ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 29, 1, 27—37, 1981
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

PARAMETRIC OPTIMIZATION OF VISCOPLASTIC BARS UNDER DYNAMIC AXIAL LOADING(*)

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The paper formulates the problem of the parametric optimal design of a viscoplastic bar under the impact of axial force, Eq. (4.4), and gives some solutions. The dual approach to the problem has been applied, the minimal residual deflection being the design objective, under the constraint of a constant volume of the bar. The governing equation (2.4) has been derived under the assumption of a power physical law and arbitrarily variable cross section; however, effective calculations have been performed for the linear law ($\delta=1$) and for bars of truncated cone shape. Two parameters describe the cross section of this bar, Eq. (3.1), but one of them can be determined from the condition of constant volume of the bar. Hence the parameter of tapering of the cone (A) remains the only design variable in this case. The influence of various parameters which describe the shape of the force impulse (Fig. 2) on the parameter A_{opt} has been investigated (Figs. 3, 4 and 5).

1. Introduction

The literature devoted to optimization of structures under impact loadings is rather scarce. D. Wallace and A. Seireg [16] discussed parametric optimization of plastic bars with piecewise constant cross section under longitudinal impact. Optimization of beams and frames in the range of elastic-plastic deformation was discussed by I. M. Rabinovitch [10]. An approximated approach to rigid-plastic beams based on Martin's variational principle was suggested by M. I. Reytman [11]. Optimization of viscoplastic structures subjected to impact loading has not been treated in the literature though such problems were mentioned in the general paper by M. Życzkowski [17].

In the present paper we formulate and give some solutions to the problem of parametric optimal design of a viscoplastic bar under axial impact with a certain prescribed function of force in time, $P=P(t)$, $0<t<T$, $P=0$ for $t<0$ and $t>T$. In principle, the minimal volume of the bar is regarded as the design objective under the constraint of a given residual deflection (shortening of the bar). However, it is more convenient to use dual formulation and to look for minimal residual deflection under a prescribed volume of the bar. We confine the shape of the bar to truncated cones (or pyramids) and the parameter of tapering of the cone will be the only design variable. Due to inertia forces, a prismatic bar is not optimal in the case under consideration. The material of the bar will be assumed to be rigid-

(*) Grant No. 05.12—13.2 is gratefully acknowledged

-viscoplastic described by a nonlinear (power) constitutive equations of viscoplasticity; however, effective solutions will be given for a linearly viscoplastic material only.

Viscoplastic bars under dynamic axial loading were first discussed by V. V. SOKOLOVSKY [12] and L. E. MALVERN [8]. Relatively simple rigid-viscoplastic solutions were given by P. S. SYMONDS and T. C. T. TING for a linear [15] and a power law of viscoplasticity [13], [14]. An elastic/viscoplastic solution for a nonhomogeneous bar was presented by V. N. KUKUDZHANOV and L. V. NIKITIN [7]; Kukudzanov studied also unloading waves [6]. An impact of a bar on a rigid wall was discussed by G. I. BARENBLAT and A. YU. ISHLINSKY [5], T. C. T. TING [14], T. HAYASHI, H. FUKUOKA and H. TODA [5] (allowing for linear hardening of the material). More extensive surveys are given by N. CRISTESCU [4] and R. J. CLIFTON [3].

In the present paper we derive the equations of propagation of viscoplastic longitudinal waves in an arbitrarily non-prismatic bar and then apply these equations to the shape optimization of the bar. Numerical integration and numerical optimization will be employed.

