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DEFORMATIONS OF ANISOTROPIC LAYERED MATERIALS

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The problem of determining the stress and displacement fields in an inhomogeneous material consisting of three bounded anisotropic layers is considered. Numerical values for the stress on the interface are obtained and the effect of a crack on the stress field is examined.

1. Introduction

In a previous paper Clements [1] considered the problem of determining the stress field round a crack in an infinite layered anisotropic material. The problem was reduced to three simultaneous Fredholm integral equations which were solved numerically in order to determine the crack energy for some particular materials.

In the present paper the problem of determining the displacement and stress fields in an anisotropic layered material of finite width is considered. The mateilal is made up of three layers with the central layer containing a crack of finite width (Fig. 1). Also the problem of a three-layered material without a crack undergoing

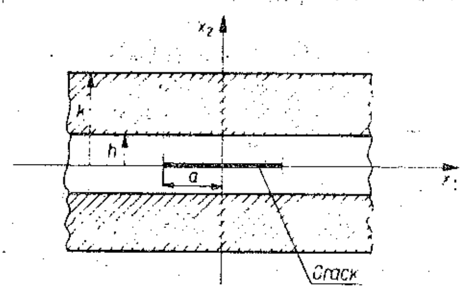

Fig. 1.

flexure due to applied loads is considered (Fig. 2). The flexure problem is reduced to solving linear simultaneous algebraic equations and evaluating definite integrals for the displacement and stress. The crack problem is an extension of this work and reduces to the solution of simultaneous Fredholm integral equations which are solved through a numerical iteration procedure. Numerical values for the stress at points in the layered material are given and the crack energy is determined in some particular cases.

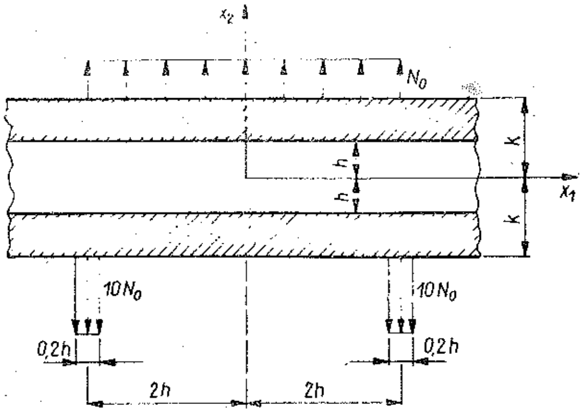

FIG. 2.

2. STATEMENT OF PROBLEMS AND BASIC EQUATIONS

Referring to the Cartesian coordinates x_1, x_2, x_3 consider the anisotropic elastic layered material depicted in Fig. 1. The layers $-k<x_2<-h$, $-h<x_2<h$ and $h<x_2<k$ are occupied by different materials. The stress and displacement fields are required for specified loadings and/or displacements on the boundaries $x_2=\pm k$. A second problem to be considered concerns finding the stress and displacement in the same layered material with a crack in the plane $x_2=0$ in the region $|x_1|<a$, $-\infty<x_3<\infty$.

It is assumed that the stress and displacement are independent of the Cartesian coordinate x_3 so that the basic equations for the displacements u_k and stresses σ_{ij} can be written (see CLEMENTS [2]) as

$$u_k = 2\mathscr{R} \sum_{\alpha} A_{k\alpha}\, \chi_\alpha(z_\alpha), \tag{2.1}$$

$$\sigma_{ij} = 2\mathscr{R} \sum_{\alpha} L_{ij\alpha}\, \chi'_\alpha(z_\alpha), \tag{2.2}$$

where $\mathscr{R}$ denotes the real part of a complex number, the $\chi_\alpha(z_\alpha)$ ($\alpha=1,2,3$) are analytic functions of the complex variable $z_\alpha = x_1 + \tau_\alpha x_2$ and the primes denote derivatives with respect to the argument in question. Also, in Eq. (2.1) the $A_{k\alpha}$ satisfy the equations

$$(c_{i1k1} + \tau_\alpha c_{i1k2} + \tau_\alpha c_{i2k1} + \tau'_2 c_{i2k2}) A_{k\alpha} = 0, \tag{2.3}$$

where the convention of summing over a repeated Latin suffix is used and the τ_α are the roots with a positive imaginary part of the sextic equation

$$|c_{i1k1} + \tau c_{i1k2} + \tau c_{i2k1} + \tau^2 c_{i2k2}| = 0, \tag{2.4}$$

where the C_{ijkl} are the elastic constants. Finally, the $L_{ij\alpha}$ occurring in Eq. (2.2) are related to the $A_{k\alpha}$ by the equation

