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RHEOLOGY OF PRESTRESSED CIRCULARLY SYMMETRIC DISCS

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The paper presents the solution of the problem of creep of an annular disc prestressed by a system of concentric cables. Assuming the distribution of cables to be uniform, the magnitudes of individual prestressing forces are determined. The problem of interaction of the cables is solved by the method of iterations.

1. Introduction

Theoretical analysis of prestressed concrete discs involves the necessity of taking into account the rheological effects. In the case of discs working in normal temperatures, these effects are connected with the appearance of creep and shrinkage of concrete [1]. Especially strains in concrete caused by creep both in a prestressing stage and in exploitation time, can make a considerable loss of prestress [2]. In the case of discs, a multi-parameter course of phenomenon accounts for the fact that even total elimination of stretching stresses in an initial stage of prestressing does not exclude the possibility of zones appearing where the stretching stresses will exceed admissible magnitudes during the later phase of work of construction.

A solution of a problem of creep for a ring disc with prestressing cables lying along concentric circles will be shown below. Magnitudes characterizing disc geometry, function of external load, a number and way of arrangement of cables, physical properties of concrete and prestressing steel and initial stresses in the cables are free parameters of the problem.

The problem of an optimal choice of prestressing [3] for fixed disc geometry and function of load remains unsolved. Assuming a uniform arrangement of the cables, prestressing forces in numerical examples have been accepted according to the criterion of restricted prestress; therefore, stretching stresses in concrete should not be greater than admissible magnitudes during exploitation of the disc [4]. A problem of interaction of the cables has been solved by an iterative method.

2. Formulation of the problem

The problem of creep of a disc will be solved on the basis of a physical equation of the Boltzman, infinitesimal theory [5, 6]

$$\mathbf{T}(t)=\Big\{E_0(t)\,\mathrm{tr}\,\mathbf{E}(t)-\int_{t_0}^{t}\mathrm{tr}\,\mathbf{E}(t')\,E_0(t')\,R_0(t,t')\,dt'\Big\}\mathbf{1}+ \\ +2G(t)\,\mathbf{E}'(t)-2\int_{t_0}^{t}\mathbf{E}'(t')\,G(t')\,R_c(t,t')\,dt', \tag{2.1}$$

where $\mathbf{T}(t)$ is Cauchy's stress tensor, $\mathbf{E}(t)$ is the strain tensor, $\mathbf{E}'(t)$ is the strain deviator and $E_0(t)$, $G(t)$, $R_0(t, t')$, $R_c(t, t')$ are scalar material functions of time t and t'.

In the initially-boundary problem formulated further the tensors $\mathbf{T}(t)$ and $\mathbf{E}(t)$ will be written in a polar coordinate system $\{r, \theta\}$. With respect to assumed axial symmetry the radial r and circumferential θ directions are simultaneously the main directions and all the inquired magnitudes will be functions of the radius r and time t. Introducing the dimensionless magnitudes.

$$s_\rho = \frac{\sigma_r}{s_0}, \quad s_\theta = \frac{\sigma_\theta}{s_0}, \quad \rho = \frac{r}{R_1}, \quad \tau = \frac{t}{t_0}, \quad \tau' = \frac{t'}{t_0}, \tag{2.2}$$

the governing set of equations in the Euler description contains:

the equilibrium equation

$$s_{\rho,\rho} + \frac{1}{\rho}(s_\rho - s_\theta) = 0,$$

and the constitutive law (2.1) which will be written in the reversed form

$$\varepsilon_\rho = \frac{1}{6} L_c(2s_\rho - s_\theta) + \frac{1}{9} L_0(s_\rho + s_\theta), \tag{2.4}$$

$$\varepsilon_\theta = \frac{1}{6} L_c(2s_\theta - s_\rho) + \frac{1}{9} L_0(s_\rho + s_\theta), \tag{2.5}$$

where for concise notation the following integral operators

$$L_c(s) = \frac{1}{\bar{G}(\tau)}\left[s(\tau) + \int_1^\tau s(\tau')\bar{K}_c(\tau, \tau')\,d\tau'\right], \tag{2.6}$$

$$L_0(s) = \frac{1}{\bar{E}_0(\tau)}\left[s(\tau) + \int_1^\tau s(\tau')\bar{K}_0(\tau, \tau')\,d\tau'\right] \tag{2.7}$$

have been defined. The dimensionless material functions $\bar{G}(\tau)$, $\bar{E}_0(\tau)$ and kernels of nondilatational $\bar{K}_c(\tau, \tau')$ and dilatational $\bar{K}_0(\tau, \tau')$ strains are given by,

$$\bar{G}(\tau) = \frac{\bar{E}(\tau)}{2[1-\mu(\tau)]}, \quad \bar{E}_0(\tau) = \frac{\bar{E}(\tau)}{3-6\mu(\tau)}, \tag{2.8}$$

$$\bar{K}_c(\tau, \tau') = \frac{1+\nu(\tau, \tau')}{1+\mu(\tau)}\bar{K}(\tau, \tau'), \quad \bar{K}_0(\tau, \tau') = \frac{1-2\nu(\tau, \tau')}{1-2\mu(\tau)}\bar{K}(\tau, \tau'), \tag{2.9}$$

where $\bar{E}(\tau) = \frac{1}{s_0}E(\tau)$ — dimensionless Young's modulus, $\mu(\tau)$ and $\nu(\tau, \tau')$ — Poisson's ratios, and $\bar{K}(\tau, \tau') = t_0 K(\tau, \tau')$ is the dimensionless general kernel of the constitutive equation (2.1). The analytical form of this kernel will be referred to commonly applied theories of concrete creep in a section treating of displacement function. The function σ_r, σ_θ and ε_r, ε_θ denote the components: radial and circumferential of stress $\mathbf{T}(t)$ and strain $\mathbf{E}(t)$ tensors. The constants s_0 and R_1 have been introduced for dimensional purposes.