2. GOVERNING EQUATIONS

Consider first an active, viscoplastic zone of the bar. The material is assumed to be described by the following uniaxial power law (P. PERZYNA [9]):

$$\dot{\varepsilon}=\gamma\left(\frac{\sigma}{\sigma_0}-1\right)^{\delta}, \qquad \sigma\geqslant\sigma_0, \tag{2.1}$$

where σ_0 denotes the yield-point stress, γ and δ are constants, and a dot denotes differentiation with respect to time t. Inverting (2.1) to find the stress σ,

$$\sigma=\sigma_0\left[1+\left(\frac{\dot{\varepsilon}}{\gamma}\right)^{1/\delta}\right] \tag{2.2}$$

and substituting (2.2) into the equation of uniaxial motion of a nonprismatic bar we get

$$\frac{\partial}{\partial x}(F\sigma)-\rho F\frac{\partial^2 u}{\partial t^2}=0, \tag{2.3}$$

where $F=F(x)$ denotes the cross-sectional area, ρ is mass density, and $u=u(x, t)$ denotes axial displacement. Bearing in mind $\varepsilon=\partial u/\partial x$ we obtain

$$\left(\frac{\partial^2 u}{\partial x\,\partial t}\right)^{\frac{1-\delta}{\delta}}\frac{\partial^3 u}{\partial x^2\,\partial t}+\frac{\delta\gamma^{1/\delta}}{F}\frac{dF}{dx}\left[1+\frac{1}{\gamma^{1/\delta}}\left(\frac{\partial^2 u}{\partial x\,\partial t}\right)^{1/\delta}\right]-\frac{\rho\delta\gamma^{1/\delta}}{\sigma_0}\frac{\partial^2 u}{\partial t^2}=0. \tag{2.4}$$

This third-order partial differential equation may easily be reduced to a second-order equation by the substitution $\partial u/\partial t=\bar{v}$, where $\bar{v}$ denotes axial velocity. Moreover, introducing dimensionless variables

$$\xi=\frac{x}{l},\qquad \tau=\frac{\sigma_0}{\delta\rho\gamma l^2}\,t,$$
(2.5)
$$v=\frac{\bar{v}}{\gamma l}=v(\xi,\tau),\qquad \varphi=\frac{1}{F}\frac{dF}{d\xi}=\varphi(\xi),$$

where l denotes the length of the bar, Eq. (2.4) can be reduced to a much simpler form

$$v'^{\frac{1-\delta}{\delta}}v''+\varphi\delta\left(1+v'^{\frac{1}{\delta}}\right)-\dot{v}=0. \tag{2.6}$$

In this equation prime and dot denote respectively differentiation with respect to ξ, and τ.

The dimensionless form of the Eq. (2.2) is

$$s=1+v'^{1/\delta}, \tag{2.7}$$

where $s=\sigma/\sigma_0$.

In what follows, we confine effective calculations to the physically linear case $\delta=1$ and integrate the parabolic equation

$$v''+\varphi(1+v')-\dot{v}=0. \tag{2.8}$$

In the case of a prismatic bar, $\varphi=1$, we obtain an equation analogous to that describing the conduction, and hence the velocity of propagation is infinitely large; this propagation starts when $s=1$, since that is the condition of the validity of (2.1). By contrast, in a nonprismatic bar the viscoplastic-rigid interface described by $s=1$ moves along the bar and the velocity of propagation is finite.

Within the rigid zones (plastically passive) we have obviously

$$u=u(\tau)=\text{const}(\xi),\qquad v=v(\tau)=\text{const}(\xi). \tag{2.9}$$

Equation of motion (2.3) expressed in terms of dimensionless quantities (2.5) takes the form

$$s'+\varphi s-\dot{v}=0. \tag{2.10}$$

Since $\dot{v}=\text{const}(\xi)$, we may integrate Eq. (2.10) with respect to ξ and obtain

$$s(\xi,\tau)=\frac{C(\tau)}{F(\xi)}+\frac{\dot{v}(\tau)}{F(\xi)}\int F(\xi)\,d\xi, \tag{2.11}$$

where the function $C(\tau)$ is to be found from the boundary conditions.

We consider here a compressive impact, and hence it is reasonable to understand the compressive strains ε and stresses σ as positive. In the present formulation possible buckling of the bar will not be analysed.