$$L_{ij\alpha} = (c_{ij}\, c_l + \tau_\alpha c_{ijk2}) A_{k\alpha}. \tag{2.5}$$

In order to distinguish between the equations for the three layers, the superscripts L and R will be used to denote the regions $-k<x_2<-h$ and $h<x_2<k$ respectively. Thus the basic equations for $h<x_2<k$ are

$$u_k^L=2\mathscr{R}\sum_\alpha A_{k\alpha}^L\chi_\alpha^L(z_\alpha^L),\qquad \sigma_{ij}^\alpha=2\mathscr{R}\sum_\alpha L_{ij\alpha}^L\chi_\alpha^L(z_\alpha^L)$$

with the same expressions with the superscript L replaced by R for the region $-k<x_2<-h$.

3. Solution of the first problem

It is convenient to choose the following representations for the stress and displacement:

In the region $h<x_2<k$ let

$$\chi_\alpha^L(z_\alpha)=\frac{1}{2\pi}\int_0^\infty\{E_\alpha^L(p)\exp(ipz_\alpha)+F_\alpha^L(p)\exp(-ipz_\alpha)\}\,dp. \tag{3.1}$$

Hence Eqs. (2.1) and (2.2) yield

$$u_k^L=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha A_{k\alpha}^L\{E_\alpha^L(p)\exp(ipz_\alpha)+F_\alpha^L(p)\exp(-ipz_\alpha)\}\,dp, \tag{3.2}$$

$$\sigma_{ij}^L=\frac{1}{\pi}\mathscr{R}\int_0^\infty\sum_\alpha L_{ij\alpha}^L\{E_\alpha^L(p)\exp(ipz_\alpha)-F_\alpha^L(p)\exp(-ipz_\alpha)\}\,ip\,dp, \tag{3.3}$$

where the functions $E_\alpha^L(p)$ and $F_\alpha^L(p)$ will be determined from boundary conditions. The expressions for the displacement u_k and stress σ_{ij} in the central layer are just Eqs. (3.2) and (3.3) with the superscipt L removed. Similarly, in $-k<x_2<-h$ the expressions are Eqs. (3.2) and (3.3) with the superscipt L replaced by R. The stresses σ_{i2} and displacements u_k must be continuous across the interfaces $x_2=\pm h$. This condition will be satisfied if the E_α and F_α are constrained by the equations

$$\sum_\alpha[L_{i2\alpha}E_\alpha(p)\exp(ip\tau_\alpha h)+\bar{L}_{i2\alpha}\bar{F}_\alpha(p)\exp(ip\bar{\tau}_\alpha h)]=$$
$$=\sum_\alpha[L_{i2\alpha}^LE_\alpha^L(p)\exp(ip\tau_\alpha^Lh)+\bar{L}_{i2\alpha}^L\bar{F}_\alpha^L\exp(ip\bar{\tau}_\alpha^Lh)], \tag{3.4}$$

$$\sum_\alpha[L_{i2\alpha}E_\alpha(p)\exp(-ip\tau_\alpha h)+\bar{L}_{i2\alpha}\bar{F}_\alpha(p)\exp(-ip\bar{\tau}_\alpha h)]=$$
$$=\sum_\alpha[L_{i2\alpha}^RE_\alpha^R(p)\exp(-ip\tau_\alpha^Rh)+\bar{L}_{i2\alpha}^R\bar{F}_\alpha\exp(-ip\bar{\tau}_\alpha^Rh)], \tag{3.5}$$

$$\sum_\alpha[A_{k\alpha}E_\alpha(p)\exp(ip\tau_\alpha h)+\bar{A}_{k\alpha}\bar{F}_\alpha(p)\exp(ip\bar{\tau}_\alpha h)]=$$
$$=\sum_\alpha[A_{k\alpha}^LE_\alpha^L(p)\exp(ip\tau_\alpha^Lh)+\bar{A}_{k\alpha}^L\bar{F}_\alpha^L(p)\exp(ip\bar{\tau}_\alpha^Lh)], \tag{3.6}$$

(3.7) $$\sum_{\alpha} [A_{k\alpha} E_\alpha (p) \exp(-ip\tau_\alpha h) + \bar{A}_{k\alpha} \bar{F}_\alpha (p) \exp(-ip\bar{\tau}_\alpha h)] = \sum_{\alpha} [A^R_{k\alpha} E^R_\alpha (p) \exp(-ip\tau^R_\alpha h) + \bar{A}^R_{k\alpha} \bar{F}^R_\alpha (p) \exp(-ip\bar{\tau}^R_\alpha h)].$$