The solution of the initially-boundary problem reduces to the solution of a system of three differentially-integral equations (2.3)–(2.5) with respect to the components of stress s_ρ, s_θ and to the dimensionless radial displacement $\bar{u}(\rho, \tau)$ by which the components of the strain tensor are expressed:

(2.10) $$\varepsilon_\rho = \bar{u}_{,\rho}(\rho, \tau), \quad \varepsilon_\theta = \frac{1}{\rho}\bar{u}(\rho, \tau),$$

where

(2.11) $$\bar{u}(\rho, \tau) = \frac{u(\rho, \tau)}{R_1}.$$

3. State of stress

In the considered problem of statical equilibrium the components of the strain tensor ε_ρ and ε_θ satisfy the compatibility condition

(3.1) $$\varepsilon_{\theta,\rho} + \frac{1}{\rho}(\varepsilon_\theta - \varepsilon_\rho) = 0$$

which, after substitution of Eqs. (2.3)–(2.5), takes the form

(3.2) $$L_c(\rho s_{\rho,\rho\rho} + 3s_{\rho,\rho}) + \frac{1}{3}L_0(\rho s_{\rho,\rho\rho} + 3s_{\rho,\rho}) = 0.$$

According to the theorems for linear integral operators, Eq. (3.2) can be written in the following form:

(3.3) $$\rho s_{\rho,\rho\rho} + 3s_{\rho,\rho} = 0.$$

Despite the lack of assumption that the material is quasi-elastic, it follows from Eq. (3.3) that an elastic-viscous-elastic analogy holds in the plane axially-symmetrical problem [7]. Up to the present moment one has not had a displacement function, and only a formal presentation of the boundary problem for the radial (3.3) and circumferential (2.3) stresses is possible.

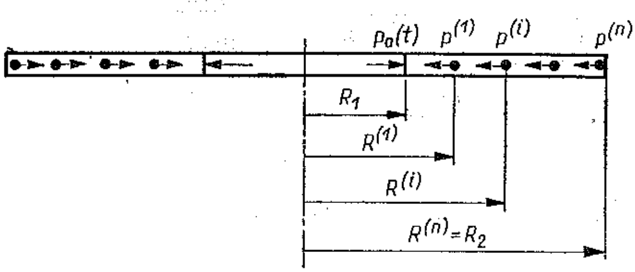

FIG. 1.

An analysis of the process of creep will be connected with a concrete ring disc subjected to the action of multiple circumferential prestressing. A diagrammatic action of prestressing cables is shown in Fig. 1. A uniform distribution of a load $p_0(t)$ along an edge of a hole (radius R_1) will be treated as a pay load. A function $p^{(i)}$ denotes a pressure deriving from the i-th prestressing cable ($i=1, 2, \ldots, n$) whose route coincides with a circle of radius $R^{(i)}$. In the particular case for $R^{(i)}=R_2$ it will be considered as prestressing the external edge of the disc.

The pressure $p^{(i)}(t)$ is a compound function of time which depends on the displacement $u^{(i)}(t)$ of disc particles in a place of localization of the prestressing cable

$$p^{(i)}(t)=p^{(i)}[u^{(i)}(t)], \tag{3.4}$$

where

$$u^{(i)}(t)=u(R^{(i)}, t). \tag{3.5}$$

The assumption that the cables work at the linear elastic region leads to the dependence

$$p^{(i)}(t)=a_1^{(i)}+a_2^{(i)}u^{(i)}(t). \tag{3.6}$$

The coefficients $a_1^{(i)}$, $a_2^{(i)}$ will be calculated according to localization, cross-section and Young's modulus of the cable and to the initial prestress.

Equation (3.6) in dimensionless variables can be written as

$$\bar{p}^{(i)}(\tau)=\bar{a}_1^{(i)}+\bar{a}_2^{(i)}\bar{u}^{(i)}(\tau). \tag{3.7}$$

Dimensionless magnitudes are here expressed by

$$\begin{aligned} &\bar{p}^{(i)}(\tau)=\frac{1}{s_0}p^{(i)}(\tau), \qquad \bar{a}_1^{(i)}=\frac{1}{s_0}a_1^{(i)},\\ &\bar{a}_2^{(i)}=\frac{1}{s_0}R_1 a_2^{(i)}, \qquad \bar{u}^{(i)}(\tau)=\frac{1}{R_1}u^{(i)}(\tau), \end{aligned} \tag{3.8}$$

where

$$\bar{u}^{(i)}(\tau)=\bar{u}(\rho^{(i)}, \tau), \qquad \rho^{(i)}=\frac{R^{(i)}}{R_1}. \tag{3.9}$$

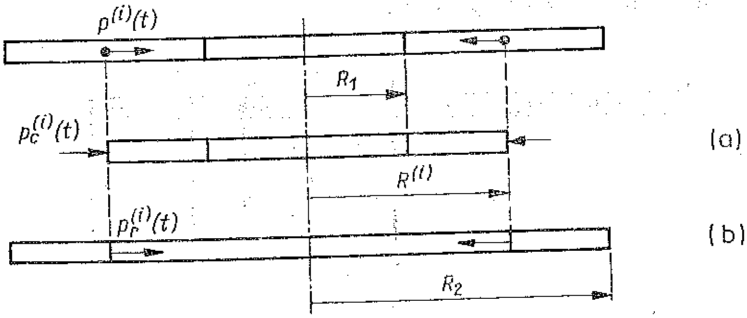

FIG. 2.