The boundary conditions are as follows (Fig. 1). At the left hand end, $\xi=0$, we have $v=0$. At the opposite end, $\xi=1$, we may describe the impact by $P=P(\tau)$, or $v=v(\tau)$, or $f[P(\tau), v(\tau)]=0$. We consider here only the case $P=P(\tau)$. If the adjacent zone is rigid, then from the condition $s(1,\tau)=P(\tau)/\sigma_0 F_1$, where $F_1=F(1)$, we may determine $C(\tau)$ in Eq. (2.11). If the zone is viscoplastic, then Eq. (2.7) gives

$$v'(1,\tau)=\frac{P(\tau)}{F_1\sigma_0}-1. \tag{2.12}$$

At the rigid-viscoplastic interface $\xi=\xi_{bi}$ we require the continuity of stresses s and velocity v; on the other hand, acceleration $\dot{v}$ may be discontinuous (V. N. Kukudzhanov [7]). In general, we admit several such interfaces $\xi=\xi_{bi}$, $i=1, 2, \ldots, n$.

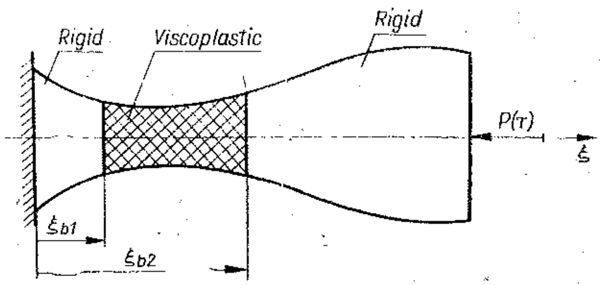

Fig. 1.

We assume that the function $P=P(\tau)$ is increasing monotonically from zero; in this case the initial condition $v(\xi,0)=0$ is justified.

3. Numerical integration and optimization

As we mentioned before, we look for optimal truncated cones, and hence we substitute into Eq. (2.8)

$$F=r_0^2\,\pi\,(1+A\xi)^2, \qquad \varphi=\frac{2A}{1+A\xi}. \tag{3.1}$$

The radius $r(0)=r_0$ may be found from the condition of the constant volume V:

$$V=\frac{\pi}{3}r_0^2\,l\,(A^2+3A+3), \qquad r_0=\sqrt{\frac{3V}{\pi l\,(A^2+3A+3)}}, \tag{3.2}$$

and hence the parameter A is the only design variable.

Numerical integration of Eq. (2.8) with substitution of Eq. (3.1) was performed using the computer CYBER 72. Spatial derivatives were replaced by fourth-order finite difference formulae of the type

$$v_{n+1}=v_n+h_\xi v_n'+\frac{1}{2}h_\xi^2 v_n''+\frac{1}{6}h_\xi^3 v'''+\frac{1}{24}h_\xi^4 v^{IV}, \tag{3.3}$$

where h_ξ denotes the space mesh size; higher derivatives were calculated directly from Eq. (2.8). The time derivatives were calculated from a simple Euler formula

$$v_{m+1}=v_m+h_t\,\dot{v}_m, \tag{3.4}$$

where h_t is the time step. In general in the case of parabolic equations a careful analysis of numerical stability of the procedure must be performed to ensure negligible errors after a longer time (W. F. AMES [1]). However in the problem under consideration one observes full rigidification of the bar after a relatively short time close to T and hence the discussion of longer times is not necessary at all. In any case, it has been checked that the condition of stability quoted by Ames

$$\frac{h_t}{h_\xi^2}\leqslant\frac{1}{2} \tag{3.5}$$

resulted in smaller numerical errors.

At the beginning of the process the entire bar is rigid and Eq. (2.10) was used. If, at any node, s exceeds unity, then Eqs. (2.8) and (2.7) are employed. Conversely, if after some time we find that $s<1$ then Eqs. (2.10) or (2.11) are used. The quantity $u(1)$ after full rigidification was calculated for various values of A, and the smallest value of $u(1)$ served as the design objective.