These equations may be written in the form

(3.8) $$NE + \bar{R}\bar{F} = N^L E^L + \bar{R}^L \bar{F}^L,$$

(3.9) $$RE + \bar{N}\bar{F} = R^R E^R + \bar{N}^R \bar{F}^R,$$

(3.10) $$UE + \bar{V}\bar{F} = U^L E^L + \bar{V}^L \bar{F}^L,$$

(3.11) $$VE + \bar{U}\bar{F} = V^R E^R + \bar{U}^R \bar{F}^R,$$

where

$$N = [L_{i2\alpha} \exp(ip\tau_\alpha h)] \qquad R = [L_{i2\alpha} \exp(-ip\tau_\alpha h)],$$

$$U = [A_{k\alpha} \exp(ip\tau_\alpha h)], \qquad V = [A_{k\alpha} \exp(-ip\tau_\alpha h)],$$

$$E = [E_j], \qquad F = [F_j]$$

with similar definitions for the superscripted matrices. Also let

(3.12) $$\mathcal{N}^L E^L + \bar{\mathcal{R}}^L \bar{F}^L = \mathcal{P},$$

(3.13) $$\mathcal{R}^R E^R + \bar{\mathcal{N}}^R \bar{F}^R = \mathcal{Q},$$

(3.14) $$\mathcal{U}^L E^L + \bar{\mathcal{V}}^L \bar{F}^L = \mathcal{M},$$

(3.15) $$\mathcal{V}^R E^R + \mathcal{U}^R \bar{F}^R = \mathcal{H},$$

where

$$\mathcal{N} = [L_{i2\alpha} \exp(ip\tau_\alpha k)], \qquad \mathcal{R} = [L_{i2\alpha} \exp(-ip\tau_\alpha k)],$$

$$\mathcal{U} = [A_{k\alpha} \exp(ip\tau_\alpha k)], \qquad \mathcal{V} = [A_{k\alpha} \exp(-ip\tau_\alpha k)]$$

with appropriate superscripts inserted. The $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{M}$ and $\mathcal{H}$ are column matrices whose elements will be determined by the boundary conditions on $x_2 = \pm k$. In general, only two of these four column matrices will be known corresponding to either the stress or displacement being specified on the boundaries $x_2 = \pm k$. Equations (3.8)–(3.15) thus provide six equations for E, F, E^L, F^L, E^R and F^R. Once these equations have been solved Eqs. (3.2) and (3.3) (and the similar equations for the other two regions) provide the stress and displacement throughout the material.

As a specific example consider the case when the layered material is loaded as shown in Fig. 2. In this case the matrices $\mathcal{P}$ and $\mathcal{Q}$ are known while $\mathcal{M}$ and $\mathcal{H}$ are not known. Use of the inversion formula for Fourier transforms shows that $\mathcal{P}$ and $\mathcal{Q}$ adopt the forms

$$\mathcal{P} = \begin{bmatrix} 0 \\ 2iN_0 \sin(2ph)/p^2 \\ 0 \end{bmatrix},$$

$$\mathcal{Q}=\begin{bmatrix} 0 \\ 2iN_0\,[\sin(2.1ph)-\sin(1.9ph)]/p^2 \\ 0 \end{bmatrix}.$$

Thus, in this case, Eqs. (3.8)–(3.11) together with Eqs. (3.12) and (3.13) provide six equations from which E, F, E^L, F^L, E^R and F^R may be determined.

4. Solution of the crack problem

In this case it is convenient to use the representations (3.2) and (3.3) for the displacement and stress in the layer $h<x_2<k$ with similar expressions with L replaced by R for these quantities in the layer $-k<x_2<-h$. For the layer $-h<x_2<h$ we consider the regions $-h<x_2<0$ and $0<x_2<h$ separately. Guided by the analysis in Clements [1] we obtain the following expressions for the displacement and stress in these two regions.

In $0<x_2<h$:

$$(4.1)\qquad u_k=\frac{1}{\pi}\mathcal{R}\int_0^\infty \sum_\alpha A_{k\alpha}\,\{[E_\alpha(p)+M_{\alpha i}\,\psi_i(p)]\exp(ipz_\alpha)+ +F_\alpha(p)\exp(-ipz_\alpha)\}\,dp,$$

$$(4.2)\qquad \sigma_{ij}=\frac{1}{\pi}\mathcal{R}\int_0^\infty \sum_\alpha L_{ij\alpha}\,\{[E_\alpha(p)+M_{\alpha i}\,\psi_i(p)]\exp(ipz_\alpha)- -F_\alpha(p)\exp(-ipz_\alpha)\}\,ip\,dp.$$