A method of the solution will be based on the principle of superposition; therefore, it is sufficient to present the solution for the disc loaded only with one cable. A static scheme of division of the disc along the i-th cable route is shown in Fig. 2. Motions of external (a) and internal (b) edges of the disc $(r=R^{(i)})$ are described by the same displacement function $u^{(i)}(t)$. This function can be written as a sum of

1) a function $u_{RC}^{(i)}(t)$ describing creep of the disc under the influence of one cable whose route coincides with a circle of radius $r=R^{(i)}$ and

2) a displacement function $u_R^{(i)}(t)$ caused by common action of other cables.

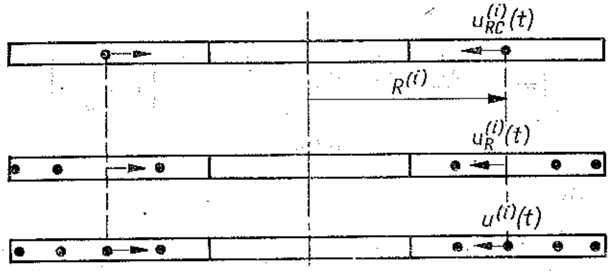

FIG. 3.

A diagrammatic sum of the displacement vector $u^{(i)}(t)$ is presented in Fig. 3

$$\bar{u}^{(i)}(\tau)=\bar{u}_{RC}^{(i)}(\tau)+\bar{u}_R^{(i)}(\tau), \tag{3.10}$$

where dimensionless functions of displacement are taken as

$$\bar{u}_{RC}^{(i)}(\tau)=\frac{1}{R_1}u_{RC}^{(i)}, \quad \bar{u}_R^{(i)}(\tau)=\frac{1}{R_1}u_R^{(i)}. \tag{3.11}$$

The load functions of the external edge (a) and hole (b) of the disc can be written then in the form

$$\bar{p}_c^{(i)}(\tau)=\bar{a}_{21}^{(i)}+\bar{a}_{22}^{(i)}[\bar{u}_{RC}^{(i)}(\tau)+\bar{u}_R^{(i)}(\tau)], \tag{3.12}$$

$$\bar{p}_r^{(i)}(\tau)=\bar{a}_{11}^{(i)}+\bar{a}_{12}^{(i)}[\bar{u}_{RC}^{(i)}(\tau)+\bar{u}_R^{(i)}(\tau)]. \tag{3.13}$$

Comparing Eqs. (3.12), (3.13) and (3.7) and taking into account

$$\bar{p}^{(i)}(\tau)=\bar{p}_c^{(i)}(\tau)+\bar{p}_r^{(i)}(\tau),$$

the following two algebraic relations for the coefficients $\bar{a}_{kl}^{(i)}$ can be written:

$$\bar{a}_{11}^{(i)}+\bar{a}_{21}^{(i)}=\bar{a}_1^{(i)}, \quad \bar{a}_{12}^{(i)}+\bar{a}_{22}^{(i)}=\bar{a}_2^{(i)}. \tag{3.14}$$

Two further equations for the coefficients $\bar{a}_{kl}^{(i)}$ follow from a continuity condition for the displacement function at the points $\rho=\rho^{(i)}$. If the values of these coefficients are treated as known, the static boundary conditions can be presented separately for

the disc (a)

$$s_\rho^{(i)}(1,\tau)=0, \quad s_\rho^{(i)}(\rho^{(i)},\tau)=-\bar{p}_c^{(i)}, \tag{3.15}$$

and the disc (b)

$$(3.16)\qquad s_\rho^{(i)}(\rho^{(i)},\tau)=\bar{p}_r^{(i)}(\tau),\quad s_\rho^{(i)}(\rho_2,\tau)=0,$$

where ρ_2 is the dimensionless diameter of the disc

$$(3.17)\qquad \rho_2=\frac{R_2}{R_1}.$$

The solution of the differential equation (3.3) with the boundary conditions (3.15) and (3.16) gives the radial and circumferential stresses at the region of the disc (a)

$$(3.18)\qquad \left.\begin{matrix} s_\rho \\ s_\theta \end{matrix}\right\}=\frac{-\bar{a}_{21}^{(i)}-\bar{a}_{22}^{(i)}\left[\bar{u}_{RC}^{(i)}(\tau)+\bar{u}_R^{(i)}(\tau)\right]}{1-\left(\dfrac{1}{\rho^{(i)}}\right)^2}\left[1\mp\frac{1}{\rho^2}\right],$$

and the disc (b)

$$(3.19)\qquad \left.\begin{matrix} s_\rho \\ s_\theta \end{matrix}\right\}=\frac{-\bar{a}_{11}^{(i)}-\bar{a}_{12}^{(i)}\left[\bar{u}_{RC}^{(i)}(\tau)+\bar{u}_R^{(i)}(\tau)\right]}{1-\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2}\left[1\mp\left(\frac{\rho^{(i)}}{\rho}\right)^2\right]+\bar{a}_{11}^{(i)}+$$
$$+\bar{a}_{12}^{(i)}\left[\bar{u}_{RC}^{(i)}(\tau)+\bar{u}_R^{(i)}(\tau)\right].$$