4. Discussion of the results

The optimal shape of the bar depends, in the case under consideration, on the shape of the impulse, i.e. on the function $P=P(\tau)$. Triangular or quasi-triangular (curvilinear) impulses were only discussed. Since

$$F_1=r_0^2\,\pi\,(1+A)^2=\frac{3V}{l}\,\frac{(1+A)^2}{3+3A+A^2}, \tag{4.1}$$

the stress at the end $\xi=1$ equals

$$s_1=\frac{P(t)\cdot l}{3V\sigma_0}\,\frac{3+3A+A^2}{(1+A)^2}. \tag{4.2}$$

We introduce the dimensionless force $p(\tau)$

$$p(\tau)=\frac{l}{V\sigma_0}P(\tau) \tag{4.3}$$

and consider the following impulse shapes

$$p(\tau)=\begin{cases} 0 & \text{for} \quad \tau\leqslant 0 \text{ and } \tau\geqslant T,\\ \left(\dfrac{\tau}{\mu T}\right)^m p_{\max} & \text{for} \quad 0\leqslant\tau\leqslant\mu T,\\ \left[\dfrac{T-\tau}{(1-\mu)T}\right]^m p_{\max} & \text{for} \quad \mu T\leqslant\tau\leqslant T. \end{cases} \tag{4.4}$$

These impulse shapes are characterized by four dimensionless parameters: m, μ, p_{ma} and T (Fig. 2). In the case $m=1$ we obtain usual triangles, if $m \neq 1$ — curvilinear triangles. Substituting $\mu=1/2$ we have symmetric impulses (e.g. an isosceles trianglr in the case $m=1$). The parameter p_{max} determines the maximum force; it shoule be larger than unity, since $p_{max}=1$ gives $s_{1\,max}=1$ for $A=0$, and no deformatiod process takes place at all. Finally, the parameter T characterizes the duration on the impulse; in fact, bearing in mind the definition of the dimensionless time τ,f Eq. (2.5), we see that T comprises also several other parameters, namely, for a given physical duration of the impulse, T increases with σ_0 and decreases with ρ, γ, and l.

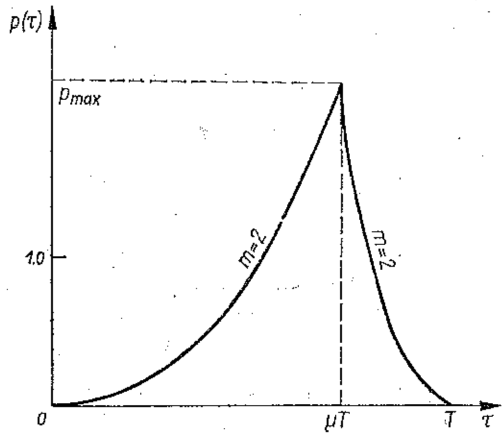

FIG. 2.

In all cases the process was examined until full rigidification and the residual displacement $u(1)$ was calculated and minimized. The profit z was defined as

$$z \overset{\text{def}}{=} \frac{u_p(1) - u_{min}(1)}{u_p(1)} \cdot 100\%, \tag{4.5}$$

where $u_p(1)$ denotes the residual displacement of a prismatic bar, $A=0$.

Figures 3–5 show the results of optimization of the parameter A for various forms of loading impulses. The presented curves show the influence of the dimensionless duration of impulses T as well as of the parameters m and μ (Fig. 2) on the optimal value of A.

The common assumption in all calculations was that the area under the curve $P=P(t)$ (t — being the dimensional quantity) was the same.

The dotted line shows "the profit" z which amounted on the average to about 12 per cent. The maximum calculated value of z has reached 32 per cent for $T=0.0629$.