In $-h<x_2<0$:

$$(4.3)\qquad u_k=\frac{1}{\pi}\mathcal{R}\int_0^\infty \sum_\alpha A_{k\alpha}\,\{E_\alpha(p)\exp(ipz_\alpha)+[M_{\alpha i}\,\bar{\psi}_i(p)+ +F_\alpha(p)]\exp(-ipz_\alpha)\}\,dp,$$

$$(4.4)\qquad \sigma_{ij}=\frac{1}{\pi}\mathcal{R}\int_0^\infty \sum_\alpha L_{ij\alpha}\,\{E_\alpha(p)\exp(ipz_\alpha)-[M_{\alpha i}\,\bar{\psi}_i(p)+ +F_\alpha(p)]\exp(-ipz_\alpha)\}\,ip\,dp,$$

where the $M_{\alpha j}$ are defined by

$$(4.5)\qquad \sum_\alpha L_{i2\alpha}\,M_{\alpha j}=\delta_{ij}.$$

In these equations the unknown functions $E_\alpha(p)$, $F_\alpha(p)$ and $\psi_i(p)$ will be determined from the boundary conditions on $x_2=\pm h$.

From Eqs. (4.2) and (4.4) it follows that σ_{i2} is continuous across $x_2=0$. The difference in displacement across $x_2=0$ is, from Eqs. (4.1) and (4.3),

$$(4.6)\qquad \Delta u_k=\frac{1}{\pi}\mathcal{R}\,(B_{kj}-\bar{B}_{kj})\int_0^\infty \psi_j(p)\exp(ipx_1)\,dp,$$

where

$$B_{kj}=\sum_{\alpha} A_{k\alpha} M_{\alpha j}.$$

Now Δu_k must be zero outside the crack and this condition together with the stress boundary conditions on the crack face $\sigma_{i2}(x_1, 0)=-P_i(x_1)$ for $|x_1|<a$ yield

$$\mathscr{R}(B_{kj}-\bar{B}_{kj})\int_0^\infty \psi_j(p)\exp(ipx_1)\,dp=0 \quad \text{for} \quad |x_2|>a, \tag{4.7}$$

$$\frac{1}{\pi}\mathscr{R}\int_0^\infty \Big[\psi_j(p)+\sum_{\alpha}\{L_{j2\alpha}E_\alpha(p)+\bar{L}_{j2\alpha}\bar{F}_\alpha(p)\}\Big]\, ip\exp(ipx_1)\,dp= =-P_j(x_1) \quad \text{for} \quad |x_1|<a. \tag{4.8}$$

The displacement u_k and stresses σ_{i2} must be continuous across $x_2=\pm h$. This requirement will be satisfied if

$$\sum_{\alpha}[L_{i2\alpha}\{E_\alpha(p)+M_{\alpha j}\psi_j(p)\}\exp(ip\tau_\alpha h)+\bar{L}_{i2\alpha}\bar{F}_\alpha(p)\exp(ip\bar{\tau}_\alpha h)]= =\sum_{\alpha}[L^L_{i2\alpha}E^L_\alpha(p)\exp(ip\tau^L_\alpha h)+\bar{L}^L_{i2\alpha}\bar{F}^L_\alpha(p)\exp(ip\bar{\tau}^L_\alpha h)], \tag{4.9}$$

$$\sum_{\alpha}[L_{i2\alpha}E_\alpha(p)\exp(-ip\tau_\alpha h)+\bar{L}_{i2\alpha}\{\bar{F}_\alpha(p)+\bar{M}_{\alpha j}\psi_j(p)\}\exp(-ip\bar{\tau}_\alpha h)]= =\sum_{\alpha}[L^R_{i2\alpha}E^R_\alpha(p)\exp(-ip\tau^R_\alpha h)+\bar{L}^R_{i2\alpha}\bar{F}^R_\alpha\exp(-ip\bar{\tau}^R_\alpha h)], \tag{4.10}$$

$$\sum_{\alpha}[A_{k\alpha}\{E_\alpha(p)+M_{\alpha j}\psi_j(p)\}\exp(ip\tau_\alpha h)+\bar{A}_{k\alpha}\bar{F}_\alpha(p)\exp(ip\bar{\tau}_\alpha h)]= =\sum_{\alpha}[A^L_{k\alpha}E^L_\alpha(p)\exp(ip\tau^L_\alpha h)+\bar{A}^L_{k\alpha}\bar{F}^L_\alpha(p)\exp(ip\bar{\tau}^L_\alpha h)], \tag{4.11}$$

$$\sum_{\alpha}[A_{k\alpha}E_\alpha(p)\exp(-ip\tau_\alpha h)+\bar{A}_{k\alpha}\{\bar{F}_\alpha(p)+\bar{M}_{\alpha j}\psi_j(p)\}\exp(-ip\bar{\tau}_\alpha h)]= =\sum_{\alpha}[A^R_{k\alpha}E^R_\alpha(p)\exp(-ip\tau^R_\alpha h)+\bar{A}^R_{k\alpha}\bar{F}^R_\alpha\exp(-ip\bar{\tau}^R_\alpha h)]. \tag{4.12}$$