The stress function from the pay load is shown separately $(s_\rho(1,\tau)=-\bar{p}_0(\tau)$, $s_\rho(\rho_2,\tau)=0)$:

$$(3.20)\qquad \left.\begin{matrix} s_\rho \\ s_\theta \end{matrix}\right\}=\frac{\bar{p}_0(\tau)}{1-\left(\dfrac{1}{\rho_2}\right)^2}\left[1\mp\frac{1}{\rho^2}\right]-\bar{p}_0(\tau),$$

where

$$(3.21)\qquad \bar{p}_0(\tau)=\frac{1}{s_0}p_0(\tau).$$

Equations (3.18) and (3.19) indicate how the stresses depend on the displacement functions $\bar{u}_{RC}^{(i)}(\tau)$ and $\bar{u}_R^{(i)}(\tau)$ at the particles of the disc where the cables are placed. These functions will be determined by means of the constitutive equation (2.1) written for the fixed radial variable $\rho=\rho^{(i)}$.

4. State of strain

A narrow range of experimental studies on Poisson's ratios, Eqs. (2.8) and (2.9), as well as their negligible dependence on τ and τ' for concrete, induce to accept the following assumption:

$$(4.1)\qquad \mu(\tau)=\nu(\tau,\tau')=\mu=\text{const}.$$

Not decreasing in this way a practical application of the solutions, the constitutive equation (2.1) can be written in dimensionless shorter form:

$$\bar{E}(\tau)\,\varepsilon_{ij}(\rho,\tau)=\bar{t}_{ij}(\rho,\tau)+\int_1^{\tau}\bar{t}_{ij}(\rho,\tau)\,\bar{K}(\tau,\tau')\,d\tau' \tag{4.2}$$

where, according to the definition, it has been introduced:

$$\bar{t}_{ij}(\rho,\tau)=(1+\mu)\,\bar{s}_{ij}+(1-2\mu)\,\bar{\sigma}_m\,\delta_{ij}. \tag{4.3}$$

The magnitudes $\bar{s}_{ij}$ and $\bar{\sigma}_m$ denote the dimensionless stress deviator and dimensionless mean stress

$$\bar{s}_{ij}=\frac{1}{s_0}\,s_{ij},\qquad \bar{\sigma}_m=\frac{1}{s_0}\,\sigma_m. \tag{4.4}$$

Equation (4.2) along with Cauchy's relations (2.10) and the stress functions (3.18)–(3.20) given earlier make a starting point to determine the displacements $\bar{u}(\rho,\tau)$.

The function $\bar{K}(\tau,\tau')$ is usually found to be nonlinear [8, 9]. In this paper it is taken as a degenerated function:

$$\bar{K}(\tau,\tau')=f_1(\tau)\,g_1(\tau')+f_2(\tau)\,g_2(\tau'), \tag{4.5}$$

where

$$f_1(\tau)=\bar{E}(\tau),\qquad f_2(\tau)=\bar{E}(\tau)\,e^{-\gamma\tau},$$
$$g_1(\tau')=\frac{1}{\bar{E}^2(\tau')}\frac{d\bar{E}(\tau')}{d\tau'}-\frac{d\bar{\varphi}(\tau')}{d\tau'},\qquad g_2(\tau')=\left[\frac{d\bar{\varphi}(\tau')}{d\tau'}+\gamma\bar{\varphi}(\tau')\right]e^{\gamma\tau'}. \tag{4.6}$$

The dimensionless ageing function $\bar{\varphi}(\tau')$ and Young's modulus $\bar{E}(\tau)$ which appear in the formulas (4.6) will be taken in calculations as follows:

$$\bar{\varphi}(\tau')=C_0+\frac{A}{\tau'},\qquad \bar{E}(\tau)=E_0\,(1-\beta e^{-\alpha\tau}). \tag{4.7}$$

The physical properties of concrete are identified by means of the material constants C_0, A, E_0, α, β and γ.

The integral physical relation (4.2), in the case of the kernel determined by the formula (4.5) after differentiating twice, can be replaced by a differential equation which, after substitution of Eq. (4.6), has the form

$$\partial_\tau^2\,\bar{t}_{ij}(\rho,\tau)+\left\{\gamma\,[1+\bar{E}(\tau)\,\bar{\varphi}(\tau)]-\frac{1}{\bar{E}(\tau)}\,\partial_\tau\bar{E}(\tau)\right\}\partial_\tau\,\bar{t}_{ij}(\rho,\tau)=$$
$$=\bar{E}(\tau)\,[\partial_\tau^2\,\varepsilon_{ij}(\rho,\tau)+\gamma\partial_\tau\,\varepsilon_{ij}(\rho,\tau)], \tag{4.8}$$

where the following denotations have been introduced for the first and second time derivatives:

$$\partial_\tau\,\bar{t}_{ij}(\rho,\tau)=\frac{\partial\bar{t}_{ij}(\rho,\tau)}{\partial\tau}\qquad \partial_\tau^2\,\bar{t}_{ij}(\rho,\tau)=\frac{\partial^2\,\bar{t}_{ij}(\rho,\tau)}{\partial\tau^2}. \tag{4.9}$$