Figure 6 points out the difference between residual strains ε_r in the prismatic and optimal bars calculated for the following values of parameters: $T=0.251$, $\mu=0.5$, $m=1$. Figures 7 and 8 show curves of the stress s plotted against ξ and τ, respectively and the Fig. 9 curves of the acceleration of the end of the bar $\dot{v}(1)$ against τ for the same value of the parameters T, μ and m.

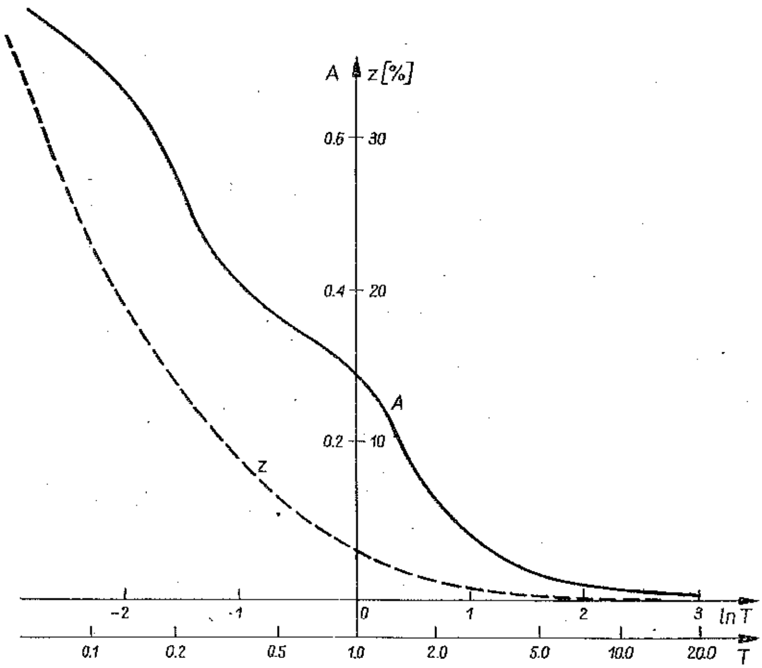
A
z [%]
0.6
30
0.4
20
0.2
10
A
z
-2
-1
0
1
2
3
ln T
0.1
0.2
0.5
1.0
2.0
5.0
10.0
20.0
T

FIG. 3.

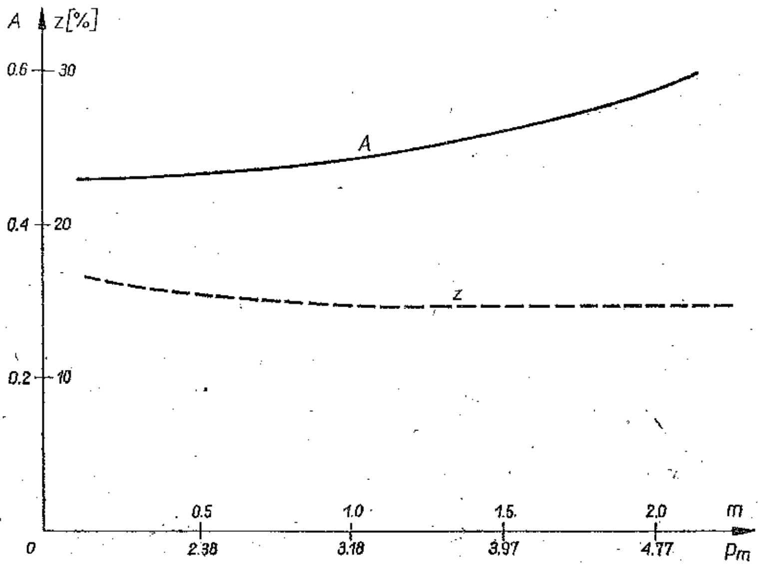
A
z [%]
0.6
30
0.4
20
0.2
10
A
z
0.5
1.0
1.5
2.0
m
0
2.38
3.18
3.97
4.77
p_m

FIG. 4.

