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THEORETICAL FOUNDATIONS OF THE HYPOTHESES OF MATERIAL EFFORT

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This year in June, the Swiss Association of Material Testing for Technology (*Schweiz. Verband für die Materialprüfungen der Technik)* arranged two interesting lectures about material effort and buckling. Even the sole course of the discussion speaks for topical interest of both problems; it is enough to say that it proceeded for over five hours in a tightly filled Auditorium I of the Zurich Polytechnic (*Eidgenössische Technische Hochschule*). The chairman was Prof. Dr. Eng. h.c. M. Roš, the director of the Confederate Material Testing Laboratory (*Eidgenössische Materialprüfungsanstalt – EMPA*).

Personally, I was more interested in the first lecture "Theoretical foundations of the investigations carried out at *EMPA* to elucidate the question of risk of fracture" (*Die theoretische Grundlagen zu den Versuchen der EMPA zur Klärung der Frage der Bruchgefahr*), delivered by a scientist associated with the said laboratory, Eng. A. Eichinger. We can find the content of the lecture in the Chapters I and IV, published respectively in *EMPA Bericht*: No. 28 from 1928 and No 34. from 1929. The whole content can be summarized in the following way. *Critical material effort of a large number of plastic metals obeys quantitatively the Huber-von Mises-Hencky hypothesis. The behaviour of all other materials is relatively best explained, even though not precisely, by Mohr's hypothesis. The idea inherent in the Huber-Hencky theory had until now a hypothetical character; only EMPA has managed – but still with the conservation of Mohr's main idea – to conduct a clear (plausible), convincing and exact proof of rightness of the hypothesis. As a result, the theory of constant critical energy of distortion is nothing else but a generalization of Mohr's concept of an envelope.*

It cannot be denied that the four-year-long series of fine experiments on elucidating the enigma of material effort, conducted with great expenditure of work and money, has a considerable significance. However, it cannot be denied also that the interpretation of the obtained results, used by *EMPA*, leaves much to be desired. The idea inherent in the Mohr theory is outdated and what even worse – erroneous; there exists a general hypothesis which in the right way and with satisfactory exactness comprises the results of all, without exception, experiments done by the Laboratories in Zürich and Göttingen as well as by many other groups interested in the discussed problem. Similarly, the theoretical efforts of the researches in *EMPA* went the wrong way. The pride of *EMPA*, the meticulous conversion of the Huber-Hencky theory to the area of Mohr's concept, underlined by Roš and Eichinger on nearly every page of the said bulletin, contains a series of errors in the principal matter.

In reply to Eng. Eichinger's lecture I addressed the meeting and in an over one-hour-long speech I tried to explain my view on this question. In the speech I kept, above all, to the outline marked by the title of the preceding speaker's lecture. In the first place then, I explained the theoretical side of the hypotheses of material effort, leaving the experimental aspect of the matter in the background. Referring to the present state of affairs, I limited myself, of necessity, to discuss problems of local and static material effort only.

Foreign countries do not know most of the critical arguments known in Poland, as I had learned on the occasion of delivering a similar lecture in Göttingen. The hypotheses of material effort are treated there – to a certain degree even rightly – only as hypotheses; all novelties in this field, are often studied in laboratories straightaway, without prior insight into elementary theoretical details. Owing to this fact, every couple of years there arises the need for a new general hypothesis, since the old one fails. In such state of affairs only the calculus of probability can tell, by examining all the existing typical groupings of components, how many new and useless theories we will be seeing.

Coming back to the said lecture, I have to admit with satisfaction that it met with a great interest, which reflected at least in Professor Roš's request for a written copy of it; surely, a relevant article will soon appear in German in print, edited by *EMPA*[1)]. Before it happens though, it will be good to acquaint Polish readers with this topical question. The present article extends the mentioned speech by 25%.

Questions of applied mechanics seldom have such a rich history to their credit as the theories of material effort do[2)]. The question started in Galileo[3)] and Leibniz[4)] times, it outlived Coulomb[5)] and Navier[6)] as well as de Saint Venant[7)] and Rankine[8)], Clebsch[9)] and Beltrami[10)], and went through the hands of many later, distinguished scholars and in the present day – the time of competition between reliability and economics – it is, next to the problem of buckling, the most topical scientific issue in the theory of elasticity as well as in the very popular plasticity and, finally, in the strength of materials. Rarely can one meet with so many different views and inconsistencies as in this field. The hypotheses of material effort were first transferred from their birthplace – the domain of stress – to the ground of strain theories wherefrom, after not very good results, they were moved to their last resort – the energy-based approach, in the state of which they have been remaining till the present day; however, their mathematical form does not guarantee any general reliable theory and clearly does not satisfy the needs and interest of practice. But even though, such divergence of views has never before produced so much benefit as in this field. A thorough review of the existing material allows one to criticize it, to judge its bad and good parts, reject the first and use the latter; in consequence, it leads to a general hypothesis which has a very good chance of success. For a better understanding of its sense and, in case of need, quality, I will give at first a brief list of parameters of calculation of the discussed matter and also a critical draft of the existing hypotheses.

The material effort of a certain point of a body is a physical state closely related to the state of stress of this point, or its strain. The value of the material effort is described, irrespectively of the choice of a coordinate system, by six components of the state of strain ε_x, ε_y, ε_z, $\frac{1}{2}\gamma_x$, $\frac{1}{2}\gamma_y$, $\frac{1}{2}\gamma_z$ or equivalently – by six components of the state of stress[11)] σ_x, σ_y, σ_z, τ_x, τ_y, τ_z. The effects of material effort, especially the manner of passing through the characteristic limits (the proportional limit, the limit of elasticity, the yield point and the ultimate strength), the pace of passing from one limit to the next and changes of behaviour in-between, depend on individual properties of the body. Material efforts of two points subjected to two different states of strain or stress are equal when their physical effects are equal; the function

$$f\left(\varepsilon_x,\ \varepsilon_y,\ \varepsilon_z,\ \frac{1}{2}\gamma_x,\ \frac{1}{2}\gamma_y,\ \frac{1}{2}\gamma_z\right) = a \tag{1}$$

or equivalently

$$g\left(\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_x,\ \tau_y,\ \tau_z\right) = b, \tag{2}$$

expresses mathematically the consistency of the effects.

We assume that the sets of critical components (as we are going to call them hereafter), which cause equal material effort, change in a continuous manner; we demand the same from the above equations. It is basically guaranteed by the continuity of the material considered; in case of its lack, even the simplest considerations fail in the area of strain as well as of stress.

The highlighted above independence of the material effort of a certain point upon the chosen frame of reference at this point, allows one to simplify the functions f and g. The mechanics of continuum allows, to one's considerable advantage, replace a set of six arbitrary components by the three principal ones. Their values ε_1, ε_2, ε_3 or σ_1, σ_2, σ_3 result from the cubic equation:

$$\varepsilon^3 - 3\pi_1\varepsilon^2 + 3\left(\pi_1^2 - \frac{1}{2}\pi_2^2\right)\varepsilon - \pi_3^3 = 0 \tag{3}$$

or accordingly

$$\sigma^3 - 3\omega_1\sigma^2 + 3\left(\omega_1^2 - \frac{1}{2}\omega_2^2\right)\sigma - \omega_3^3 = 0, \tag{4}$$

where the homogeneous expressions

$$\begin{aligned}
\pi_1 &= \frac{1}{3}\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right),\\
\pi_2^2 &= \frac{1}{9}\left[(\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + (\varepsilon_x - \varepsilon_y)^2 + \frac{3}{2}\left(\gamma_x^2 + \gamma_y^2 + \gamma_z^2\right)\right],\\
\pi_3^3 &= \varepsilon_x\varepsilon_y\varepsilon_z + \frac{1}{4}\gamma_x\gamma_y\gamma_z - \frac{1}{4}\left(\varepsilon_x\gamma_x^2 + \varepsilon_y\gamma_y^2 + \varepsilon_z\gamma_z^2\right),
\end{aligned} \tag{5}$$

or accordingly

$$\begin{aligned}
\omega_1 &= \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right),\\
\omega_2^2 &= \frac{1}{9}\left[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\sigma_x - \sigma_y)^2 + \frac{3}{2}\left(\tau_x^2 + \tau_y^2 + \tau_z^2\right)\right],\\
\omega_3^3 &= \sigma_x\sigma_y\sigma_z + \frac{1}{2}\tau_x\tau_y\tau_z - \left(\sigma_x\tau_x^2 + \sigma_y\tau_y^2 + \sigma_z\tau_z^2\right)
\end{aligned} \tag{6}$$

are evidently independent of the choice of the six groups of components. They are also mutually independent invariants of the state of stress or strain. The simplest way to calculate their values is to reject the shear (tangent) components and replace the normal components with the principal ones.

If we know the principal components, we can easily define the components ε, $\frac{1}{2}\gamma$ or σ, τ, for the directions ϕ, χ, ψ referred to the orthogonal coordinate system $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ or $(\sigma_1, \sigma_2, \sigma_3)$, by the formulae:

$$\begin{aligned}
\varepsilon &= \varepsilon_1\cos^2\phi + \varepsilon_2\cos^2\chi + \varepsilon_3\cos^2\psi,\\
\left(\frac{1}{2}\gamma\right)^2 &= (\varepsilon_2 - \varepsilon_3)^2\cos^2\chi\cos^2\psi + (\varepsilon_3 - \varepsilon_1)^2\cos^2\psi\cos^2\phi\\
&\quad + (\varepsilon_1 - \varepsilon_2)^2\cos^2\phi\cos^2\chi
\end{aligned} \tag{7}$$

or

$$\begin{aligned} \sigma &= \sigma_1 \cos^2\phi + \sigma_2 \cos^2\chi + \sigma_3 \cos^2\psi, \\ \tau^2 &= (\sigma_2-\sigma_3)^2 \cos^2\chi \cos^2\psi + (\sigma_3-\sigma_1)^2 \cos^2\psi \cos^2\phi \\ &\qquad + (\sigma_1-\sigma_2)^2 \cos^2\phi \cos^2\chi\,. \end{aligned} \tag{8}$$

There is only one condition demanded for the existence of these formulae – the material continuity. The orientations $\phi = \psi = \frac{\pi}{4}$, $\chi = \frac{\pi}{2}$, and $\phi = \chi = \psi = \arccos\frac{1}{\sqrt{3}}$ are worthy of consideration. We obtain successively

$$\begin{aligned} \varepsilon &= \varepsilon_{II} = \frac{\varepsilon_1+\varepsilon_3}{2}, \\ \frac{1}{2}\gamma &= \frac{1}{2}\gamma_{II} = \frac{\varepsilon_1-\varepsilon_3}{2}, \end{aligned} \tag{9}$$

or

$$\begin{aligned} \sigma &= \sigma_{II} = \frac{\sigma_1+\sigma_3}{2}, \\ \tau &= \tau_{II} = \frac{\sigma_1-\sigma_3}{2}, \end{aligned} \tag{10}$$

for them; moreover:

$$\begin{aligned} \varepsilon &= \frac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3) = \pi_1, \\ \frac{1}{2}\gamma &= \frac{1}{3}\sqrt{(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_3-\varepsilon_1)^2+(\varepsilon_1-\varepsilon_2)^2} = \pi_2, \end{aligned} \tag{11}$$

or

$$\begin{aligned} \sigma &= \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \omega_1, \\ \tau &= \frac{1}{3}\sqrt{(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+(\sigma_1-\sigma_2)^2} = \omega_2. \end{aligned} \tag{12}$$

For arbitrary directions the formulae for ε, $\frac{1}{2}\gamma$ or σ, τ can be represented in a very fair, developed by Mohr[12)] and well-known graphical form of circles for strains and stresses. A system of three circles for strains is homothetic to a system of three circles for stresses; it occurs not only in elastic regions but also in plastic ones.