The initial conditions can be written in the form of two relations:

$$\bar{E}(1)\,\varepsilon_{ij}(\rho, 1)=f_{ij}(\rho, 1), \tag{4.10}$$

$$\bar{E}(1)\,\partial_\tau \varepsilon_{ij}(\rho, 1)=\partial_\tau f_{ij}(\rho, 1)+\varepsilon_{ij}(\rho, 1)\,\gamma\bar{\varphi}(1)\,\bar{E}^2(1). \tag{4.11}$$

The first one will be obtained putting formally in Eq. (4.2) $\tau=1$, where $\tau=1$, time determining the instant of the loading. The strains obtained here correspond to the instant elastic reaction of the material. The other relation (4.11) follows from Eq. (4.2) after time differentiating once and substitution $\tau=1$.

To determine the displacement functions $\bar{u}_{RC}^{(i)}(\tau)$ at the particles belonging simultaneously to the external and internal edges of the discs (a) and (b) (Fig. 2), the initial problem (4.8), (4.10) and (4.11) for the circumferential direction can be written as

$$\partial_\tau^2 s_\theta^{(i)}(\rho, \tau)-\mu\partial_\tau^2 s_\rho^{(i)}(\rho, \tau)+\Big\{\gamma\,[1+\bar{E}(\tau)\bar{\varphi}(\tau)]-\frac{1}{\bar{E}(\tau)}\,\partial_\tau \bar{E}(\tau)\Big\}[\partial_\tau s_\theta^{(i)}(\rho, \tau)-\mu\partial_\tau s_\rho^{(i)}(\rho, \tau)]=\bar{E}(\tau)\left[\frac{1}{\rho}\,\partial_\tau^2 \bar{u}^{(i)}(\rho, \tau)+\frac{\gamma}{\rho}\,\partial_\tau \bar{u}^{(i)}(\rho, \tau)\right], \tag{4.12}$$

$$\bar{E}(1)\,\frac{1}{\rho}\,\bar{u}^{(i)}(\rho, 1)=s_\theta^{(i)}(\rho, 1)-\mu s_\rho^{(i)}(\rho, 1), \tag{4.13}$$

$$\bar{E}(1)\,\frac{1}{\rho}\,\partial_\tau \bar{u}^{(i)}(\rho, 1)=\partial_\tau s_\theta^{(i)}(\rho, 1)-\mu\partial_\tau s_\rho^{(i)}(\rho, 1)+\frac{1}{\rho}\,\gamma\bar{\varphi}(1)\,\bar{E}^2(1)\,\bar{u}^{(i)}(\rho, 1). \tag{4.14}$$

The stress functions $s_\rho^{(i)}$ and $s_\theta^{(i)}$ occurring in Eqs. (4.12)–(4.14) have been written separately for region (a) Eq. (3.18) and (b) Eq. (3.19) of the disc. These stresses, similarly like the equations of the above given system for the fixed radial variable $r=R^{(i)}$, contain the same unknown function, namely, the displacement function $\bar{u}_{RC}^{(i)}(\tau)$. For this function two independent initial problems are formulated by substituting to the system (4.12)–(4.14) (for $\rho=\rho^{(i)}$) the stresses expressed separately by the formulas (3.18) and (3.19). As a result of the substitution of Eq. (3.18), the initial problem from the region of the disc (a) can be written in the form

$$\partial_\tau^2 \bar{u}_{RC}^{(i)}=\frac{1}{\bar{a}_{22}^{(i)}A^{(i)}-\dfrac{\bar{E}(\tau)}{\rho^{(i)}}}\left\{\partial_\tau \bar{u}_{RC}^{(i)}\left[\frac{\gamma\bar{E}(\tau)}{\rho^{(i)}}-\bar{a}_{22}^{(i)}A^{(i)}f(\tau)\right]-\bar{a}_{22}^{(i)}A^{(i)}\,[\partial_\tau^2 \bar{u}_R^{(i)}+\partial_\tau \bar{u}_R^{(i)} f(\tau)]\right\}, \tag{4.15}$$

$$(4.16)\qquad \partial_\tau \bar{u}_{RC}^{(i)}(1)=\frac{1}{\dfrac{\bar{E}(1)}{\rho^{(i)}}-\bar{a}_{22}^{(i)}A^{(i)}}\left\{\frac{g(1)\,[\bar{a}_{21}^{(i)}+\bar{a}_{22}^{(i)}\,\bar{u}_R^{(i)}(1)]\,A^{(i)}}{\rho^{(i)}\left[\dfrac{\bar{E}(1)}{\rho^{(i)}}-\bar{a}_{22}^{(i)}A^{(i)}\right]}+\bar{a}_{22}^{(i)}A^{(i)}\,\partial_\tau \bar{u}_R^{(i)}(1)\right\},$$

$$(4.17)\qquad \bar{u}_{RC}^{(i)}(1)=\frac{A^{(i)}}{\dfrac{\bar{E}(1)}{\rho^{(i)}}-\bar{a}_{22}^{(i)}A^{(i)}}\,[\bar{a}_{21}^{(i)}+\bar{a}_{22}^{(i)}\,\bar{u}_R^{(i)}(1)],$$

where

$$(4.18)\qquad A^{(i)}=\mu\frac{1+\left(\dfrac{1}{\rho^{(i)}}\right)^2}{1-\left(\dfrac{1}{\rho^{(i)}}\right)^2},$$

$$(4.19)\qquad f(\tau)=\gamma\left[1+E_0\left(C_0+\frac{A}{\tau}\right)(1-\beta e^{-\alpha\tau})\right]-\frac{\alpha\beta e^{-\alpha\tau}}{1-\beta e^{-\alpha\tau}},$$

$$(4.20)\qquad g(\tau)=\gamma E_0^2\left(C_0+\frac{A}{\tau}\right)(1-\beta e^{-\alpha\tau}).$$