Let $\Psi=[\psi_i]$ so that, using the notation of the previous section, these equations may be written in the matrix form:

$$NE+\bar{R}\bar{F}+NM\psi=N^L E^L+\bar{R}^L\bar{F}^L, \tag{4.13}$$

$$RE+\bar{N}\bar{F}+\bar{N}\bar{M}\psi=R^R E^R+\bar{N}^R\bar{F}^R, \tag{4.14}$$

$$UE+\bar{V}\bar{F}+UM\psi=U^L L^L+\bar{V}^L\bar{F}^L, \tag{4.15}$$

$$VE+\bar{U}\bar{F}+\bar{U}\bar{M}\psi=V^R E^R+\bar{U}^R\bar{F}^R. \tag{4.16}$$

In addition, Eqs. (3.12)–(3.15) provide constraints on the matrices E^L, F^L, E^R and F^R through the conditions on the boundaries $x_2=\pm k$. Equations (4.13)–(4.16) together with two equations of the set (3.12)–(3.15) (depending on the precise form of the boundary conditions on $x_2 = \pm k$) provide six equations which may be used to express E, F, E^L, F^L, E^R and F^R in terms of ψ. Specifically, it is possible to obtain two equations of the form

(4.17) $$E = Q^{(1)}\psi + X,$$

(4.18) $$F = Q^{(2)}\overline{\Psi} + Y,$$

where the exact form for $Q^{(1)}$, $Q^{(2)}$, X and Y depends on which of the equations of the set (3.12)–(3.15) are applicable.

Now Eq. (4.7) will be satisfied if $\psi_j(p)$ is taken in the form

(4.19) $$\psi_j(p) = \int_0^a s_j(t) J_1(pt)\, dt + i \int_0^a r_j(t) J_0(pt)\, dt,$$

where $r_j(t)$ and $s_j(t)$ for $j=1, 2, 3$ are real functions to be determined and J_0 and J_1 are Bessel functions of orders zero and one respectively.

Substitution of Eqs. (4.16) and (4.18) into Eq. (4.8) yields

(4.20) $$\mathscr{R}\int_0^\infty [\psi_j(p) + T_{jk}(p)\psi_k(p)]\, ip \exp(ipx_1)\, dp = \mathscr{P}_j(x_1) \quad \text{for} \quad |x_1| < a,$$

where

(4.21) $$\mathscr{P}_j(x_1) = -\pi P_j(x_1) - \mathscr{R}\int_0^\infty \sum_\alpha [L_{j2\alpha} X_\alpha(p) + \bar{L}_{j2\alpha} \bar{Y}_\alpha(p)]\, ip \exp(ipx_1)\, dp$$

and

(4.22) $$T_{jk}(p) = \sum_\alpha L_{j2\alpha} Q^{(1)}_{\alpha k}(p) + \sum_\alpha \bar{L}_{j2\alpha} \bar{Q}^{(2)}_{\alpha k}(p).$$

Use of Eq. (4.19) in Eq. (4.20) yields

(4.23) $$\int_0^\infty p\cos(px_1)\, dp \int_0^a r_j(t) J_0(pt)\, dt + \int_0^\infty T^{(1)}_{jk}(p)\, p\cos(px_1)\, dp \times$$
$$\times \int_0^\infty r_k(t) J_0(pt)\, dt + \int_0^\infty T^{(2)}_{jk}(p)\, p\cos(px_1)\, dp \int_0^a s_k(t) J_1(pt)\, dt =$$
$$= -\frac{1}{2}[\mathscr{P}_j(x_1) + \mathscr{P}_j(-x_1)],$$

(4.24) $$\int_0^\infty p\sin(px_1)\, dp \int_0^a s_j(t) J_1(pt)\, dt + \int_0^\infty T^{(1)}_{jk}(p)\, p\sin(px_1)\, dp \times$$
$$\times \int_0^a s_k(t) J_1(pt)\, dt - \int_0^\infty T^{(2)}_{jk}(p)\, p\sin(px_1)\, dp \int_0^a r_k(t) J_0(pt)\, dt =$$
$$= -\frac{1}{2}[\mathscr{P}_j(x_1) - \mathscr{P}_j(-x_1)],$$

where $T_{jk} = T^{(1)}_{jk} + iT^{(2)}_{jk}$ with $T^{(1)}_{jk}$ and $T^{(2)}_{jk}$ real.