The following facts deserve notice: the sum of circumferences of three principal circles, on the assumption that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ or $\sigma_1 > \sigma_2 > \sigma_3$, is equal to

$$U_\varepsilon = 2\pi(\varepsilon_1-\varepsilon_3) = 2\pi\gamma_{II} \quad \text{or} \quad U_\sigma = 2\pi(\sigma_1-\sigma_3) = 2\pi\tau_{II}. \tag{13}$$

Similarly, the total area of these circles is equal to

(14)
$$F_\varepsilon = \frac{\pi}{4}\left[(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_3-\varepsilon_1)^2+(\varepsilon_1-\varepsilon_2)^2\right] = \frac{9\pi}{4}\pi_2^2,$$
or
$$F_\sigma = \frac{\pi}{4}\left[(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+(\sigma_1-\sigma_2)^2\right] = \frac{9\pi}{4}\omega_2^2.$$

When the principal components are used, the critical material effort is defined by the functions

(15)
$$f(\varepsilon_1,\varepsilon_2,\varepsilon_3) = a$$

or

(16)
$$g(\sigma_1,\sigma_2,\sigma_3) = b.$$

The functions f and g in the earlier introduced, general forms as well as in the above particular ones are mutually dependent, since there exist close correlations between the components of strain and the components of stress.

Nowadays we hear more and more often that the study of material effort has to be necessarily based on investigation of the critical values of strain as well as stress; this opinion is quite right; however, only from the experimental point of view, since a properly constructed hypothesis serves its turn equally well on the ground of strain as on the ground of stress. In the first case, function f plays the role of the third equation, which in a correlation with an a priori assumed double proportion $\varepsilon_1 : \varepsilon_2 : \varepsilon_3$ allows one to calculate the critical values ε_1, ε_2, ε_3. On the other hand, the function g plays equivalently the unique role for a sequential ratio $\sigma_1 : \sigma_2 : \sigma_3$. On this occasion we have additionally defined a mathematical role of hypotheses of material effort: they are supposed to enable one to calculate correctly the limit (allowable, critical) values of components for their assumed ratio. If only a so-called inhomogeneity of matter could be defined precisely each time, the inhomogeneity would not constitute much difficulty in the interpretation of experimental results and in additional correction of the theories of material effort. However, this is not true in the real terms; facing this fact we have to assume that technically important materials are perfectly homogenous and that the hypotheses of material effort are developed for such materials. It partly explains the deviations between experiment and theory. The lack of uniformity of a strain state, or a state of stress, during laboratory tests, becomes an equally important factor; to obtain uniformity is practically impossible.

In this way, the interpretation of the experimental results presents immense difficulties for the theory; hence, we are being forced to mingle two notions:

the total and the local material effort. All theoretical calculations are based on the material effort of a point (e.g. an outer fibre of a beam in the intersection where a maximum bending moment occurs), while all the experimental data come from the material effort of a certain volume of continuum. This is the next cause of the deviations between the theory and the experiments. However, the most important factor is unquestionably the isotropy – or the quasi-isotropy (in the meaning established by VOIGT[13)]) – of materials. We are compelled to decide against a great number of all the various types of anisotropy (defined by 21 elastic constants in a general case; however, still in a quite narrow range). We assume, of necessity, that the investigated materials are isotropic; in a large number of cases we cannot guarantee the latter or determine the corresponding deviations in a precise manner (vibrating of concrete, rolling of steel, internal stresses – to some extent, repeated loading and so on).

The isotropy is expressed in a particularly simple, though highly restricted in its application case: the linear Hooke's law:

$$\begin{aligned} \sigma_{x,y,z} &= 2G\left(\varepsilon_{x,y,z} + \frac{\mu e}{1-2\mu}\right), \\ \tau_{x,y,z} &= G\gamma_{x,y,z} \end{aligned} \tag{17}$$

or

$$\begin{aligned} \varepsilon_{x,y,z} &= \frac{1}{E}\left[(1+\mu)\,\sigma_{x,y,z} - \mu s\right], \\ \gamma_{x,y,z} &= \frac{1}{G}\tau_{x,y,z}, \end{aligned} \tag{18}$$

where E, G, μ are commonly known elastic constants; additionally: $e = 3\pi_1$, $s = 3\omega_1$. The usage of the principal components eliminates the shear ones; for the normal components the indices *1, 2, 3* should be used instead of the x, y, z.

The relation between the components of both states has a secondary character; the energy of strain, or elasticity, of an assumed unit of volume

$$\Phi = \Phi_v + \Phi_f \tag{19}$$

has the primary significance in this aspect, though still with the identical restrictions. The first term Φ_v stands for the energy due to the change of volume, the other term Φ_f – for the energy due to the change of shape; they are shortly expressed by:

$$\begin{aligned} \Phi_v &= \frac{3}{2}\frac{E}{1-2\mu}\pi_1^2 = \frac{3}{2}\frac{1-2\mu}{E}\omega_1^2, \\ \Phi_f &= 3G\pi_2^2 = \frac{3}{4G}\omega_2^2. \end{aligned} \tag{20}$$

Here, the components of one state are partial derivatives of the energy Φ with respect to the components of the other state. The energy Φ expressed in such way has an approximate character, since it consists of the first terms of a general expansion in power series. Basing on the Stickelberg theorem on the structure of an rational integer function, which is independent of the choice of a reference system, we can easily give a general expression for the energy Φ. This is, in general, a series formed from the terms: π_1, π_1^2, π_2^2; π_1^3, $\pi_1\pi_2^2$, π_3^3; π_1^4, $\pi_1^2\pi_2^2$, $\pi_1\pi_3^3$, π_2^4;.... and an appropriate number of elastic constants m as coefficients. A similar series can be formed from the three invariants ω_1, ω_2^2, ω_3^3. In the above expansion, the free term and the term of the first grade is disregarded due to the commonly known reasons. The terms π_3^3; π_1^4, $\pi_1\pi_3^3$, π_2^4;.... are disregarded owing to the unique dependence between the components of strain and stress states; then the invariant π_3^3, does not play any essential role in the theory of elasticity.

The described above decomposition of the energy Φ into the two parts Φ_v and Φ_f, done by STOKES[14)] for the first time and a little later by HELMHOLTZ[15)], is in this particular case right; however, in general, a linear superposition does not simply lead to a superposition of squares. There arises a thought that this particular feature generally characterizes isotropic bodies (enabling the possibility of transition from solids to liquids). Supposing then the possibility of splitting the energy into the two characteristic parts also in the case considered here, we see that in the general expansion the mixed terms, namely π_1, π_1^2, $\pi_1^2\pi_2^2$,...., are disregarded. Finally, we obtain the result (19), where

$$(22)\qquad \begin{aligned} \Phi_v &= m_{2v}\pi_1^2 + m_{3v}\pi_1^3 + m_{4v}\pi_1^4 + \ldots, \\ \Phi_f &= m_{2f}\pi_2^2 + m_{4f}\pi_2^4 + \ldots . \end{aligned}$$

The series written above are of course convergent; their sums indeed present the energies Φ_v and Φ_f.

We obtain the components of the state of stress by differentiating Φ with respect to the components of the strain state. Using them, and even in a general case, that is without reducing the energy terms, one can easily prove that the invariants of one state are functions of the invariants of the other state. In general then, we have

$$(24)\qquad \pi_i = \pi_i\left(\omega_1,\, \omega_2,\, \omega_3\right) \quad \text{and in reverse} \quad \omega_i = \omega_i\left(\pi_1,\, \pi_2,\, \pi_3\right),$$

where $i = 1, 2, 3$.

In this way we have finished reviewing the theoretical means used in developing the hypotheses of material effort. Let us now look closely at the arduous way they had to go. The whole effort and wit of the authors of the hypotheses of material effort were focused on producing from the six components of strain or stress states a single mathematical expression that would experimentally characterize itself with constancy or at least possibly simple, definable variability. That moment there arose an idea that one handles three components more easily than six. Basing on this, the principal components were introduced. This smooth simplification caused a new necessity: to introduce the inconvenient inequality $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$, or $\sigma_1 \geq \sigma_2 \geq \sigma_3$. We do not realize in general what a great disadvantage in hypotheses of material effort this inequality presents. Let us suppose that one uses such a hypothesis in a certain particular case. The theory of elasticity or the science of strength of materials provides us with the solution of the particular problem by giving six components as functions of spatial coordinates; external loads and dimensions of the investigated element play the role of parameters while the elastic constants are coefficients. We obtain the three principal components from the cubic Eqs. (3) or (4); since, in general, these components are different and real, the solution must be obtained with the use of transcendental functions: in such a solution all the details of calculation are lost. Finally, to crown it all, one has to fix the order $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ or $\sigma_1 > \sigma_2 > \sigma_3$, which is simply impossible; the hypothesis is of no general use. That these details escape the common notice is due to the fact that we have accustomed ourselves to simplifying some problems to their two-dimensional form, where the above difficulties cancel to a considerable degree.

In search for a possibly simple combination of components defining the material effort, primitive means were initially used. In this way the hypothesis of a constant tension σ_1 disregarded the two remaining principal stresses σ_2 and σ_3 and, respectively, the hypothesis of a constant elongation ε_1 neglected the two principal strains ε_2 and ε_3. Astonishing! The stress hypothesis was quickly denied acceptance in favour of the strain hypothesis; an exact physical justification was being seen in the expression: a system of stresses which causes the greatest elongation is a measure of critical material effort (18). If the stress hypothesis had been put into the following words: a system of strains which causes the greatest tension (17) is a measure of critical material effort, such hypothesis would certainly linger till nowadays in various textbooks and almanacs, just like its rival. But still they are both asymmetrical and both of them are erroneous. Later corrections, that is lower limits introduced with the help of constant pressure σ_3 or constant contraction ε_3, are not worthy much more. A properly constructed hypothesis should express itself with the same groupings of components on both grounds; while here the equalities $\varepsilon_1 = a_1$, $\varepsilon_3 = a_3$ are not corresponded by $\sigma_1 = b_1$, $\sigma_3 = b_3$ and the same in the reverse way (17), (18). All attempts of further resort in this direction failed; the dubious worth of BACH'S coefficient α_0 is at present widely known[16)].

Employing single components did not cause the desired result, so the use of simple sets of groupings of strains or stresses was introduced. The hypothesis of constant shear stress τ_{II} (10) (Coulomb[5)], Guest[17)])has indisputably a fundamental significance in the development of the study of material effort[18)]; only very precise experiments proved its very good, though approximate character. It is the effect of the mathematical form of the hypothesis; one should pay careful attention to the fact that the equality $\tau_{II} = b$ corresponds to by the equality $\frac{1}{2}\gamma_{II} = a$ and the same in the reverse direction (17), (18). The substance of the hypothesis is then independent of the choice of the units of measurement. Its minor errors are caused by the absence of the intermediate component. That these errors did not go far beyond the practical limit is only due to the fact that the range of application of this hypothesis is very narrow; *viz.* it refers only to materials whose behaviour in critical tension k_t and compression k_c is expressed by the equality $k_t = k_c = k$.

The matter with the Duguet[19)] and Mohr[20)] theory presents itself a bit worse. This hypothesis in the general form:

$$g\left(\sigma_{II},\, \tau_{II}\right) = b \tag{25}$$

makes a clear progress in the development of the study of material effort. A generalisation inherent in the theory (25) can be even to a certain degree well explained, since the stresses σ_{II} and τ_{II} belong to the same orientation; on a pictorial scheme the three stress circles move along the σ axis and change their total perimeter (13) in a continuous manner. But this interpretation fails in the ground of strain, since, even though τ_{II} corresponds with the magnitude $\frac{1}{2}\gamma_{II}$, the stress σ_{II} is not corresponded by ε_{II} but by (in a linear approximation): $\varepsilon_{II} + \dfrac{\mu e}{1-2\mu} = \dfrac{\varepsilon_1 + 2\mu\varepsilon_2 + \varepsilon_3}{2\,(1-2\mu)}$ (17). It is similar in a reversed procedure. The hypothesis is then asymmetrical; in the ground of stress it expresses itself differently than in the ground of strain.