In turn the substitution of Eq. (3.19) leads to the initial problem from the region of the disc (b):

$$(4.21)\qquad \partial_\tau^2\bar{u}_{RC}^{(i)}=\frac{1}{\bar{a}_{12}^{(i)}B^{(i)}-\dfrac{\bar{E}(\tau)}{\rho^{(i)}}}\left\{\partial_\tau \bar{u}_{RC}^{(i)}\left[\frac{\gamma\bar{E}(\tau)}{\rho^{(i)}}-\bar{a}_{12}^{(i)}B^{(i)}f(\tau)\right]-\bar{a}_{12}^{(i)}B^{(i)}\,[\partial_\tau^2\bar{u}_R^{(i)}+\partial_\tau\bar{u}_R^{(i)}f(\tau)]\right\},$$

$$(4.22)\qquad \partial_\tau \bar{u}_{RC}^{(i)}(1)=\frac{1}{\dfrac{\bar{E}(1)}{\rho^{(i)}}-\bar{a}_{12}^{(i)}B^{(i)}}\left\{\frac{g(1)\,[\bar{a}_{11}^{(i)}+\bar{a}_{12}^{(i)}\,\bar{u}_R^{(i)}(1)]\,B^{(i)}}{\rho^{(i)}\left[\dfrac{\bar{E}(1)}{\rho^{(i)}}-\bar{a}_{12}^{(i)}B^{(i)}\right]}+\bar{a}_{12}^{(i)}B^{(i)}\,\partial_\tau\bar{u}_R^{(i)}(1)\right\},$$

$$(4.23)\qquad \bar{u}_{RC}^{(i)}(1)=\frac{B^{(i)}}{\dfrac{\bar{E}(1)}{\rho^{(i)}}-\bar{a}_{12}^{(i)}B^{(i)}}\,[\bar{a}_{11}^{(i)}+\bar{a}_{12}^{(i)}\,\bar{u}_R^{(i)}(1)],$$

where

$$(4.24)\qquad B^{(i)}=\frac{2}{\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2-1}+1-\mu.$$

The same displacement function $\bar{u}_{RC}^{(i)}(\tau)$, for the particles belonging simultaneously to the external edge of the disc (a) and hole edge of the disc (b) (consequence of the fictitious division, Fig. 2), should result from both initial problems. It involves the necessity of the identical equality of the right hand sides of Eqs. (4.15) and (4.21) and the initial conditions (4.16) and (4.22), and Eqs. (4.17) and (4.23). Then such relations are valid:

$$\bar{a}_{22}^{(i)} A^{(i)} = \bar{a}_{12}^{(i)} B^{(i)}, \qquad \bar{a}_{21}^{(i)} A^{(i)} = \bar{a}_{11}^{(i)} B^{(i)} \tag{4.25}$$

constituting with Eqs. (3.14) an algebraic system of four equations for the unknown coefficients $\bar{a}_{kl}^{(i)}$ $(k, l=1, 2)$. The solution of this system takes the form

$$\begin{aligned} \bar{a}_{11}^{(i)} &= \frac{A^{(i)}}{A^{(i)}+B^{(i)}}\,\bar{a}_1^{(i)}, & \bar{a}_{12}^{(i)} &= \frac{A^{(i)}}{A^{(i)}+B^{(i)}}\,\bar{a}_2^{(i)}, \\ \bar{a}_{21}^{(i)} &= \frac{B^{(i)}}{A^{(i)}+B^{(i)}}\,\bar{a}_1^{(i)}, & \bar{a}_{22}^{(i)} &= \frac{B^{(i)}}{A^{(i)}+B^{(i)}}\,\bar{a}_2^{(i)}. \end{aligned} \tag{4.26}$$

Integration of the differential equation of second order (4.21) (or Eq. (4.15)) will be replaced by integration of the adjoint system of the differential equations of first order with separated initial conditions:

$$\partial_\tau \bar{v}_{RC}^{(i)}(\tau) = \frac{1}{\bar{a}_{12}^{(i)} B^{(i)} - \dfrac{\bar{E}(\tau)}{\rho^{(i)}}} \left\{ \bar{v}_{RC}^{(i)} \left[\frac{\gamma \bar{E}(\tau)}{\rho^{(i)}} - \bar{a}_{12}^{(i)} B^{(i)} f(\tau) \right] - \right. \\ \left. - \bar{a}_{12}^{(i)} B^{(i)} \left[\partial_\tau^2 \bar{u}_R^{(i)} + \partial_\tau \bar{u}_R^{(i)} f(\tau) \right] \right\}, \tag{4.27}$$