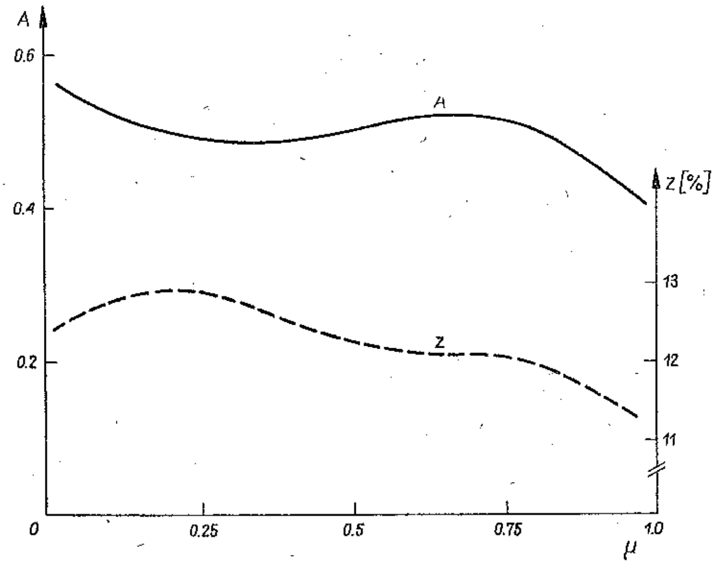
A
0.6
0.4
0.2
0
A
Z
Z [%]
13
12
11
0.25
0.5
0.75
1.0
μ

FIG. 5.

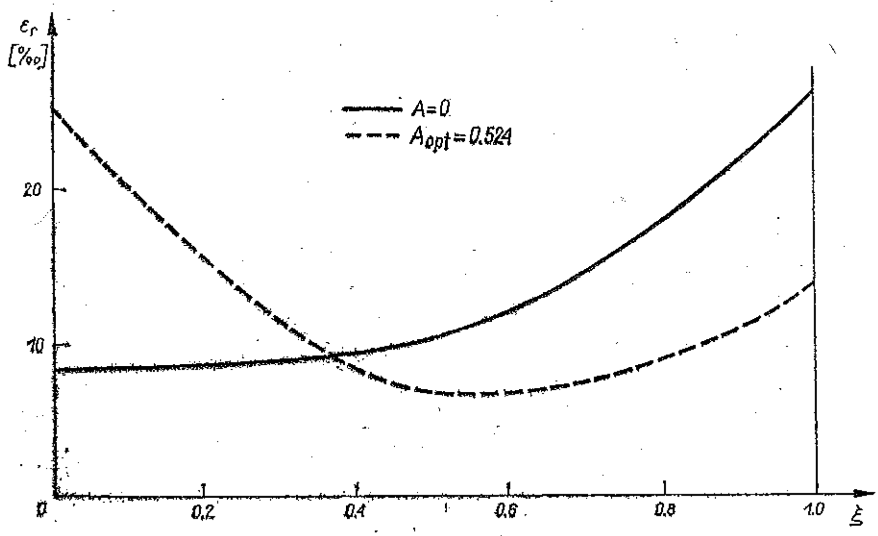
ε_r [‰]
A=0
A_{opt}=0.524
20
10
0
0.2
0.4
0.6
0.8
1.0
ξ

FIG. 6.