As for the details, one should take both perspectives – from Duguet's position and from Mohr's position. The first one was calculating, the other one was drawing – both nearly in the same time; the stranger it seems then, that the first one was forgotten so quickly. Duguet puts Coulomb's premises into a mathematical form; he assumes that the critical effects occur as a result of overcoming friction (of the coefficient $f = \tan\beta$) and cohesion d, namely, in such a two-dimensional orientation ξ, η, ζ for which the left-hand side of the equation $\tau + f\sigma = d$ reaches its maximum; σ and τ are defined by the formulae (8). I am not sure if it is commonly known that the intermediate stress σ_2 is cancelled from Duguet's calculation only due to a mathematical coincidence, since the critical orientation turned out to be the direction $\xi = \frac{\pi}{4} - \frac{\beta}{2}$, $\eta = \frac{\pi}{2}$, $\zeta = \frac{\pi}{4} + \frac{\beta}{2}$. Mohr proceeds inversely and incorrectly; he assumes in advance and without appropriate justification, the independence of material effort of the intermediate stress σ_2 and, disregarding the two stress circles, he surrounds the series of circles $(\sigma_1,\, \sigma_3)$ by an envelope and in consequence, with the help of the consideration of not the material effort itself but the geometrical features of the pictorial scheme of the stress circles, he comes to the same critical orientation and the same scheme of envelope as the one (but not called this name) found by Duguet. These details have been overlooked in the literature; Mohr's unquestionably justified authority weighed here more than his theory of material effort.

This theory is then unfit for a general use for the reasons explained above (the inconvenient inequality) and incorrect due to the proved asymmetry. Also experiments do not support it completely; envelopes for different groupings of components do not and cannot mutually cover; the influence of the intermediate stress σ_2 cannot be omitted. And even if such a particular envelope existed, it would not suit characterizing material effort in this case; the experimental details as well as the mathematical arguments and to some extent – even the hypothesis itself, demand and prove that Mohr's envelope does not surround all possible critical circles, since some of them are hidden inside the envelope without touching it.

This detail in the case of existence of the said envelope does not prove fundamentally the incorrectness of the Duguet-Mohr theory; it only indicates that the coordinate system $(\sigma,\, \tau)$ is inadequate for expressing the substance of this hypothesis. The appropriate system for this hypothesis would be the system $(\sigma_{II},\, \tau_{II})$, in which all the critical states find their place in the form of the points $(\sigma_{II},\, \tau_{II})$. This subtlety has also escaped notice, which can be proved by the graphical schemes found in the immense number in various publications and drawn exactly due to Mohr's recipe. Meanwhile, it is clear that it is easier to put the point in the system $(\sigma_{II},\, \tau_{II})$ than to draw a circle with the center coordinates $(\sigma_{II}, 0)$ and the radius τ_{II} in the system $(\sigma,\, \tau)$. As for the angles – a simple relation $\sin\beta = \tan\alpha$ between the slope of the tangent $\beta = \arctan f$ in the system $(\sigma,\, \tau)$ and α in the system $(\sigma_{II},\, \tau_{II})$ (α, β measured from the negative sense of the axis of abscissae) has also escaped notice. The whole misunderstanding lies in the fact that it is commonly assumed that the envelope is the essence of Mohr's hypothesis while it is untrue: the essence of the hypothesis is the assumption of existence of the function (25).

There is no need for proving that every hypothesis can be graphically illustrated by a single-parameter set of envelopes; it is enough to make this parameter (c) dependent on the value σ_2 in the following way:

$$\sigma_2 = \frac{1+c}{2}\sigma_1 + \frac{1-c}{2}\sigma_3 = \sigma_{II} + c\tau_{II}, \tag{26}$$

where c is limited by the inequality $-1 \le c \le 1$; each c corresponds one envelope. This can be clearly seen in the illustrations to experiments on brittle materials for which, due to technical difficulties, laboratories apply the two extreme cases

$c = -1$, $c = 1$, or rarely $c = 0$. These envelopes – as I have mentioned – are not identical. In this way, the general incorrectness of all hypotheses which disregard the intermediate stress σ_2, and in particular the Duguet-Mohr one, is being stated.

The attempt at mathematical justification of the theory of material effort made by Duguet, close to correctness and the only one known in the literature, failed. This attempt shows clearly that there exist no *theories* of material effort, there are only *hypotheses* of material effort. In this field one must not prove but should only verify.

With passing of time, the failures, which I have presented, have forced one to a considerable carefulness in formulating hypotheses. The generality of the hypothesis discussed above is already a proof of this fact. HERTZ[21)] went even further. He suggested namely to represent each of the three components by a point in the coordinate system (σ_1, σ_2, σ_3) and to obtain the equation of the surface constructed in this manner as the desired function. If we look closely at Hertz's concept, we see that there is a lot of – let us say – practical sense in it, but nothing more. Hertz did not know really what to think about the material effort. HAIGH[22)], an Englishman, took up Hertz's suggestion (independently of him, it seems) and developed it with reference to all hypotheses known by himself; we did not witness any significant progress in the field of material effort due to this fact.

In the meantime, the needs of theory and practice were insistently demanding some plausible hypothesis. Sometimes this pressure caused desperate actions. BECKER[23)], an American, overlapped (in the exact meaning of this word) the hypothesis $\tau_{II} = b$ on the hypothesis $\varepsilon_1 = a_1$, $\varepsilon_3 = a_3$. Then, he multiplied the first one by 1.2 (Bach's 1.3 reminds itself) and, finally, in the system (σ_1, σ_3) he obtained for plane states a decagon whose outline agreed with the experiments. WESTERGAARD[24)] drew this outline along the line $\sigma_1 = \sigma_2 = \sigma_3$ and has obtained in the Hertz-Haigh system a prism deceptively similar to the Huber-Mises-Hencky cylinder; their cross-sections differed only by 8 per cent. To use such hypothesis is simply impossible.

The hypothesis developed in the doctor's dissertation (TH Stuttgart) by SANDEL[25)] is more demanding in this aspect; it is a classic example of ... a false theory. According to this hypothesis, in case when $\kappa = \frac{k_c}{k_t} > 3$ (and we know technical materials for which κ reaches even the value 20), the three following states should be considered to be equal in terms of material effort $\sigma_1 = -\infty$, $\sigma_2 = -\infty$, $\sigma_3 = -\infty$; $\sigma_1 = 0$, $\sigma_2 = -\infty$, $\sigma_3 = -\infty$; $\sigma_1 = +\infty$, $\sigma_2 = -\infty$, $\sigma_3 = -\infty$; it is completely sufficient to demonstrate the value of this theory. However, the example of Sandels hypothesis is very worthy. His theory relates linearly maximum shear strain $\frac{1}{2}\gamma_{II}$ with a volumetric strain $e = 3\pi_1$; it is then symmetrical since in terms of stress the above expressions (18) correspond to magnitudes τ_{II} and $s = 3\omega_1$ constructed in an analogous way. Where to search for the cause of the incorrectness? The answer is simple: the magnitudes $\frac{1}{2}\gamma_{II}$ and π_1 are not connected (9), (11); similarly to τ_{II} and ω_1 (10), (12). Since the first ones are the tangent components of the orientation $\phi = \frac{\pi}{4} = \psi$, $\chi = \frac{\pi}{2}$, the other ones express the normal components of the orientation $\phi = \chi = \psi = \arccos\frac{1}{\sqrt{3}}$, that is of a totally different one. It is – as we see – a case opposite to the Duguet-Mohr's one, where the referred expressions were used, but moving from one system of units (*variables – ed. note*) to the other induced the asymmetric results. A correctly constructed hypothesis should satisfy both conditions.

I have noticed with surprise (*EMPA Diskussionsbericht Nr. 28*) that Sandel, without explaining reasons, abandoned the theory he had been previously working on and privately informed about the development of a new one about the constant "resultant" strain $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = a^2$. This hypothesis has a much narrower range of application, namely only for the cases where $\kappa = 1$; it does not lead then to so many distinct contradictions as the former one. However, it is still incorrect: the sum of squares of principal components does not have any known significance. The hypothesis is asymmetrical: the expression $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$ corresponds, according to Hooke's law (18), to a more extent expression in stress components: $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \dfrac{2\mu\,(1-2\mu)}{1+3\mu-\mu^2}\,(\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2)$, which reduces to the symmetrical hypothesis expressed in stress components only for the theoretical value $\mu = \frac{1}{2}$, possible in materials with $\kappa = 1$ only in plastic regions. One cannot develop every couple of years any new, completely different hypotheses without exposing oneself to suspicion that one has recognized the former one as incorrect.

It will not be irrelevant here to state a certain general remark induced by the review of all the known hypotheses: all theories involving Poisson's ratio μ in their mathematical form are more or less incorrect. For, elastic constants have nothing in common with the material effort or, more precisely: it is not the material effort that depends on them, but they depend on the material effort. If we are considering the dimensions of a bar in tension, we do not think of the constant E or μ, but of the allowable stress $\sigma_1 = k_t$; if we are designing a twisted shaft, we do not ask about the constant G or μ but about the admissible effort, whose measure is $\tau_{II} = k_s$. It must be similar in a general three-axial case. Calibrating the empirical facts by the constant μ completely misses the point and is only an unfortunate burden to the form of the theory of material effort. It is a detail which, apart from other facts, speaks against Sandel's new concept.

A great step forward in the development of the hypotheses was assuming by Beltrami[8], Huber[26] and Haigh[22] the energy $\Phi = \dfrac{3}{2}\dfrac{1-2\mu}{E}\omega_1^2 + \dfrac{3}{4G}\omega_2^2$ (19), (20), (21) as a measure of critical material effort. The energy-based theories have a lot of characteristics of being probable. Material effort is a directionless magnitude (*by the assumption of material isotropy – ed. note*) – a scalar, and energy is also a scalar. The energy is the expression of independence of the state of strain, or stress, of our fancy of choosing directions of reference; material effort is also independent of the choice of a coordinate system. A change of units of calculation (*variables - ed. note*) does not reflect on the shape of the theories, since we also have: $\Phi = \dfrac{3}{2}\dfrac{E}{1-2\mu}\pi_1^2 + 3G\pi_2^2$ (19), (20), (21). The hypothesis expresses itself in the most general way by six components; conversion to principal components consists in deleting the tangent ones. The inequality, which was used in other hypotheses precluding their further discussion and application, here cancels out from the calculation completely. The magnitudes π_1, π_2 or equivalently ω_1, ω_2, are not in conflict with each other; they hold distinct positions on the field of strain as well as stress (11), (12).

It was difficult at first to find any better means. And, in spite of all, there was a great need to find such, since the new hypothesis – even in the narrow range of its application – did not satisfy our demands for exactness. If we look closer at the essence of this failure, we see that there is only one reason for this failure. The hypothesis was burdened with the presence of Poisson's ratio μ; as I have mentioned above, it is a rather general cause of the incorrectness of a considerable number of hypotheses.

There exists in the literature a faint, probably unconscious, attempt to remove this constant. From WEHAGE'S[27] theory (1905) it follows that he considers the expression $\sigma_1^2+\sigma_2^2+\sigma_3^2 = b^2$ as a measure of material effort; it is nothing else but Beltrami's hypothesis for $\mu = 0$. Sandel's new approach then consists only in replacing the units σ with ε. One can advance identical critical arguments against Wehage's theory.

Only HUBER[26] (1904) gave the matter a favorable turn by taking under consideration the decomposition of energy introduced by STOKES[12] and HELMHOLTZ[13]. His hypothesis, set in a letter to FÖPPL[28], contains an unusual idea. This hypothesis has all the advantages of the theory discussed before and apart from that, it gives a certain generalization by introducing two critical regions: $\omega_1 \geq 0$ and $\omega_1 \leq 0$. Unfortunately, the range of the hypothesis still remains narrow: $1 \leq \kappa \leq 1.225$, owing to the incomplete omission of the constant μ. Apart from that, the hypothesis comes from one region to the other one in a discontinuous manner.