These Abel integral equations may be inverted to yield

$$r_j(t)+\frac{2t}{\pi}\int_a^t \frac{du}{(t^2-u^2)^{\frac{1}{2}}}\int_0^\infty T_{jk}^{(1)}(p)\,p\cos(pu)\,dp\int_0^a r_k(q)\,J_0(pq)\,dq+ \tag{4.25}$$

$$+\frac{2t}{\pi}\int_0^t \frac{du}{(t^2-u^2)^{\frac{1}{2}}}\int_0^\infty T_{jk}^{(2)}(p)\,p\cos(pu)\,dp\int_0^a s_k(q)\,J_1(pq)\,dq=$$

$$=-\frac{t}{\pi}\int_{-t}^t \frac{P_j(u)\,du}{(t^2-u^2)^{\frac{1}{2}}}\quad \text{for}\quad 0<t<a;$$

$$s_j(t)+\frac{2}{\pi}\int_0^t \frac{udu}{(t^2-u^2)^{\frac{1}{2}}}\int_0^\infty T_{jk}^{(1)}(p)\,p\sin(px_1)\,dp\int_0^a s_k(q)\,J_1(pq)\,dq- \tag{4.26}$$

$$-\frac{2}{\pi}\int_0^t \frac{udu}{(t^2-u^2)^{\frac{1}{2}}}\int_0^\infty T_{jk}^{(2)}(p)\,p\sin(px_1)\,dp\int_0^a r_k(q)\,J_0(pq)\,dq=$$

$$=-\frac{1}{\pi}\int_{-t}^t \frac{uP_j(u)\,du}{(t^2-u^2)^{\frac{1}{2}}}\quad \text{for}\quad 0<t<a.$$

Interchanging the order of integration and using the results

$$\frac{2}{\pi}\int_0^t \frac{\cos(pu)\,du}{(t^2-u^2)}=J_0(pt), \tag{4.27}$$

$$\frac{2}{\pi}\int_0^t \frac{u\sin(pu)\,du}{(t^2-u^2)^{\frac{1}{2}}}=tJ_1(pt), \tag{4.28}$$

it follows that

$$r_j(t)+t\int_0^a K_{jk}^{(0,0)}(u,t)\,r_k(u)\,du+t\int_0^a K_{jk}^{(0,1)}(u,t)\,s_k(u)\,du= \tag{4.29}$$

$$=-\frac{t}{\pi}\int_{-t}^t \frac{P_j(u)\,du}{(t^2-u^2)^{\frac{1}{2}}}\quad \text{for}\quad 0<t<a.$$

$$s_j(t)+t\int_0^a K_{jk}^{(1,1)}(u,t)\,s_k(u)\,du-t\int_0^a K_{jk}^{(1,0)}(u,t)\,r_k(u)\,du= \tag{4.30}$$

$$=-\frac{1}{\pi}\int_{-t}^t \frac{uP_j(u)\,du}{(t^2-u^2)^{\frac{1}{2}}}\quad \text{for}\quad 0<t<a,$$

where

$$K_{jk}^{(M,N)}(u,t)=\int_0^\infty T_{jk}^{(N+1)}J_N(pu)\,dp. \tag{4.31}$$

Equations (4.29) and (4.30) constitute six simultaneous integral equations for the $r_j(t)$ and $s_j(t)$, $j=1, 2, 3$. These equations may be solved numerically by iteration.

Once this has been done Eqs. (4.17)–(4.19) yield $\psi_j(p)$, $E_\alpha(p)$ and $F_\alpha(p)$. Equations (4.1)–(4.4) and equations of the type (3.2) and (3.3) may then be used to calculate the stress and displacement throughout the material.

5. Numerical results

For the purpose of calculating specific numerical values it will be convenient to consider the class of anisotropic materials which are transversely isotropic. Such materials may be characterized by five elastic constants which will be denoted by A, N, F, C and L. The relationship between these constants and the c_{ijkl} is given in detail in Clements [1].

Consider a composite made up of material I, say, with constants $A=5.96$, $N=2.57$, $F=2.14$, $C=6.14$ and $L=1.64$ in the middle layer and material II, say,

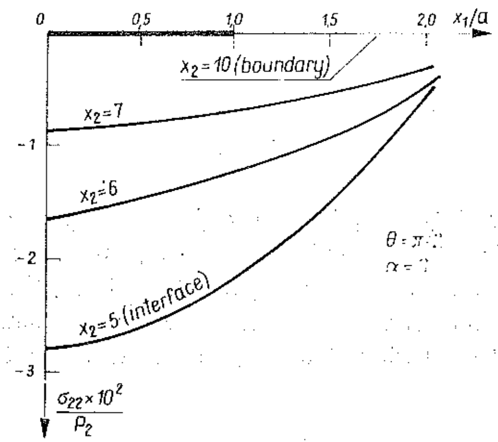

Fig. 3.