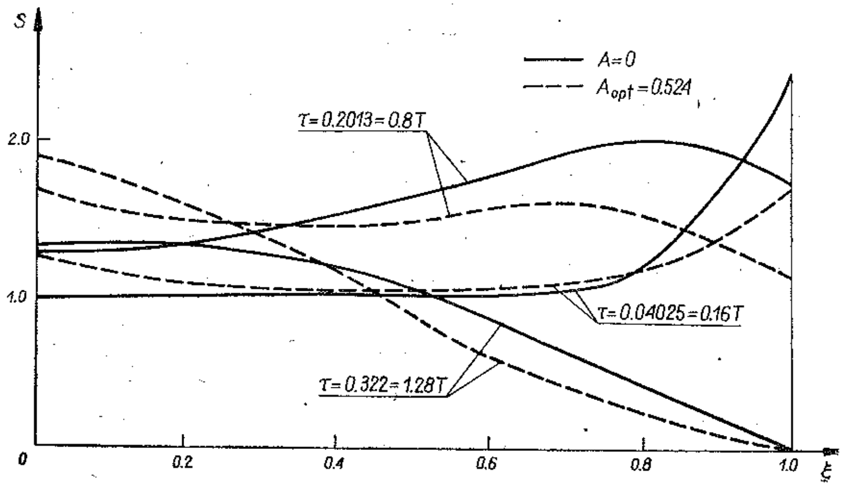
S
A=0
A_{opt}=0.524
τ=0.2013=0.8T
2.0
1.0
τ=0.04025=0.16T
τ=0.322=1.28T
0
0.2
0.4
0.6
0.8
1.0
ξ

FIG. 7.

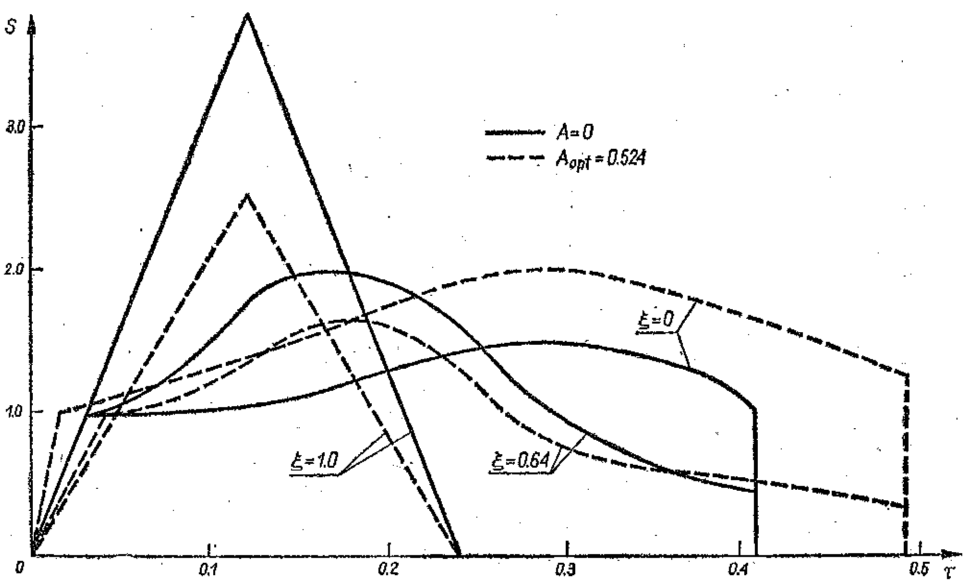
S
3.0
2.0
1.0
A=0
A_{opt}=0.524
ξ=0
ξ=1.0
ξ=0.64
0
0.1
0.2
0.3
0.4
0.5
τ

FIG. 8.

