*, Stahlbau **35**/9, 257–269, 1966.
6. W. Prager, J. L. Synge, *Approximations in elasticity based on the concept of function space*, Quart. Appl. Math., **5**, 241–269, 1947–1948.
7. W. Prager, *The extremum principles of the mathematical theory of elasticity and their use in stress analysis*, Univ. Washington Engng. Exp. Station, Bull, No 113, Univ. of Washington Press, Seattle 1950/51.
8. J. L. Sygne, *The hypercircle in mathematical physics. A method for the approximation of the solution of boundary value problems*, University Press, Cambridge 1957.
9. G. Rieder, *Über Eingrenzungsverfahren und Integralgleichungsmethoden für elastische Scheiben*, Platten und verwandte Probleme, Wiss. Z. der Hochschule für Architektur und Bauwesen Weimar, 19, 2, 217–222, 1972.
10. G. Rieder, *Eingrenzungen in der Elastizitäts- und Potentialtheorie*, ZAMM, **52**, T340–T347, 1972.
11. G. Rieder, *On bounds on linear elasticity and potential theory*, in: Continuum mechanics and related problems of analysis, Muskhelishvili Anniversary Volume, Nauka Publ. House, 409–417, Moscow 1972.
12. W. Prange, *Das Extremum der Formänderungsarbeit*, Habilitationsschrift, Techn. Hochschule Hannover 1916.
13. W. Ameling, *Grundlagen der Elektrotechnik I*, Bertelsmann Universitätsverlag, Düsseldorf 1973.
14. J. C. Maxwell, *A treatise on electricity and magnetism*, Clarendon Press, Oxford 1873.
15. M. August, *Theorie statischer Versetzungen*, Teubner Verlag, Leipzig 1966.
16. E. Kröner, *Kontinuumstheorie der Versetzungen und Eigenspannungen*, Springer-Verlag, Berlin 1958.
17. K.-H. Anthony, *Die Reduktion von nichteuklidischen geometrischen Objekten in eine euklidische Form und physikalische Deutung der Reduktion durch Eigenspannungszustände in Kristallen*, Arch. Rat. Mech. Anal., **37**, 161–180, 1970.
18. K.-H. Anthony, *Die Theorie der Disklinationen*, Arch. Rat. Mech. Anal., **39**, 43–88, 1970.
19. F. R. N. Nabarro, *Theory of crystal dislocations*, Claredon Press, Oxford 1967.
20. J. C. Maxwell, *The scientific papers of James Clerk Maxwell*, Vol. I, II, Cambridge Univ. Press 1890.
21. P. Klein, K. Wieghardt, *Uber Spannungsflächen and reziproke Diagramme, mit besonderer Berücksichtigung der Maxwell'schen Arbeiten*, Archiv der Math. und Physik, **8**,**3**, 1–10, 95–119, 1905.
22. K. Wieghardt, *Über einen Grenzübergang der Elastizitätslehre und seine Anwendung auf hochgradig statisch unbestimmte Fachwerke*, Verhandl. d. Ver. zur Beförd. d. Gewerbfleisses, Verlag L. Simon, **85**, 139–176, Berlin 1906.

23. K. Wieghardt, *Über Nebenspannungen gewisser hochgradig statisch unbestimmter Fachwerke*, Z. f. Math. Phys., **53**, 113–143, 1906.

24. P. Funk, *Die linearen Differenzengleichungen und ihre Anwendung in der Theorie der Baunkonstruktionen*, Springer, Berlin 1920.

25. G. Grioli, *Struttura della funzione di Airy nei sistemi molteplicemente connessi*, Giornale di Matematiche di Battaglini, **77**, 119–144, Napoli 1947.

26. G. Rieder, *Mechanik, Lectures at RWTH Aachen*, Printed manuscript.

27. G. Rieder, U. Zastrow, *Eine Variante des Kraftgrössenverfahrens für ebene Rahmentragwerke*, Paper read at the 7th IKM, Weimar 1975.