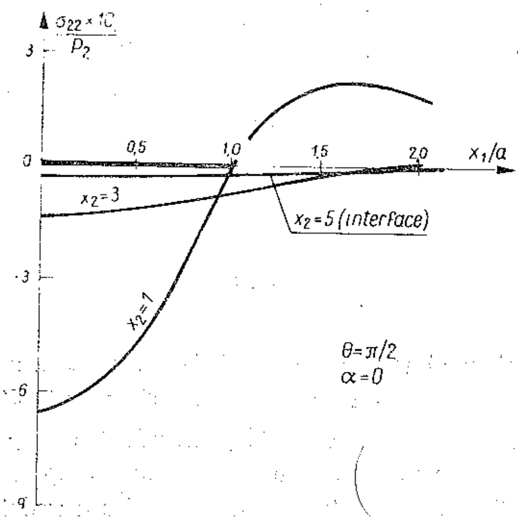

Fig. 4.

with constants $A=16.2$, $N=9.2$, $F=6.9$, $C=18.1$ and $L=4.67$ for the upper and lower layers. If each of these constants is multiplied by 10^{11}, then the units for the constants are dynes/cm^2.

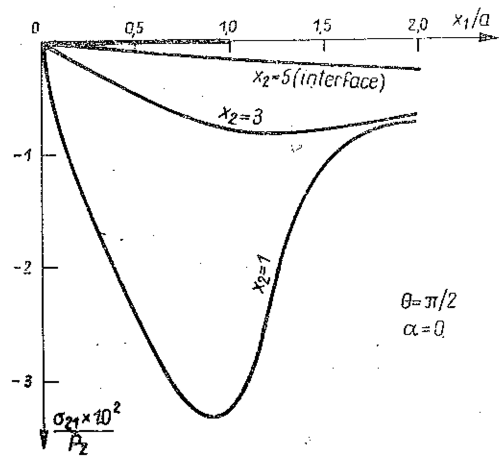

FIG. 5.

Figures 3–6 show plots of σ_{22}/P_2 and σ_{21}/P_2 for the case of a layered material with a crack with the applied tractions P_1 and P_3 over the crack face both zero. They are plotted at various values of x_2/h; $x_2/h=5$ being the interface value and $x_2/h=10$ being the boundary value. All the results given in these figures were calculated for $\theta=\pi/2$ and $\alpha=0$ where these angles are defined in CLEMENTS [1].

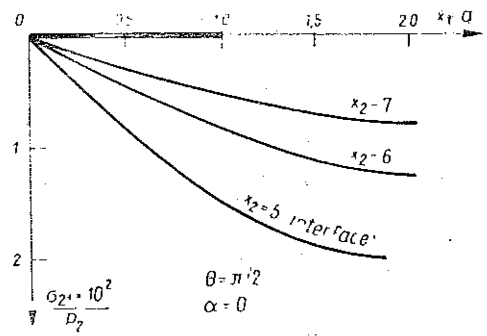

FIG. 6.

Figures 7 and 8 show plots of σ_{22}/P_2 for different values of α and θ in the middle slab (the values $\alpha=0$ and $\theta=\pi/2$ being maintained for the outer slab).

The effect on the crack energy of making the middle stronger was also considered (see CLEMENTS [1[for an expression for the crack energy). The constant C was increased while all the other constants and the angles α and θ were held at their original values. The results are given in Table 1.

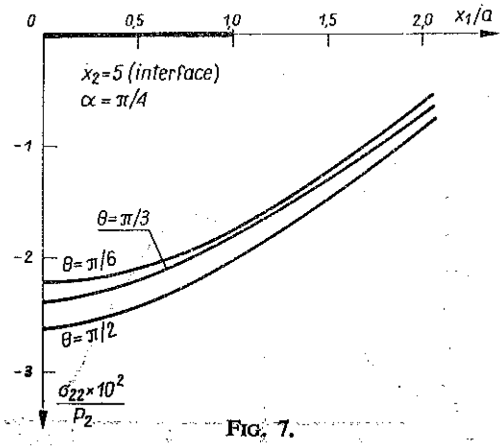

FIG. 7.