According to the above reasons, only the second part of the Huber hypothesis $\Phi_f = \frac{3}{4G}\omega_2^2 = b^2$, and without the restrictions concerning ω_1, has been accepted in the literature. Tests in the first place and HENCKY'S papers[29] from the field of theory of plasticity in the second case contributed to this fact. MISES'S[30] (1913) solely graphical argument of a sphere as a surface of critical states in the system of tangent principal components had a considerably smaller influence. Based on erroneous mathematical and logical premises, Roš and Eichinger's efforts to present the hypothesis as a general expansion of Mohr's idea have completely no meaning. It is enough to say that the referred proof describes the strain state of a body with the help of only three, and moreover – tangent components, and that it neglects difference between a vector and a tensor (by introducing a geometrical summation of components of strain state without regard to their respective two-dimensional orientation) and that, finally, as a consequence of the above mistakes, it describes isotropy by the material constants characterizing material that depend on the direction. On the other hand, among experiments, one should put Roš and Eichinger's[31)] tests on the first place. From a whole series of facts stated in *EMPA*, one fact demands a special attention; Roš proved empirically the mathematical equality $\Phi_f = \frac{3}{4G}\omega_2^2 = 3G\pi_2^2$ (21) in the meaning that one can use ω_2 equally well as π_2 as a measure of material effort of plastic materials $\kappa = 1$. It clearly confirms my supposition that the change of variables should not reflect in any way on the form of a hypothesis.

Huber-Hencky's theory discredited a whole series of other hypotheses at one go. Final accounts with a very near Coulomb-Guest's theory were made easier to settle by Roš[31)], Ensslin, Lode[32)] and many other researchers' great experiments. Presently, no one doubts its truthfulness for plastic metals characterized by $\kappa = 1$. In this area the question of material effort has been definitely solved. If further, verifying tests are being performed, it is only due to extend the interval of its validity with respect to the types of stress states, namely to possibly advance the two border limits: hydrostatic uniform tension on the one hand and the corresponding pressure on the other.

However, the matter has not been settled in general; there remain all the materials $\kappa > 1$, that is a great majority of technically important materials. There were attempts to solve this difficult task, yet still simplified by a large amount of the existing material. A careful reader guesses at once what such a theory should look like in its general shape. It should relate to Huber-Hencky theory like the Duguet-Mohr hypothesis relates to Coulomb-Guest hypothesis. One should be a generalization of the other; from the more general one, there should follow a more specific one as a particular case.

Unfortunately, this direction has not been exactly followed. Schleicher[33)] (and Mises at the same time, it seems) picked up Huber's general idea; instead of – as primarily Huber had done – dividing the critical groupings into the two regions $\omega_1 \geq 0$ and $\omega_1 \leq 0$, he made a division into an infinitely large number of infinitesimally small areas, he fixed a different measure of material effort in each of them and moved on to a limit in a purely mathematical sense. To get rid of the suspicion of the deciding meaning of the energy – based idea or maybe to attain originality, he did not introduce into the calculation the essential components of Huber's general theory, that is Φ_v and Φ_f, but $\omega_1 = \sqrt{\dfrac{2E\Phi_v}{3\,(1-2\mu)}}$ and $\sigma_0 = \sqrt{2E\Phi}$, which are still in a close relation to the latter. And this was the critical mistake of his theory $g\,(\omega_1,\,\sigma_0) = b$, since in this manner it lost the influence of the energy of distortion Φ_f and shut itself the way to transition to Huber-Hencky hypothesis in a specific case $\kappa = 1$. This hypothesis flashed up for a moment and then vanished dumped by... its author (*Bauingenieur, 1928*); for, on the occasion of calibration of elementary experiments on shear, the hypothesis demanded for concrete $\mu = 2.8$ or 5.4, which was too far beyond the possibility $0 \le \mu \le 0.5$. The number μ led the hypothesis to catastrophe. The curve $g\,(\omega_1,\,\sigma_0) = b$, assumed by Schleicher to be typical exclusively for plastic materials, can be obtained from his theory for brittle materials, as marble, and inversely. A comprehensive critics of the discussed hypothesis can be found in my article[1)]. One should not be surprised then that Schleicher abandoned his theory, for the sake of the rests of appearances he introduced a new hypothesis only under the name of a different mathematical form. The error lies in the false assumption of the relation between Φ and Φ_f, that is – the relation which is never fulfilled by brittle materials and by plastic ones – only approximately.

The review of the above remarks allows one to judge the mistakes in the existing hypotheses and draw conclusions concerning the correctness of the theoretical construction of hypotheses of local material effort. They can be summarized in the following way:

1. A mathematical form of a hypothesis of material effort should be characterized by continuity and simplicity.
2. The hypothesis should be expressed in general by six components.
3. The choice of units of calculation (*variables – sci. ed. note*) should not influence the substance of the hypothesis.
4. The use of principal components should not be restricted by any numerical sequence of them.
5. Single terms built from the components must have a mechanical sense (from the point of view of continuum mechanics).
6. Sets of such terms must have a distinct and concrete meaning.
7. Such set cannot be calibrated by elastic constants (as e.g. μ).
8. In the particular case of plastic materials of the characteristic $k_t = k_c = k$, that is $\kappa = \frac{k_c}{k_t} = 1$, the hypothesis should transform to the Huber-Hencky's theory.
9. The number of parameters like k_t, k_c, k_s,... should be possibly small.
10. The hypothesis should correspond with the experiments.

I tried to settle a hypothesis that would meet all the requirements listed. It can be generally formulated in the following way:

I. The local material effort of isotropic bodies is expressed by a function created from three invariants of strain state, that is:

$$F(\pi_1, \pi_2, \pi_3) = A. \tag{27}$$

II. The local material effort of isotropic bodies is fully described by three invariants of stress state in the form:

$$G(\omega_1, \omega_2, \omega_3) = B. \tag{28}$$

These two hypotheses, though expressed by different variables, are not different, as it could seem to be after our experience with a great number of previous hypotheses; on account of the relations (24) they are identical.

In the above statements we give, apart from a new, very general hypothesis, also a new way of presenting all hypotheses in the orthogonal reference system (π_1, π_2, π_3) or equivalently $(\omega_1, \omega_2, \omega_3)$. All critical states, e.g. plane states, are referred by points lying on the plane (π_1, π_2) or equivalently (ω_1, ω_2); uniform hydrostatic states find their representation here in points lying on the plane (π_1, π_3) or (ω_1, ω_3). The Huber-Hencky hypothesis is here illustrated by a plane parallel to the plane (π_1, π_3) or (ω_1, ω_3). If particular points (π_1, π_2, π_3) or $(\omega_1, \omega_2, \omega_3)$ in a particular series of experiments generate a surface with two finite curvatures, the hypothesis should remain valid and should be applied in the general form (27) and (28). Though, one should expect that the invariant π_3 or equivalently ω_3, does not play any prominent role in defining the material effort. It would be represented by a cylindrical surface of generators parallel to the direction π_3 or equivalently ω_3, or by simplified equations

$$F(\pi_1, \pi_2) = A, \tag{29}$$

$$G(\omega_1, \omega_2) = B. \tag{30}$$

The simplification implies the following suppositions:

III. Energy of distortion and certain part of energy of volumetric strain, which depends on the strain state and individual characteristics of a body, are measures of the material effort.

As we have mentioned above, the invariant π_3 of necessity cancels out from the energy term. Moving on to the other variables, we can express the Statement III in the following manner:

IV. About material effort decides the function built from the energy of elasticity in general shear and hydrostatic uniform stress, namely – the Eq. (30).

The two seemingly different definitions due to (24) are also convergent here. The statements III and IV have a certain disadvantage; they use the word "energy" taken from the grounds of elasticity. Whereas, the presented hypothesis should be valid in all regions of material effort – of course, at the cost of the appropriate change of numerical parameters characterizing the material in each region. It is, to some extent, a disadvantage common for all the energy-based hypotheses. To avoid the above, we come to the following definition:

V. Critical material effort depends quantitatively on strains of the orientation $\phi = \chi = \psi = \arccos \frac{1}{\sqrt{3}}$ *referred to principal directions.*

As I have mentioned – those are the normal strains π_1 and the tangent ones π_2 (11). In this way we come again to the form (29) without exposing ourselves to the blame for limited validity of the hypothesis.

Statement V demands a small explanation. If we assume that the material effort is really caused by components of a certain direction then, on the assumption of isotropy of the body, it is clear that there can be only one such orientation which is neutral regarding the principal directions and it is the only one assumed above. I lay strong emphasis on the fact that one comes to this orientation exclusively in the way shown here, that is in the way of logical reasoning and assumption. The attempt existing in the literature (Roš and Eichinger) at arithmetical proof of the choice of exactly this direction among all the others, is false from the very foundations; in the science of material effort nothing can be proved, one can only experimentally verify the accuracy of some logical assumptions.

The hypothesis V can be transferred to the ground of stress, namely:

VI. Critical material effort is produced by stresses in the neutral orientation $\phi = \chi = \psi = \arccos \frac{1}{\sqrt{3}}$.

It is mathematically expressed by the Eq. (30). Presently, this orientation is being suspected to be a slip-plane. The above justification is a generalization of the Huber-Hencky theory in the sense of stress (similarly as Duguet-Mohr relates to Coulomb-Guest).

In the literature on material effort there are sometimes graphical proofs of the hypotheses given (Mohr, v. Mises). Let us also present one here:

VII. In critical states, a system of three strain circles moves along the axis of abscissae changing its total area in a continuous manner, due to its position.

As we have proved, this surface it proportional to π_2^2 (14), by introducing π_1 as the coordinate of the position of the complex of circles we come again to (29). Just as well we can agree for the following statement:

VIII. The manner of change of the magnitude of the surface and position of the three stress circles of critical states changes, that is the relation (30) is a measure of material effort.

Both graphical explanations are identical since the system of three strain circles is – independently of the existence of Hooke's law or the quality of the investigated critical state – always homothetic to the complex of stress circles.

In this manner, the discussed hypothesis has been explained in eight different ways; in particular: in a purely mathematical way (the invariants) on the grounds of strain, stress or energy and, finally, in a graphical scheme. All these ways lead to one result, what cannot be said about all the other hypotheses, known to us. The presented theory belongs to all the hitherto known groups and at the same time to none.

The above hypothesis in the incomplete form II and in the complete ones IV and VI. As well as VIII (with a great number of details, not given here due to the space restrictions) I have developed in 1927 and published in my dissertation *"A study on hypotheses of material effort"* (issued by Academy of Technical Sciences – Lwów, Jan. 7, 1928)[1)]. After myself, on the basis of an erroneous assumption, SCHLEICHER[33)] (Bauingenieur, 1928) derived the form IV from his primary hypothesis. Generalization of the so-called degree of reliability given by him can be found also in my work.

The presented theory satisfies all the recently listed conditions. The actual presence of uniformity of material and state of strain (stress) and additionally – the isotropy of a body, are guarantee of the laboratory success of the theory. If the above conditions are not fulfilled, the theory has in the highlighted sense only an approximate meaning; and speaking precisely – the verifying tests have an approximate sense in this case. But even then the author's hypothesis will differ from the presently launched Duguet-Mohr theory – and will differ significantly, since it will satisfy in an exact manner a whole series of remaining conditions, which cannot be stated with respect to the competitive theory.

Apart from this, deviations from the experiments here can be removed in a relatively simple way. I performed such an attempt in the work referred above; it has led to – as it will turn out – most favorable results. However, before I shall speak about this, it will not be irrelevant to discuss first some practical details concerning the conditions 9 and 10, which have been not discussed as yet.

The question of the number of constant parameters in the mathematical form of the hypothesis has not been discussed comprehensively in the literature as yet. One cannot precisely state what number of the simplest facts like:

(i) uniaxial tension $\sigma_1 = k_t, \sigma_2 = 0, \sigma_3 = 0$,
(ii) uniaxial compression $\sigma_1 = 0, \sigma_2 = 0, \sigma_3 = k_c$,
(iii) simple shear $\sigma_1 = k_s, \sigma_2 = 0, \sigma_3 = -k_s$

and so forth, the construction of the functions of material effort should be based on. This matter should be elucidated. In all publications known to myself one can see a silent suggestion of two parameters. It reveals itself evidently in the efforts to state the relation $k_s = f(k_t, k_c)$. But has a unique relation been found? Is it possible to infer the whole shape of the critical surface exclusively from the traces of coordinate axes? Is it enough to know only the behaviour of a body in one, the most simple case $\omega_1 > 0$ $(c = -1)$ and in another, equally simple one $\omega_1 < 0$ $(c = 1)$? Should not one of the critical states from the range $\omega_1 = 0$ $(c = 0)$ be also taken under consideration?