28. P. Funk, *Variationsrechnung und ihre Anwendung in Physik und Technik*, Springer-Verlag, Wien 1962.

29. J. Stickforth, *On the derivation of the conditions of compatibility from Castigliano's principle by means of three-dimensional stress functions*, J. Math. Phys., **44**/3, 214–226, 1965.

30. H. Schaefer, *Die Motorfelder des dreidimensionalen Cosserat-Kontinuums im Kalkül des Differentialformen*, Int. Centre for Mech. Sciences (CISM), Courses and Lectures, Udine 1970.

31. F. M. Dimentberg, *Vintovoe ischislenie i ego prilozheniya v mechanike* (Screw Calculus and its applications in mechanics) (in Russian), Nauka, Moscow 1965.

32. J. P. Baty, G. Williams, *Graph theoretic decomposition of large structural network systems*, Proc. of the 2nd Conf. on Graph theory and applications, Boulder, Colorado 1970.

33. K. Kuratowski, *Topologie*, vol. I, II (in French), Polskie Towarzystwo Matematyczne (Polish Mathematical Society), Warsaw 1952.

34. F. Harary, R. Z. Norman, D. Cartwright, *Structural models, introduction to the theory of directed Graphs, Wiley and Sons*, New York 1965.

35. E. Chwalla, *Einführung in die Baustatik*, Stahlbau-Verlags-GmbH, Köln 1954.

36. H. Neuber, *Technische Mechanik* (Vol. I Statik, Vol. II Elastostatik und Festigkeitslehre), Springer-Verlag, Berlin-Heidelberg-New York 1971.

37. M. M. Filonenko-Boroditsch, *Elastizitätstheorie*, VEB Fachbuchhandlung, Leipzig 1967.

38. S. Timoshenko, I. N. Goodier, *Theory of elasticity*, McGraw Hill, New York 1951.

39. I. S. Sokolnikoff, *Mathematical theory of elasticity*, McGraw Hill, New York 1965.

40. E. Trefftz, *Über den Schubmittelpunkt in einem durch eine Einzellast gebogenen Balken*, Z. Angew. Math. Mech., 15, 220–225, 1935.

41. W. Günther, C. Weber, *Torsionstheorie*, Friedr. Vieweg und Sohn, Braundschweig 1958.

42. J. H. Argyris, S. Kelsey, *Energy theorems and structural analysis*, Butterworth, London--Sydney-Wellington 1960.

43. L. M. Milne-Thomson, *Antiplane elastic system*, Springer-Verlag 1962.

44. R. D. Mindlin, M. G. Salvadori, *Analogies*, In: Handbook of experimental stress analysis, ed. M. Hetényi, Wiley and Sons, New York — Chapman and Hall, London 1950–1957.

45. K. Wieghardt, *Über ein neues Verfahren, verwickelte Spannungsverteilungen in elastischen Körpern auf experimentellen Wege zu finden*, Mitt. über Forschungsarbeiten und dem Gebiete des Ingenieurwesens, **49**, 15–30, 1908.

46. H. Schaefer, *Die vollständige Analogie Scheibe-Platte*, Abh. Braunschweig, Wiss. Ges., **8**, 142, 1956.

47. H. Schaefer, *Die drei Spannungsfunktionen des zweidimensionalen ebenen Kontinuums*, Österr. Ing.-Archiv, **10**, 267, 1956.

48. G. Rieder, *Die Randbedingungen für den Spannungsfunktionentensor an ebenen und gekrümmten belasteten Oberflächen*, Osterr. Ing.-Archiv, **18**, 208–243, 1964.

49. J. Stickforth, *On the derivation of the conditions of compatibility from Castigliano's principle by means of three-dimenional functions*, J. Math. Phys., **44**, 3, 214–226, 1965.

50. H. SCHAEFER, *Die Spannungsfunktionen des dreidimensionalen kontinuums; statische Deutung und Randwerte*, Ing.-Archiv, **28**, 281, 1959.