$$\bar{v}_{RC}^{(i)}(1) = \frac{1}{\dfrac{\bar{E}(1)}{\rho^{(i)}} - \bar{a}_{12}^{(i)} B^{(i)}} \left\{ \frac{g(1)\,[\bar{a}_{11}^{(i)} + a_{12}^{(i)} \bar{u}_R^{(i)}(1)]\,B^{(i)}}{\rho^{(i)} \left[\dfrac{\bar{E}(1)}{\rho^{(i)}} - \bar{a}_{12}^{(i)} B^{(i)} \right]} + \right. \\ \left. + \bar{a}_{12}^{(i)} B^{(i)} \partial_\tau u_R^{(i)}(1) \right\}, \tag{4.28}$$

and

$$\partial_\tau \bar{u}_{RC}^{(i)} = \bar{v}_{RC}^{(i)}, \tag{4.29}$$

$$\bar{u}_{RC}^{(i)}(1) = \frac{B^{(i)}}{\dfrac{\bar{E}(1)}{\rho^{(i)}} - \bar{a}_{12}^{(i)} B^{(i)}} \left[\bar{a}_{11}^{(i)} + \bar{a}_{12}^{(i)} \bar{u}_R^{(i)}(1) \right]. \tag{4.30}$$

The solution of the initial problem (4.27)–(4.30) for the function $\bar{u}_{RC}^{(i)}(\tau)$ will be possible after determining the function $u_{RC}^{(i)}(\tau)$ which describes the action of other cables of the disc prestressed many times. For this purpose we intend to determine the function $\bar{u}^{(i)}(\rho, \tau)$ caused by the action of one arbitrarily located prestressed cable in the whole region of the disc. The system (4.12)–(4.14) is the starting point in this case. The solution for the displacement function $\bar{u}^{(i)}(\rho, \tau)$ is again sought separately in the regions (a) and (b) of the disc. As a result of independent substitution of formulas (3.18) and (3.19) to the system (4.12)–(4.14), the initial problems in both regions will be obtained. Using the relations (4.27)–(4.30) these problems can be written as:

in region (a) $(1 \leqslant \rho \leqslant \rho^{(i)})$

$$\partial_\tau^2 \bar{u}^{(i)}(\rho, \tau) = -\gamma \partial_\tau \bar{u}^{(i)}(\rho, \tau) - F^{(i)}(\rho, \tau), \tag{4.31}$$

$$\partial_\tau \bar{u}^{(i)}(\rho, 1) = -\frac{\rho}{\bar{E}(1)} \frac{1-\mu+(1+\mu)\dfrac{1}{\rho^2}}{1-\left(\dfrac{1}{\rho^{(i)}}\right)^2} \Big\{ \bar{a}_{22}^{(i)} [\partial_\tau \bar{u}_{RC}^{(i)}(1) + \partial_\tau \bar{u}_R^{(i)}(1)] + \frac{g(1)}{\bar{E}(1)} [\bar{a}_{21}^{(i)} + \bar{a}_{22}^{(i)} (\bar{u}_{RC}^{(i)}(1) + \bar{u}_R^{(i)}(1))] \Big\}, \tag{4.32}$$

$$\bar{u}^{(i)}(\rho, 1) = -\frac{\rho}{\bar{E}(1)} \frac{1-\mu+(1+\mu)\dfrac{1}{\rho^2}}{1-\left(\dfrac{1}{\rho^{(i)}}\right)^2} [\bar{a}_{21}^{(i)} + \bar{a}_{22}^{(i)} (\bar{u}_{RC}^{(i)}(1) + \bar{u}_R^{(i)}(1))], \tag{4.33}$$

and in region (b) $(\rho^{(i)} \leqslant \rho \leqslant \rho_2)$

$$\partial_\tau^2 \bar{u}^{(i)}(\rho, \tau) = -\gamma \partial_\tau \bar{u}^{(i)}(\rho, \tau) - G^{(i)}(\rho, \tau), \tag{4.34}$$

$$\partial_\tau \bar{u}^{(i)}(\rho, 1) = -\frac{\rho}{\bar{E}(1)} \frac{(1+\mu)\left(\dfrac{\rho^{(i)}}{\rho}\right)^2 + (1-\mu)\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2}{1-\left(\dfrac{1}{\rho^{(i)}}\right)^2} \times \Big\{ \bar{a}_{12}^{(i)} (\partial_\tau \bar{u}_{RC}^{(i)}(1) + \partial_\tau \bar{u}_R^{(i)}(1)) + \frac{g(1)}{E(1)} [\bar{a}_{11}^{(i)} + \bar{a}_{12}^{(i)} (\bar{u}_{RC}^{(i)}(1) + \bar{u}_R^{(i)}(1))] \Big\}, \tag{4.35}$$

$$\bar{u}^{(i)}(\rho, 1) = -\frac{\rho}{\bar{E}(1)} \frac{(1+\mu)\left(\dfrac{\rho^{(i)}}{\rho}\right)^2 + (1-\mu)\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2}{1-\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2} [\bar{a}_{11}^{(i)} + \bar{a}_{12}^{(i)} (\bar{u}_{RC}^{(i)}(1) + \bar{u}_R^{(i)}(1))]. \tag{4.36}$$