The answer can be delivered by a whole series of not too complicated experiments on various materials conducted only for the three first states (i), (ii) and (iii). The following, rather general formulae

$$k_s = s_a \frac{k_t + k_c}{2\sqrt{3}}, \qquad k_s = s_g \sqrt{\frac{k_t k_c}{3}}, \qquad k_s = s_b \frac{2}{\sqrt{3}} \frac{k_t k_c}{k_t + k_c}$$

or other, similar to these, could be used as the basis of an arithmetical verification. It is worthy of notice that the harmonic formula results mainly from linear hypotheses and the geometrical one - from square hypotheses; the arithmetical one has the least prospects for success. It is obvious that the numerical coefficient s cannot be kept absolutely constant. Maybe in can be put in the form of a function $s = s(\kappa)$; but if it would be unique, it is hard to say. However, in general, it seems that developing the hypotheses with two constants is incorrect; they can have an approximate meaning, limited to a certain range. Three constant stress parameters in general could be a starting point.
(*Sci. ed. note: In the doctoral dissertation of Burzyński, op. cit.*[2]) *pp. 111–114, the discussed above statements of material effort hypothesis are formulated in an equivalent way as an energy-based hypothesis, called by the Author the hypothesis of variable volumetric – distortional limit energy, expressed by the following equation:*

$$\text{(N1)} \qquad \Phi_f + \eta(p)\,\Phi_v = K,$$

where $\eta = w + \frac{\delta}{p}$, $0 \le \eta \le 1$, $p \equiv \omega_1$, *is a certain material function accounting for a particular material properties and the diminishing pressure sensitivity, while K is a limit material constant. The core of Burzyński's formulation of the energy-based material effort hypothesis is the exchange of three material parameters* w, δ, K *appearing in (N1) with the discussed above three material constants:* k_t, k_c, k_s, *or with the triplet* k_t, k_c, ν, *accomplished by means of the replacements:*

$$\frac{1-2\mu}{1+\mu} w = \frac{1-2\nu}{1+\nu}, \qquad \frac{1-2\mu}{1+\mu}\delta = \frac{3(k_c - k_r)}{1+\nu}, \qquad \nu = \frac{k_c k_r}{2k_s^2} - 1,$$

$$12GK = \frac{3k_c k_r}{1+\nu}, \qquad \sigma_f^2 = 12G\Phi_f,$$

what results in the following formula:

$$\text{(N2)}\qquad \frac{1+\nu}{3}\sigma_f^2 + 3\,(1-2\nu)\,p^2 + 3\,(k_c - k_r)\,p - k_c k_r = 0,$$

where ν is the so-called "plasticity coefficient" describing the degree of material ductility. For hard and brittle materials $\nu < \frac{1}{2}$, for hard but ductile materials $\nu = \frac{1}{2}$ and for soft (plastic) materials $\nu > \frac{1}{2}$, and it is assumed that $0 \le \nu \le 1$. Consequently, Equation (N2) can be transformed as follows:

$$\text{(N3)}\qquad \frac{1+\nu}{3}\sigma_f^2 + 3\,(1-2\nu)\left(p+\sigma'\right)^2 = k'^2,$$

where $\sigma' = \dfrac{k_c - k_t}{2\,(1-2\nu)}$, $k'^2 = k_c k_t + \dfrac{3}{4}\dfrac{(k_c - k_t)^2}{1-2\nu}$.
Generally it is assumed that $k_s \ge \dfrac{2}{\sqrt{3}}\dfrac{k_t k_c}{k_t + k_c}$, *due to this the required transition:* $k_t = k_c = k$, $\sqrt{3}\,k_s = k$ *is provided. Equation (N3) describes in the plane* (p, σ_f) *curves of the second degree. In particular:*
(i) for $\nu < \frac{1}{2}$, *i.e. if* $k_s \ge \sqrt{\dfrac{k_s k_r}{3}}$, $1 - 2\nu > 0$ *and* $k'^2 > 0$ *we get an ellipse,*
(ii) for $\nu < \frac{1}{2}$ *and* $\frac{k_c}{k_t} > 1$ *equation (N3) describes a parabola of the second degree, while for* $\frac{k_c}{k_t} = 1$ *two lines parallel to the axis p are obtained,*
(iii) for $\nu > \frac{1}{2}$, *i.e. if* $\dfrac{2}{\sqrt{3}}\dfrac{k_c k_t}{k_c + k_t} < k_s < \sqrt{\dfrac{k_c k_t}{3}}$ *we have* $1 - 2\nu < 0$ *and* $k'^2 < 0$, *Equation (N3) describes a hyperbola, only one branch of which has a physical meaning.*
(iv) in particular for $k_s = \dfrac{2}{\sqrt{3}}\dfrac{k_c k_t}{k_c + k_t}$, *the hyperbola degenerates into two straight lines intersecting on the axis p.).*

The author's theory can be adjusted to experiments in general by means of three parameters, in specific cases – by two, and finally – by one parameter. This cannot be said about the Duguet-Mohr theory, since, if we assume that one of the arbitrarily chosen experimental envelopes can also be represented by three parameters, we have to express all the remaining envelopes with the help of the shape set in such way. Although experiments indicate that these are homothetic curves, they at the same time prove that they are translated and rotated; then, there are at least two additional parameters (due to the symmetry of envelopes, the translation occurs only along the σ axis) required for calibration. Meanwhile, the same experiments teach that the author's theory uses none or only one additional parameter. In this way, the point 9 would speak in favor of the new hypothesis.

A couple of words about uniformity of material. A simple experiment on compression demands for brittle materials 4–6 tests, since particular results differ from each other sometimes even by 20 per cent. What to say then about complex experiments, in which the ratio $\sigma_1 : \sigma_2 : \sigma_3$ assumes an arbitrary value? Can we regard a datum resulting from one such test as reliable? Besides, can one succeed in repeating this test? For, most often in laboratory devices, due to technical difficulties, the directions σ_1, σ_2, σ_3 are not associated and the preservation of the assumed ratio $\sigma_1 : \sigma_2 : \sigma_3$ cannot be simply realized. And there are a lot of such measurements, either for the reason of the mentioned difficulties or for the reason of economy. What absolute value do the resulting numbers have?

And what about the uniformity of the state of stress? Again, the simple experiment of uniaxial compression teaches that realization of a uniform distribution of stresses in a whole body is simply impossible. However, it seemed until now that the state obtained in these conditions was at least axially symmetrical. Whereas, we learn (a private conversation with prof. Roš) that it is out of the question: careful measurements on a compressed rectangular prism show that to obtain equal stresses along four edges of a specimen, an external resultant must be placed eccentrically. Material heterogeneity causes inhomogeneity of the stress state. And if the material does not hold any flaw, a large influence is found on the side of the technical devices. It is commonly known that different machines are used for different types of stress states. Each of them causes certain experimental deviation, while moving from one device to another the deviations change not only their values but, what is worse, maybe even their signs. However what in the theory of compensation would be an advantage, here is absolutely none, since the corresponding series of points on a graph are not distributed in an arbitrary manner. One series of them goes in a continuous manner along one curve, and another series – along another curve, and so forth. And can the results of experiments be assumed as exactly certain?

Finally – the isotropy. This is absolutely out of the question, there is in fact no isotropy. So perhaps quasi-isotropy, mathematically established in Voigt's beautiful work? It could be generally taken under consideration if the dimensions of crystalline structures would be small in comparison to the dimensions of a body. Then, due to the immense quantity of them, the disorder in orientations of particular individual ones would not distinguish any direction. To obtain such a situation, the dimensions of specimens would have to be adequately large in comparison to the dimensions of crystalline structures; but, in this manner we would give way to heterogeneity of the material and additionally we would demand very precise technical devices. And still, experimental results are dependent on the size of the investigated body, which is a kind of a proof of the above. If only the anisotropy could be theoretically defined, just like for all the known crystals! Unfortunately, in the technical materials used these are impalpable influences. And again we have to ask ourselves – how big are the deviations that our measurements are burdened with?

One must always take into account the most unfavourable possibilities. Let us assume that all the briefly described effects do not mutually cancel – on the contrary: let us assume that they sum up, that they exist. The theoretical hypothesis can be in this case corrected – as I have mentioned – by one additional parameter, which unfortunately quite considerably ruins the previous harmony; we are forced in advance to assume the inequality $\sigma_1 \geq \sigma_2 \geq \sigma_3$. The correction can be introduced in two ways, namely IV or VI and still with the same result. The first is more arduous one for it requires longer considerations of energy of isotropic bodies; here we will confine ourselves to use the second manner only.

We assume that presence of the calculated deviations will make the material effort dependent on stresses of orientation slightly different from the one used till now $\phi = \chi = \psi = \arccos \frac{1}{\sqrt{3}}$, and in particular that it will distinguish one of the principal cases e.g. σ_2. For this purpose, assuming generally: $\phi^* = \arccos \sqrt{\dfrac{\lambda}{1+\lambda}} = \psi^*$, $\chi^* = \arccos \sqrt{\dfrac{1-\lambda}{1+\lambda}}$, we obtain from formulae (8) the expressions for components of this direction

$$\text{(31)} \qquad \begin{aligned} \sigma &= \frac{\lambda\sigma_1 + (1-\lambda)\,\sigma_2 + \lambda\sigma_3}{1+\lambda} = \omega_1^* \\ \tau &= \frac{\sqrt{\lambda}}{1+\lambda}\sqrt{(1-\lambda)\,(\sigma_2-\sigma_3)^2 + \lambda\,(\sigma_3-\sigma_1)^2 + (1-\lambda)\,(\sigma_1-\sigma_2)^2} = \omega_2^*. \end{aligned}$$

The corrected hypothesis presently reads:

$$\text{(32)} \qquad G\left(\omega_1^*, \omega_2^*\right) = B.$$

The additional parameter λ is limited by the theoretical inequality $0 \geq \lambda \geq 1$. The lower limit $\lambda = 0$ is – as it seems – of no significance, the upper limit $\lambda = 1$ reduces the expressions (31) to formulae (10) and in consequence of the theory (32) we obtain the hypothesis (25). The Duguet-Mohr theory is then comprised in the correction (32) as a specific case. We obtain a correct theory for isotropic bodies for a middle case $\lambda = \frac{1}{2}$. Experiments seem to teach that deviations which have been discussed, require a correction inherent generally in a narrower interval $\frac{1}{2} \leq \lambda \leq 1$. The values $\lambda > 1$ are theoretically impossible; their presence could be explained only by basic numerical incorrectness of the experimentally indicated stresses; this fact is possible in the case of presence of primary stresses (*internal stresses – sci. ed. note*).

The bigger is the influence of inhomogeneity of material and stress state, of experimental devices and of anisotropy is, the more λ deviates from values of the range: 0.5 to 1.0 – that is the more the author's hypothesis (32) gets closer to the Duguet-Mohr theory (25). Experimental results do not confirm the latter one; but even though, there exists presently a tendency to apply this theory – due to the absence of any better one. It is a wrong opinion. Let us suppose for example that the parameter $\lambda = 0.75$, lying in the middle between 0.5 and 1.0 is needed for the adjustment. In this case, the deviations of tests from the theories of the author (without correcting λ) and Duguet and Mohr have practically the same value; they are both from the experimental point of view to the same degree wrong and they both – apparently – have equal rights to be used as an approximate application. However, while one will be still able to raise, apart from the approximate character, a whole series of critical arguments comprised in the points 1–8 against the hypothesis (25), there will be only the argument of approximation speaking against the hypothesis (29) or (30); and it will still meet the theoretical application and simplicity requirements. These arguments have such a dominant significance that they settle the matter in favor of the author's hypothesis (29) or (30), even when $\lambda > 0.75$. And regarding a complete adjustment of the theory to the research results, it can be performed – as I have mentioned – in the discussed hypothesis by means of one additional parameter λ in Duguet-Mohr theory there are required at least two of them. This argument also speaks strongly against the theory of envelopes.