51. V. V. NOVOZHILOV, *The theory of thin shells*, Noordhoff, Groningen 1959.

52. A. I. LUR'E, *On the static geometric analogue of shell theory*, in: Problems of Continuum Mechanics, Society for Industrial and Applied Mathematics, Philadelphia 1961.

53. W. GÜNTHER, *Analoge Systeme von Schalengleichungen*, Ing.-Archiv, 30, 160, 1961.

54. G. RIEDER, *Die Berechnung des Spannungsfeldes von Einzelkräften mit Hilfe räumlicher Spannungsfunktionen und ihre Anwendung zur quellenmässigen Darstellung der Verschiebung bei Eigenspannungszuständen*, Österr. Ing.-Archiv., **18**, 173–201, 1964.

55. G. RIEDER, *Topologische Fragen in der Theorie der Spannungsfunktionen*, Abh. Braunschweig. Wiss. Ges., **12**, 4–65, 1960.

56. G. RIEDER, *On Indenbom's Extension of Maysel's theorem*, Bull. Acad. Polon. Sci., Série Sci. Tech., **12**, 287–294, 1964.

57. E. MARTENSEN, *Potentialtheorie*, Teubner, Stuttgart 1968.

58. R. SAUER, *Differenzengeometrie*, Springer-Verlag, Berlin-Heidelberg 1970.

59. Z. KĄCZKOWSKI, *The grouped loads method in the statical analysis of spatial lattice structures*, Arch. Inżyn. lądow., **21**, 113–123, 1975.

60. K. CHOBOT, *Fünzing Jahre der Matrizen-Baumechanik in Tschechoslowakei*, Paper D-1, 12th Yugoslav Congress of Rational and Applied Mechanics, Ohrid 1974.

61. K. CHOBOT, *Matrix analysis of general space frameworks*, Acta Technica ČSAV, 420–473, 1971.

62. K. CHOBOT, *Allgemeine Ableitung verschiedener Arten von Nachgiebigkeits- und Steifigkeitsmatrizen*, Berichte VI. IKM Weimar 1972.

Streszczenie

O FUNKCJI NAPRĘŻEŃ MAXWELLA, KLEINA I WEIGHARDTA UKŁADÓW NIECIĄGŁYCH

Rozważając plany sił i przemieszczeń, nazwane później planami Cremony, Maxwell zauważył, że statyka płaskich kratownic przy braku sił zewnętrznych może być przedstawiona w postaci odcinkowo-liniowej nieciągłej funkcji naprężeń typu funkcji Airy'ego. Funkcje te Klein i Wieghardt przedstawili w postaci «powierzchni naprężeń» degenerującej się do «powierzchni płytkowej». W roku 1920 Funk zaproponował aby zastosować praktycznie tę koncepcję razem z zasadą Castigliano. Opierając się na analogii elektrycznej podanej przez Krona, Kardestuncera i innych wykazano, że wielkości statyczne opisujące każdą z «płytek» odpowiadają prądom kołowym w sieciach elektrycznych w podobny sposób, jak przemieszczenia i obroty węzłów kratownic odpowiadają potencjałom. Przedyskutowano uogólnienie dla układów przestrzennych.

Резюме

О ФУНКЦИИ НАПРЯЖЕНИЙ МАКСВЕЛЛА, КЛЕЙНА И ВИГАРДТА ДИСКРЕТНЫХ СИСТЕМ

Рассматривая планы сил и перемещений, названные позже планами Кремона, Максвелл заметил, что статика плоских ферм, при отсутствии внешних сил, может быть представлена в виде кусочно-линейной разрывной функции пнапряженуй типа функции Эри. Эти функции Клейн и Вигардт представили в виде „поверхности напряжений", вырождающейся в „пластинчатую поверхность". В 1920 году Функ предложил применять практически эту концепцию совместно с принципом Кастильяно. Опираясь на электрическую аналогию, приведенную Кроном, Кардестунцером и другими, показано, что статические величины, описывающие каждую из „пластинок", отвечают круговым токам в электрических цепях аналогичным образом, как перемещения и вращения узлов ферм отвечают потенциалам. Обсуждено обобщение для пространственных систем.

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