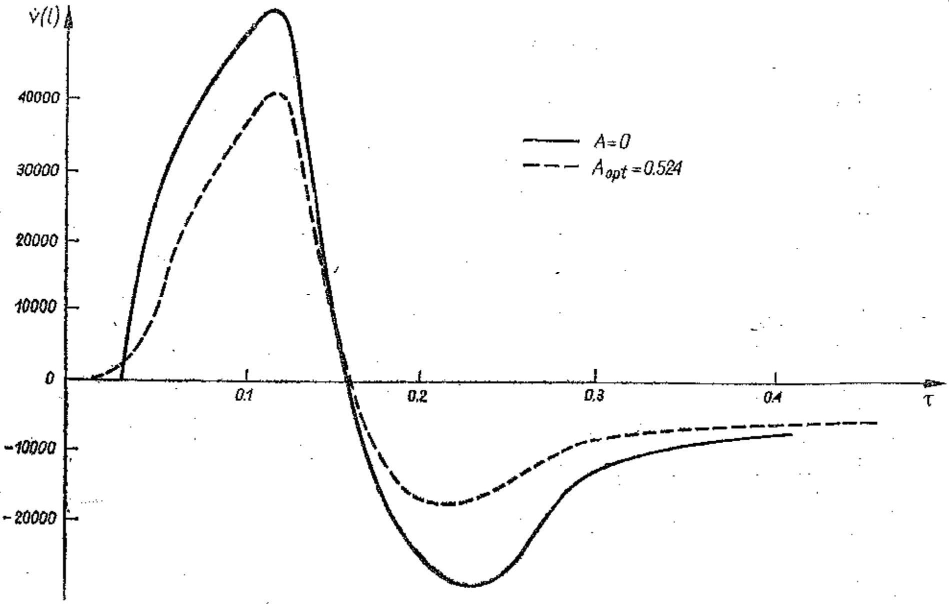

Fig. 9.

One can easily see that the time of viscoplastic flow of the optimal bar is about 20 per cent longer than the flow time of the prismatic bar. The profit z amounts to 12.3% in this case. It should be noted that the distribution of residual strains is quite different when compared with a prismatic bar.

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Streszczenie

PARAMETRYCZNA OPTYMALIZACJA LEPKOPLASTYCZNEGO PRĘTA PRZY DYNAMICZNYM OBCIĄŻENIU OSIOWYM

W pracy rozpatrywano problem optymalizacji parametrycznej sztywno-lepkoplastycznego pręta przy dynamicznym obciążeniu osiowym siłą zmienną w czasie (4.4). Problem sformułowano i rozwiązano w ujęciu dualnym; poszukiwano minimalnego przemieszczenia resztkowego końca pręta (skrócenia lub wydłużenia) przy danej stałej objętości pręta. Równanie ruchu (2.4) wyprowadzono dla potęgowego prawa fizycznego (2.1) i dowolnie zmiennego przekroju, jednakże dalsze rozwiązania podano dla prawa liniowego ($\delta=1$) oraz pręta w postaci stożka ściętego. Spośród dwóch parametrów opisujących zmianę przekroju takiego pręta (3.1) jeden można łatwo wyznaczyć z warunku stałej objętości (3.2). Jedyną zmienną kształtowania pozostaje wtedy parametr A (reprezentujący zbieżność stożka). Równanie ruchu całkowano numerycznie. Zbadano wpływ kształtu impulsu obciążenia (rys. 2) na wartość poszukiwanego parametru A_{opt} (rys. 3, 4, 5).

Резюме

ПАРАМЕТРИЧЕСКАЯ ОПТИМИЗАЦИЯ ВЯЗКОПЛАСТИЧЕСКОГО СТЕРЖНЯ ПРИ ДИНАМИЧЕСКОМ ОСЕВОМ НАГРУЖЕНИИ

В работе рассмотрена проблема параметрической оптимизации жестко-вязкопластического стержня при динамическом нагружении осевой силой переменной во времени (4.4). Проблема сформулирована и решена в дуальном подходе; ищется минимальное остаточное перемещение конца стержня (сокращение или удлинение) при заданном постоянном объеме стержня. Уравнение движения (2.4) выведено для степенного физического закона (2.1) и произвольно меняющегося сечения, однако дальнейшие решения приведены для линейного закона ($\delta=1$), а также для стержня в виде усеченного конуса. Среди двух параметров, описывающих изменение сечения такого стержня (3.1), один можно легко определить из условия постоянного объема (3.2). Единственной переменной формирования остается тогда параметр A (представляющий сходимость конуса). Уравнение движения интегрировано численно. Исследовано влияние формы импульса нагружения (рис. 2) на значение искомого параметра $A_{опт}$ (рис. 3, 4, 5).

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Received February 15, 1980.