As I have highlighted in the introduction, the aim of this paper is, above all, to define the theoretical foundations of hypotheses of the material effort. For this reason I confine the illustration of the point (10) only to a couple of interesting experiments.

Beautiful tests on Carrara marble indisputably belong to this account. They deserve attention even for the sole reason that they were conducted in different laboratories and by different researchers. For the reason of the already marked difficulties, these tests were hitherto conducted for two extreme types of loading (26): $\sigma_1 < \sigma_2 = \sigma_3 \ (c = -1)$ or $\sigma_1 = \sigma_2 > \sigma_3 \ (c = 1)$. (Results from Böker[34) tests on twist performed on solid samples should be regarded as uncertain.) Figures 1 and 2 present in the systems (σ_{II}, τ_{II})or (ω_1^*, ω_2^*) the results of these experiments, conducted on three different sorts of marble by Kármán, Böker and Roš and Eichinger. The hollow circular points $(c = -1)$ lie due to the Duguet-Mohr theory on a curve which is here always above the curve of solid circular points. The corresponding corrections $\lambda = 0.73, \ 0.87, \ 0.75$ (obtained not by means of fitting, but from brief considerations) get both of the two different types of experiments to a one common, gently bent curve in the author's system. It is possible that the value of the parameter λ is dependent on the crystalline structure of a material; however, in this case λ should be common for all the cited experiments. One should suspect rather that accidental effects, mentioned earlier, decide on the value of this parameter.

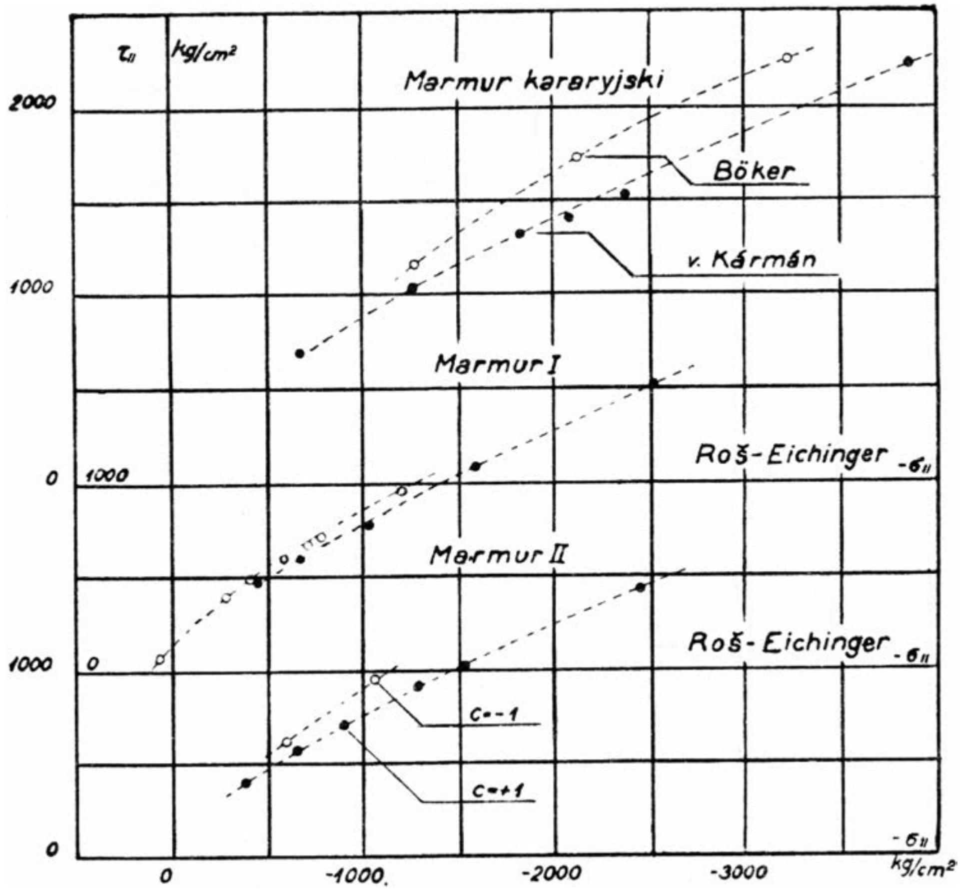

FIG. 1. The results of the tests on Carrara marble (*marmur kararyjski*), in the system (σ_{II}, τ_{II}) corresponding to the Duguet-Mohr theory, performed on three different sorts of marble (*marmur*) by Kármán, Böker and Roš and Eichinger[35)].

Experiments, also performed by Roš and Eichinger, on china, pure cement and cement mortar, cannot be used for a comprehensive discussion or comparison of hypotheses for the reason of very small number of tests (two or three pairs of points $c = \pm 1$).

Further experiments with artificial resin, though, deserve notice. It is an exceptionally uniform and certainly isotropic material. Unfortunately, we may suppose that it is burdened to a considerable degree with primary stresses (*i.e. internal stresses – sci. ed. note*), like all artificial preparations of this kind. We are confirmed with this supposition by the fact that it was required to assume $\lambda > 1$, namely $\lambda \cong 1.32$ to adjust the experimental results to author's theory (32). Results of the experiments are shown in Figs. 3 and 4, that is in the system (σ_{II}, τ_{II}) corresponding to the Duguet-Mohr theory and the author's corrected

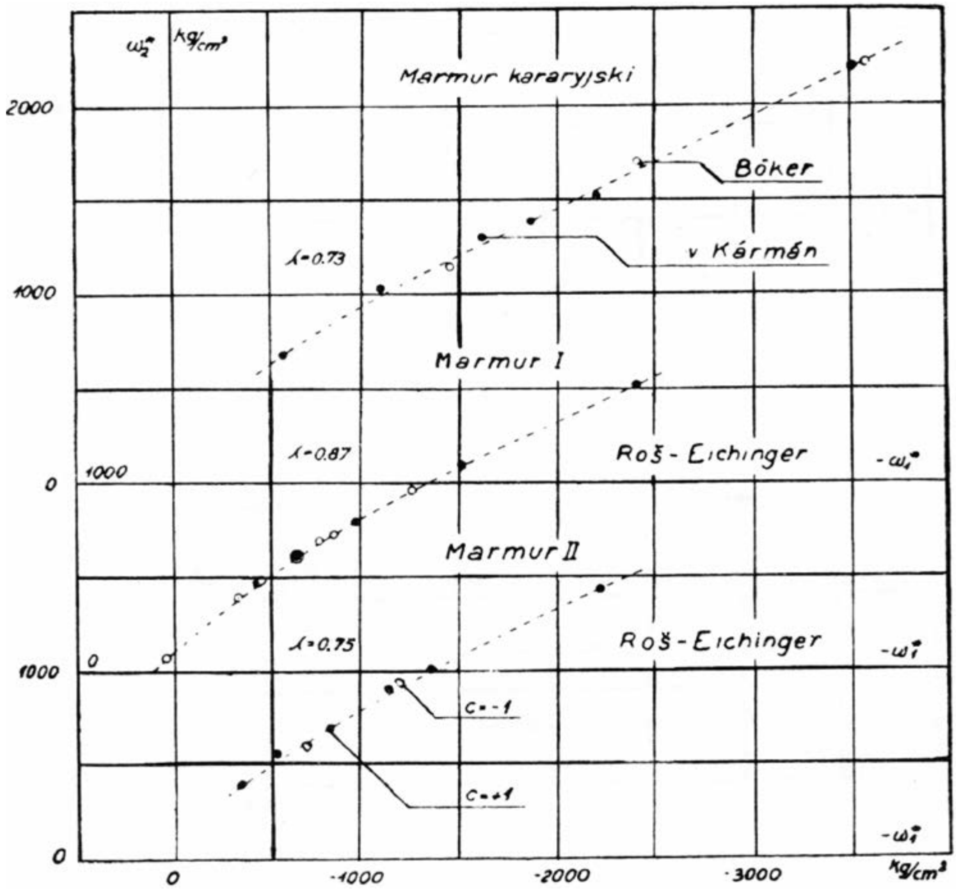

FIG. 2. The results of the tests on Carrara marble (*marmur kararyjski)*, in the author's corrected system (ω_1^*, ω_2^*), performed on three different sorts of marble (*marmur*) by Kármán, Böker and Roš and Eichinger[35].

system (ω_1^*, ω_2^*). As we can see, we have to do here with a completely opposite case; the points $c = -1$ go beneath the points $c = 1$ by Mohr, while in the author's illustration these differences vanish.

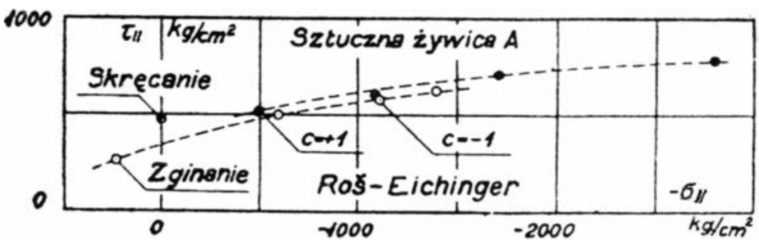

FIG. 3. The results of the experiments with artificial resin (*sztuczna żywica*) in the system (σ_{II}, τ_{II}) corresponding to the Duguet-Mohr theory[35].

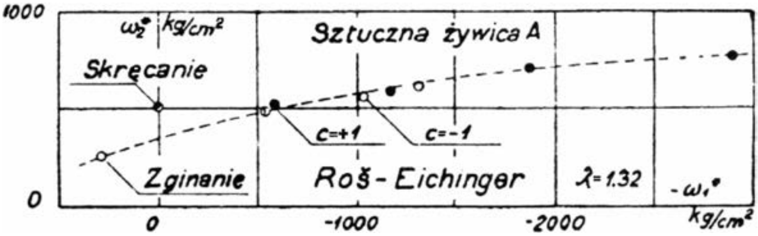

FIG. 4. The results of the experiments with artificial resin (*sztuczna żywica*) in the author's corrected system (ω_1^*, ω_2^*)[35].

From the experiments on 'brittle' metals we take into consideration the experiments by *EMPA* on cast iron (*Elektroguss EK50*) and (*Maschinenguss HS50*). They are in a much more general tone than the ones cited hitherto, for they take into account the three following types of stress states: $c \cong -1.0,\ 0.0,\ 0.3,\ 0.8C$ (as shown in Fig. 5 and Fig. 6). But unfortunately there are on the average only three points for each of these types, which is not quite enough for a relatively broad interval of stresses. And in particular it is difficult to recognize what correction λ should be assumed. For this reason it has been disregarded (or more precisely: the theoretical value $\lambda = 0.5$ has been kept) and the results have been

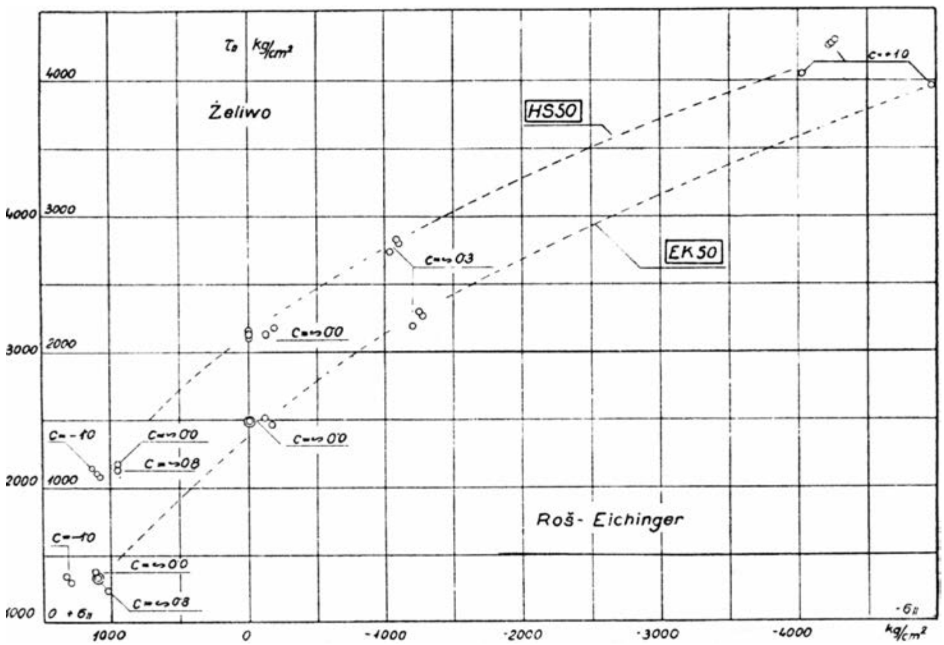

FIG. 5. The results of the experiments on "brittle" metals: experiments by *EMPA* on the cast iron (*żeliwo*) – (*Elektroguss EK50*) and (*Maschinenguss HS50*) in the system (σ_{II}, τ_{II}) corresponding to the Duguet-Mohr theory[35].