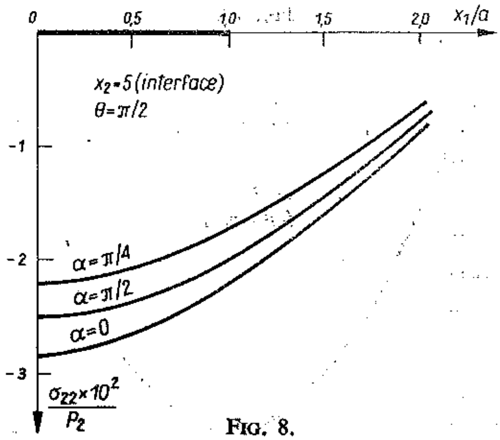

FIG. 8.

Table 1

C	10	20	30	40	50	60	70	80	90	100
Crack Energy	0.332	0.206	0.161	0.136	0.119	0.107	0.098	0.091	0.086	0.81

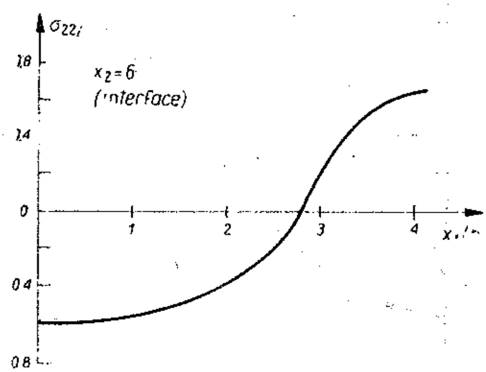

FIG. 9.

Finally, Figs. 9 and 10 contain plots of σ_{12}/N_0 and σ_{22}/N_0 along the interfaces of the uncreacked layered slab loaded as in Fig. 2, and Figs. 11 and 12 shows the plots of the displacements u_1/h and u_2/h respectively.

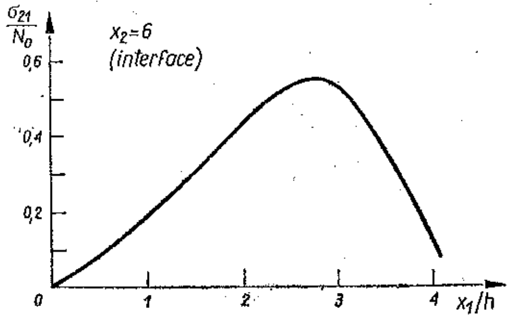

FIG. 10.

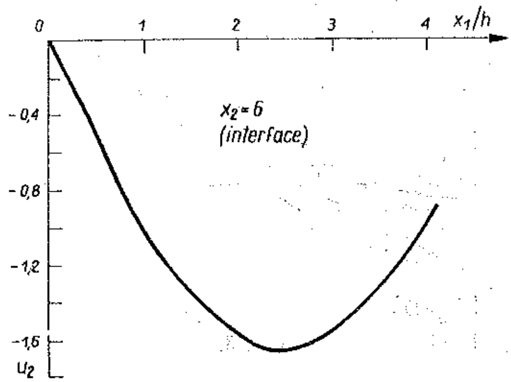

FIG. 11.

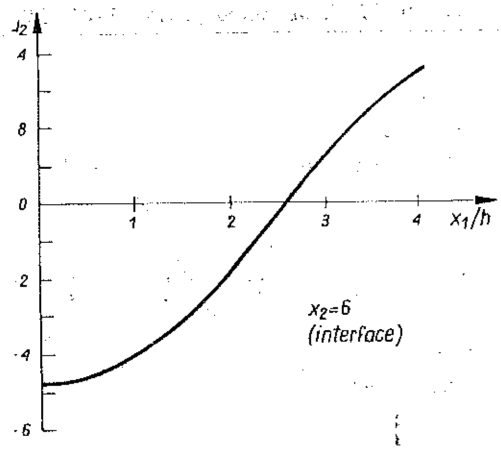

FIG. 12.

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Streszczenie

ODKSZTAŁCENIE ANIZOTROPOWYCH MATERIAŁÓW UWARSTWIONYCH

Rozważono problem określenia pól naprężeń i przemieszczeń w ciele niejednorodnym złożonym z trzech połączonych warstw anizotropowych. Otrzymane wartości liczbowe naprężeń na powierzchniach rozdziału i określono wpływ szczeliny na pole naprężenia.

Резюме

ДЕФОРМАЦИЯ АНИЗОТРОПНЫХ СЛОИСТЫХ МАТЕРИАЛОВ

Рассмотрена проблема определения полей напряжений и перемещений в неоднородноя теле, состоящем из трех соединенных анизотропных слоев. Получены числовые значения напряжений на поверхностях раздела и определено влияние щели на поле напряжений.

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