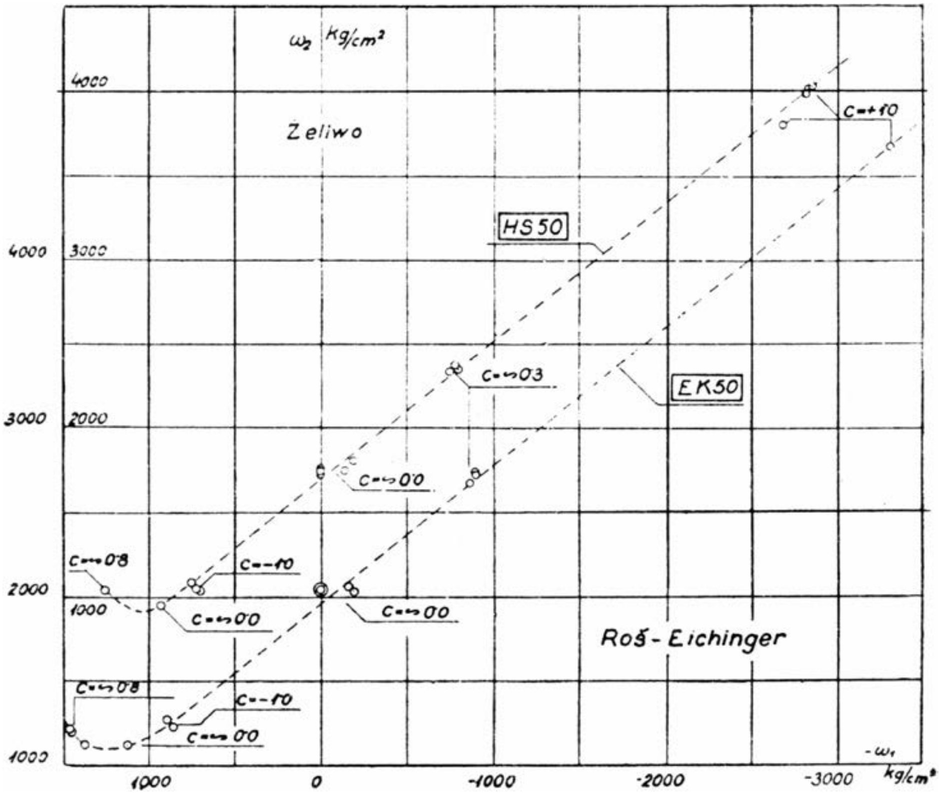

FIG. 6. The results of the experiments on "brittle" metals: experiments by *EMPA* on the cast iron (*żeliwo*) – (*Elektroguss EK50*) and (*Maschinenguss HS50*) in the author's corrected system $(\omega_1^*, \omega_2^*)^{35)}$.

presented in the theoretical system (ω_1, ω_2). However, even this is enough to recognize the superiority of the invariants theory over the one of envelopes. First of all it can be clearly seen that experimental points in the system (σ_{II}, τ_{II}) (Fig. 5) are spread more widely than in the system (ω_1, ω_2) (Fig. 6). Apart from this, it is worthy of notice that in the second system almost all experimental results lie on straight line, while by Duguet and Mohr they lie along a curve or more precisely – curves, since here, as well as in general, one curve is out of the question. In other words, the relation $k_s = \dfrac{2}{\sqrt{3}} \dfrac{k_t k_c}{k_t + k_c}$ is precisely fulfilled for cast iron. Finally, it would be good to underline one more detail, namely grouping of points in the area $\sigma_{II} > 0$ or $\omega_1 > 0$. Evidently there is a confusion in this matter in the Duguet-Mohr theory; it is not known how to lead the assumed curve. Whereas the author's theory sorts the data in a clear manner due to the value c and arranges it in a clear continuous curve; in other words, it estimates the influence of the medium stress σ_2 (*cf. Eq. (26) – sci. ed. note*), which cannot be said about the competitive theory.

Finally, let us add a couple of remarks concerning tests on plastic materials. The author's theory transforms then into the Huber-Hencky theory or other, similar to it if we assume $\lambda \neq 0.5$; in other words, the veracity of the hypothesis of energy of distortion is supported by correctness of the theory of invariants; and possible shortcomings of the energy-based theory can be removed in the theory (32) by means of the parameter λ.

As it is known, the Huber-Hencky theory refers only to plastic materials of a characteristic $k_t = k_c = k$. In this case we can use instead of the systems (σ_{II}, τ_{II}) and (ω_1, ω_2), a common one. In general there is namely:

$$\omega_1 = \sigma_{II} + \frac{c}{3}\tau_{II}, \qquad \omega_2 = \frac{\sqrt{2(c^2+3)}}{3}\tau_{II}. \tag{33}$$

The dependence of τ_{II} on σ_{II} disappears in the case of the discussed materials in the Coulomb-Guest theory $\tau_{II} = \frac{k}{2}$ (that is – the simplified Duguet-Mohr theory); the Huber-Hencky theory $\omega_2 = \frac{\sqrt{2}}{3}k$ is totally independent of the influence of ω_1. Regarding this fact and (33), at the same time we can write both hypotheses in the form of equations:

$$\tau_{II} = \frac{1}{2}, \qquad \frac{\tau_{II}}{k} = \frac{1}{\sqrt{c^2+3}}. \tag{34}$$

A corresponding author's correction, that is (32), can be analogically presented in the form:

$$\frac{\tau_{II}}{k} = \frac{1}{\sqrt{2(1+\lambda) + 2c^2(1-\lambda)}}. \tag{35}$$

The Eqs. (34) and (35) can be easily presented in the system $\left(c, \frac{\tau_{II}}{k}\right)$. Let us notice at the same moment that for the abscissa $c = \pm 1$ we obtain one and the same ordinate, namely $\frac{\tau_{II}}{k} = 0.5$, from all the three equations.

In the contemporary literature, the Roš and Eichinger experiments on Siemens-Martin's cast steel are considered to be the foundations of the Huber-Hencky theory. The numerical values of stress in the moment of going beyond the upper yield point confirm this hypothesis most exactly and it was taken under consideration. The results of these experiments are presented in Fig. 7; we have assumed $k = 2615$ kg/cm^2 as the mean value of all measurements of k_t and k_c. The experimental data indeed fall symmetrically according to

Huber-Hencky curve, disregarding the line resulting from the Coulomb-Guest theory (Duguet-Mohr). The experiments on annealed cast steel conducted by Roš and Eichinger to reinforce the hypothesis of energy of distortion have, unfortunately, hardly any value; 25 among the 32 tests were performed for $c = \pm 1$, and 3 of the remaining 7 tests are in favor of the Coulomb-Guest theory, while 4 – support the Huber-Hencky theory. The same Fig. 6 explains it clearly.

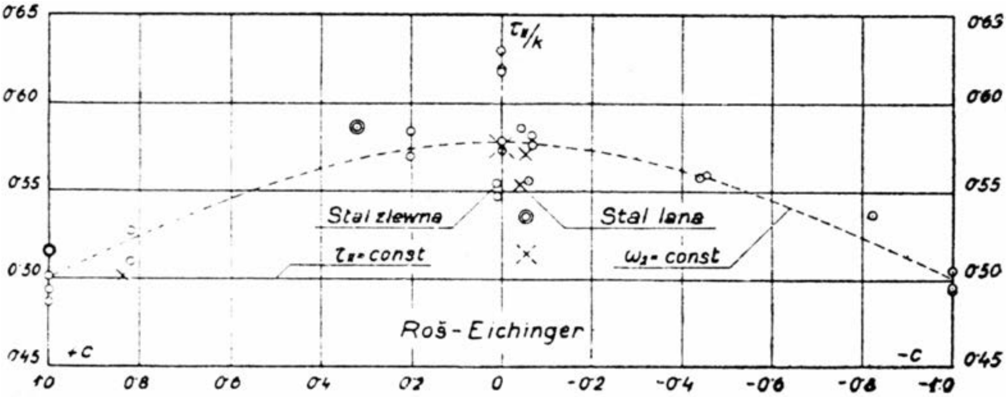

FIG. 7. The Roš and Eichinger experiments on Siemens-Martin cast steel (*stal lana*) , with assumed $k = 2615$ kg/cm^2 as the mean value of all measurements of k_t and k_c, and the experiments on annealed cast steel (*stal zlewna*)[35].

As for the selection of types of critical stress states, Lode experiments on cast iron, nickel and copper look very well in this respect; they are presented in Fig. 8; the parameter c runs here through many more values than those by Roš and Eichinger. Unfortunately, one and the same specimen was used several times in these experiments which, of course, were reflected in the results and, owing to this reason, these are less suitable for stating validity of the Huber-Hencky theory. The experiments can be best calibrated by the parameter $c = 0.6$, numerically very close to the theoretical value $c = 0.5$, supporting in this way the author's theory.

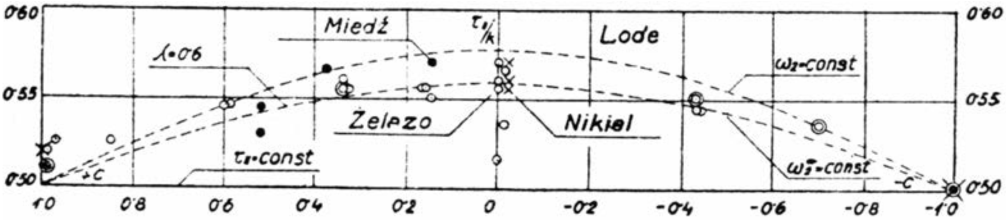

FIG. 8. The results of the Lode experiments on cast iron (*żeliwo*), nickel (*nikiel*) and copper (*miedź*)[35].

Roš and Eichinger tests on aluminum, copper and tombac cannot automatically confirm the validity of the hypothesis of distortion energy, because it has been found that $k_t \neq k_c$ for these materials. Apart from this, the mentioned experiments were conducted only for three simple states: (i), (ii) and (iii), so they are off the general, comparative considerations.

The experimental results state, as it turns out, that the author's theory with the correction λ is suitable for mathematical adjusting all former experiments; it performs this adjustment in a unique way. The parameter λ gets closer to the limit $\lambda = 0.5$ as the qualitative conditions of tests and in particular of the material itself get better. Disregarding the defects of material and all accidental sources of errors, we have to assume $\lambda = 0.5$ in (32) and in this way to accept the theory in its fundamental form (27) and (28) or (29) and (30), which shows the advantages alien to Duguet-Mohr theory. The assumption $\lambda = 0.5$ is more admissible since because brittle materials, for which the experience shows experimental value $\lambda > 0.5$, find their application in technology in dimensions much larger than those used in laboratories. In these conditions secondary effects such as inhomogeneity, incomplete isotropy, etc., lose to one's advantage their disturbing character following in the consequence the author's theory.

Some final remarks should be devoted to the so-called slip surfaces. This matter does not have any fundamental meaning for the material effort; it is only a certain side-effect, which gains significance in specific fields (problems of theory of plasticity, expert opinions in construction disasters, partially a problem of equilibrium of slopes, etc.)

Measurement of the angle ϑ between two planes of slip is a riddle, which – I honestly admit – I do not understand. This measurement should be made – strictly speaking – in one body point; since it is impossible, finite dimensions are being used. However, there is a non-uniform stress state in the range of these; so the slip does not occur on a plane but on a surface and moreover, on a non-cylindrical one. Why then slip planes are being considered, and why two of them and not three or four? What is regarded as the discussed angle in a measurement?

Without engaging closer in understanding and probability of the discussed problems and considering only the numerical results, we can state anyway that the Duguet-Mohr theory does not correspond with the measured data; the acute angle

$$\vartheta = \frac{\pi}{2} - \beta, \tag{36}$$

that is the angle between a tangent to an envelope and the axis τ, is a bit too small in the range of states $c > 0$ (in fact $c = 1$), and immeasurably large when

$c < 0$ ($c = -1$ in current tests). At the same time we see that the argument of impossibility of existence of an envelope which envelopes all circles of critical states, succeeds also at this place; since for the hydrostatic – let us say – tension, this angle obviously has to be indefinite; the referred point cannot have any definite tangent – hence, it cannot lie on the envelope of the remaining states.

More precise determination of the angle ϑ according to Mohr's recipe is impossible with the use of other hypotheses; indeed, as I have mentioned, each of them can be represented by a group of envelopes depending on the parameter $c\,(\sigma_2)$, but exactly for this reason there arises a serious problem, namely – can one in this case measure that angle from the tangent to the axis τ, or should it be measured each time to some other direction depending on the numerical value of c? Actions performed currently in laboratories are now being questionable. With reference to the author's theory the following fact can be noted: the angle ϑ calculated from the equation

$$\cos\vartheta = \cot\delta \tag{37}$$

has the same degree of approximation to the laboratory data as the angle ϑ taken from particular envelopes; δ means in the coordinate system (ω_1, ω_2) or $\left(\omega_1^*, \omega_2^*\right)$, the acute angle between a tangent and the direction ω_2 or ω_2^*. In the determination of δ it disappears the ambiguity present by the group of envelopes. I do not consider the angle δcalculated in this way as a fully correct solution, similarly as it happens in Mohr's case.

There still arise doubts and even more serious than the previous ones. The angle δ has a secondary meaning; the orientations of slip surfaces have the primary meaning, since only from these the value δ should be calculated. Mohr's concept was a kind of a stroke of genius assuming the axis σ_2 as the line of intersection of two slip planes (which was also mathematically proved by Duguet on the basis of the hypothesis of internal friction). Mohr prevented his followers from using his theory as a starting point for attempts to generalize or modify his stand. It is obvious that if we reject Mohr's assumption or Duguet's results, we face the difficult riddle. Let us pay attention to the following by the Duguet and Mohr theory: the direction σ_2 has a geometric meaning – it is the axis of symmetry. If this direction would be highlighted so much (and not ignored, as others say), indeed two slip planes should be assumed in the coordinate system of principal directions. Their normals make angles $\pm\left(\frac{\pi}{4} - \frac{\beta}{2}\right)$ and $\pm\left(\frac{\pi}{4} + \frac{\beta}{2}\right)$ with the directions σ_1 and σ_3 or – in other words – planes of the systems $\sigma_1\sigma_2$ or $\sigma_2\sigma_3$ are planes of symmetry with respect to slip. The orientations $\pm\frac{\pi}{4}$, $\frac{\pi}{2}$, $\pm\frac{\pi}{4}$ or $\pm\frac{\pi}{4}$, $\frac{\pi}{2}$, $\mp\frac{\pi}{4}$ have a significant meaning in this theory, since the Duguet-Mohr hypothesis uses the stresses (10) of these very directions. The normals of the slip planes are rotated with respect to the latter by the angle $\frac{\beta}{2}$; the direction σ_2

is then a kind of axis of rotation. Finally, this direction is perpendicular to the components σ_{II}, τ_{II} of the diagonal orientations described above. We see that so many various points of view are connected with the theories of slip planes (25); by Duguet and Mohr they all lead to one result and it is only thanks to the fact of giving σ_2 a privileged though unjustified position.

If we disregard this position – and, of course, we have to do that – each of the points of view described above will lead to different results. If we assume the existence of one arbitrary slip plane, we do not know how to reach the other one at all. Assumption a simultaneous symmetry to two planes of coordinate system fails, because in this way again one of the directions would be privileged. Assuming symmetry with respect to one of the system planes leads to a number of slip planes larger than two due to equivalence of each of the principal directions; but, if we assume only two of them a priori, the question arises how to define their edge of intersection. This question – at least by now – we cannot solve.

However, regardless of this, certain details of the Duguet-Mohr theory can be generalized in reference to one slip orientation. First of all, with respect to the remarks presented on the occasion of explaining our theory, we will assume a plane indifferent to all principal directions i.e. $\phi = \chi = \psi = \arccos \frac{1}{\sqrt{3}}$ (8) or its experimentally corrected form $\phi^* = \psi^* \neq \chi^*$ (31), instead of unreliable plane $\phi = \frac{\pi}{4} = \psi$, $\chi = \frac{\pi}{2}$. We will assume – similarly to Duguet and Mohr – that the slip plane is inclined to the above one by the angle $\frac{\beta}{2}$, but – as it has been mentioned above (36), (37) – there is: $\sin\beta = \cos\vartheta = \cot\delta$. In this assumption we leave the previous meaning of δ, but we lose the previous meaning of ϑ due to introducing the new stress orientation as a starting point for calculation; the angle δ or β is here only a measure of deviation of the slip plane with respect to the one used for calculation. Having calculated this one with the help of δ, taken from a graphical scheme or calculated from the function of material effort $\left(\cot\delta = -\frac{d\omega_2}{d\omega_1}\right)$, we are facing a simple analytical problem, which, unfortunately, can have two solutions, depending on our further considerations.

Assuming that the calculated plane due to material effort has been rotated round the axis σ_2 (as by Mohr) by the angle $\frac{\beta}{2}$, we obtain the following expressions as direction cosines of the searched slip orientation (ξ, η, ζ):

$$\begin{aligned}
\cos\xi &= \sqrt{\frac{2\lambda}{1+\lambda}}\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right), \\
\cos\chi &= \sqrt{\frac{1-\lambda}{1+\lambda}}, \\
\cos\zeta &= \sqrt{\frac{2\lambda}{1+\lambda}}\cos\left(\frac{\pi}{4}+\frac{\beta}{2}\right).
\end{aligned} \tag{38}$$

from which in particular for $\lambda = 1$ we obtain $\xi = \frac{\pi}{4} - \frac{\beta}{2}$, $\eta = \frac{\pi}{2}$, $\zeta = \frac{\pi}{4} + \frac{\beta}{2}$ as in Duguet-Mohr theory, which fact, of course, was easy to be predicted. The data obtained in this manner satisfy certain elementary experimental demands. Among other things, they e.g. confirm the result that a normal to a slip plane makes a smaller angle with the algebraically greater stress; for, one should remember that the present considerations are valid for the case $\sigma_1 > \sigma_2 > \sigma_3$. However, a serious argument speaks against the above formulae: the direction σ_2 again has played a privileged role.

In view of the above we can act in a different way: namely, let us assume that the calculated plane as the result of process has been rotated by the angle $\frac{\beta}{2}$ round the axis perpendicular to the stresses ω_1^* and ω_2^* (σ_{II} and τ_{II} by Duguet). In this way we will obtain the direction cosines of the desired orientation(ξ, η, ζ):

$$\begin{aligned}
\cos\xi &= \sqrt{\frac{\lambda}{1+\lambda}}\left[\cos\frac{\beta}{2} + \frac{\sigma_1 - \omega_1^*}{\omega_2^*}\sin\frac{\beta}{2}\right], \\
\cos\eta &= \sqrt{\frac{1-\lambda}{1+\lambda}}\left[\cos\frac{\beta}{2} + \frac{\sigma_2 - \omega_1^*}{\omega_2^*}\sin\frac{\beta}{2}\right], \\
\cos\zeta &= \sqrt{\frac{\lambda}{1+\lambda}}\left[\cos\frac{\beta}{2} + \frac{\sigma_3 - \omega_1^*}{\omega_2^*}\sin\frac{\beta}{2}\right].
\end{aligned} \tag{39}$$

The obtained expressions are very probable. The experimental fact of dependence of the position of a slip plane upon an algebraic value of principal components is here clearly confirmed. And here also – as previously – for $\lambda = 1$ we come to orientations described by the Duguet-Mohr hypothesis. Two additional facts deserve notice. The orientation (ξ, η, ζ) depends on material effort in two ways; firstly through the angle β, and secondly directly through the components σ_1, σ_2, σ_3. This seems to be right; for, it can (or rather: has to) happen that in the graph (ω_1^*, ω_2^*) two points belonging to two different types of loadings (e.g. $c = 1$ and $c = -1$) will overlap; then angle β will be common for both of them, but (ξ, η, ζ) will be not. In this way the serious shortcoming of Mohr's relations would be removed. The other fact is the indeterminacy of (ξ, η, ζ) in the case when $\sigma_1 = \sigma_2 = \sigma_3$, since then the numerator and the denominator of the multiplier of $\sin\frac{\beta}{2}$ simultaneously become zero; obviously, it is clear and convincing. Apart from this, the denominator ω_2^* can never be zero, so the given formulae do not contain any contradictions in themselves.

As I have stressed two times, the formulae (38) and (39) have been derived from two important assumptions; the first one consists in assuming the angle δ as the measure of deviation of the slip plane with respect to the plane of stresses which decide on critical material effort; and the other – in the possibility of transition of the defined orientations (ξ, η, ζ) onto the Duguet-Mohr directions in case when $\lambda = 1$. Both assumptions are of course hypothetical; the first implies the shape of the graphs (ω_1^*, ω_2^*) or – for $\lambda = 0.5$ – the curve (ω_1, ω_2); the other is probable inasmuch as it is based on a general – quoted several times – numerical dependence of the Duguet-Mohr theory (25) on the author's theory (32) provided with the correction λ. As far as I know, the formulae (38) and (39) are the only ones known in the literature attempts of this kind (of course apart from Duguet's calculation or Mohr's graphical solution). I do not attribute any distinct significance to it, anyway, not to the degree as I do in my general theory of material effort (27) and (28) or the simplification (29) and (30). The theory (32) has the meaning of a laboratory correction.

Zürich-Lwów, 1929

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The original paper in Polish was reprinted in the Selected Papers of Włodzimierz Burzyński: *Dzieła Wybrane*, tom I, Polska Akademia Nauk, PWN Warszawa, 1982, 265–303, prepared by the Editorial Committee of the Polish Academy of Sciences: Eustachy Burka (secretary), Stefan Cieśla, Marian Janusz, Igor Kisiel, Franciszek Misztal, Witold Nowacki, Zbigniew Olesiak (scientific editor), Fryderyk Staub, Robert Szewalski (chairman), Czesław Woźniak, Marek Zakrzewski, Jerzy Zawadzki.

1) The lecture was published as the paper: *Ueber die Anstrengungshypothesen* von Dr. Ing. Wladimir v. Burzyński, Assistent der Technischen Hochschule Lwów, Polen Schweizerische Bauzeitung, Bd. 94, Nr. 21, 23. November 1929, 259–262; reprinted in Włodzimierz Burzyński: *Dzieła Wybrane*, tom I, Polska Akademia Nauk, PWN Warszawa, 1982, 259–262.

2) The detailed references of papers given below, quoted in the paper only by names, were taken from the – published in Polish – doctoral dissertation of Włodzimierz Burzyński: *Studjum nad hipotezami wytężenia* (*Study on Material Effort Hypotheses*), under the imprint of the Academy of Technical Sciences, Lwów, 1928, 1–192; reprinted in: Włodzimierz Burzyński: *Dzieła Wybrane*, tom I, Polska Akademia Nauk, PWN Warszawa, 1982, 67–257.

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5) C.A. Coulomb, *Essai sur une application des régles de Maximis et Minimis à quelque problémes ...*, Mem. par divers savants, Paris, 1776.

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