The following functions have been introduced for concise notation of the differential equations (4.31) and (4.34):

$$F^{(i)}(\rho, \tau) = \bar{a}_{22}^{(i)} \frac{\rho}{\rho^{(i)}} \frac{1-\mu+(1+\mu)\left(\dfrac{1}{\rho^2}\right)}{\left[1-\left(\dfrac{1}{\rho^{(i)}}\right)^2\right]\left[\bar{a}_{22}^{(i)} A^{(i)} - \dfrac{\bar{E}(\tau)}{\rho^{(i)}}\right]} \{\partial_\tau \bar{u}_{RC}^{(i)} [\gamma - f(\tau)] - \partial_\tau^2 \bar{u}_R^{(i)} - f(\tau)\, \partial_\tau \bar{u}_R^{(i)}\}, \tag{4.37}$$

$$(4.38)\quad G^{(i)}(\rho,\tau)=\bar{a}_{12}^{(i)}\frac{\rho}{\rho^{(i)}}\frac{(1+\mu)\left(\frac{\rho^{(i)}}{\rho}\right)^2+(1-\mu)\left(\frac{\rho^{(i)}}{\rho_2}\right)^2}{\left[1-\left(\frac{\rho^{(i)}}{\rho_2}\right)^2\right]\left[\bar{a}_{12}^{(i)}B^{(i)}-\frac{\bar{E}(\tau)}{\rho^{(i)}}\right]}\{\partial_\tau\bar{u}_{RC}^{(i)}[\gamma-f(\tau)]-\partial_\tau^2\bar{u}_R^{(i)}-f(\tau)\,\partial_\tau\bar{u}_R^{(i)}\}.$$

A numerical solution of the equations of second order (4.31) and (4.34) with the possibility of using the Runge–Kutta method of integration can be obtained by integrating the substitutional system of differential equations of first order. For this purpose Eq. (4.31) can be replaced by the system

$$(4.39)\qquad \partial_\tau\bar{v}^{(i)}(\rho,\tau)=-\gamma\bar{v}^{(i)}(\rho,\tau)-F^{(i)}(\rho,\tau),$$

$$(4.40)\qquad \partial_\tau\bar{u}^{(i)}(\rho,\tau)=\bar{v}^{(i)}(\rho,\tau).$$

Similarly, a system of equations can be introduced,

$$(4.41)\qquad \partial_\tau\bar{v}^{(i)}(\rho,\tau)=-\gamma\bar{v}^{(i)}(\rho,\tau)-G^{(i)}(\rho,\tau),$$

$$(4.42)\qquad \partial_\tau\bar{u}^{(i)}(\rho,\tau)=\bar{v}^{(i)}(\rho,\tau),$$

instead of Eq. (4.34). The initial conditions (4.32), (4.33) and (4.35), (4.36) remain unchanged and, according to the definition of $\bar{v}^{(i)}(\rho,\tau)$ (4.40), (4.42), it is assumed that

$$\partial_\tau\bar{u}^{(i)}(\rho,1)=\bar{v}^{(i)}(\rho,1).$$

The displacement function from the pay load $\bar{u}^p(\rho,\tau)$ follows from the solution of the initial problem (4.12)–(4.14) if the stresses s_ρ and s_θ are expressed by the relations (3.20). The closed form for the function $\bar{u}^p(\rho,\tau)$ is obtained under the assumption $\bar{p}_0=\text{const}$

$$(4.43)\quad \bar{u}^p(\rho,\tau)=\rho\left\{\frac{\bar{p}_0}{1-\left(\frac{1}{\rho_2}\right)^2}\left[(1+\mu)\frac{1}{\rho^2}+1-\mu\right]-\bar{p}_0(1-\mu)\right\}\left\{\bar{\varphi}(\tau)\left[1-e^{-\gamma(\tau-1)}\right]+\frac{1}{\bar{E}(\tau)}\right\}.$$

The set of equations shown above, describing the problem of disc creep under the influence of the action of one arbitrarily chosen i-th cable, contains the unknown function $u_R^{(i)}(\tau)$. To calculate the functions $\bar{u}_R^{(i)}(\tau)$ we invoke an iterative procedure.

As the starting point of the iterative process the solution is taken in which the interaction of the cables is disregarded $\left(u_R^{(i)}(\tau)=0\right)$. Superposition of the displacement functions obtained separately for each cable gives the displacement function of the n-times prestressed disc

$$(4.44)\qquad \bar{u}(\rho,\tau)=\sum_{i=1}^{n}\bar{u}^{(i)}(\rho,\tau)+\bar{u}^p(\rho,\tau).$$

The second approximation for the displacement function will result from the solution of the governing set of equations (4.27)–(4.30), (4.31)–(4.33) and (4.34)–(4.36) if in place of $\bar{u}_R^{(i)}(\tau)$ it is substituted:

$$\bar{u}_R^{(i)}(\tau)=\bar{u}(\rho^{(i)},\tau)-\bar{u}_{RC}^{(i)}(\tau). \tag{4.45}$$

The successive iterative steps will consist in substituting the function $\bar{u}_R^{(i)}(\tau)$ calculated according to the displacement function $\bar{u}(\rho,\tau)$ which is taken from the former iteration to the governing set.

The numerical examples prove that the method is fast convergent. Differences between the third and fourth approximation are smaller than an error of numerical integration. The significant precision of calculations is due to the Runge–Kutta method of higher order applied to the systems of the ordinary differential equations [10].

Neglecting an effect of ageing in the description of material properties, the analytical solution for the displacement function will be obtained. Such an approach is applied to the analysis of older concrete creep. For this case the kernel form (4.5) of the integral equations (4.2) is obtainable by substituting to the formulas (4.6) boundary magnitudes of the ageing function $\bar{\varphi}(\tau)$ and Young's modulus $\bar{E}(\tau)$ (4.7) at $\tau\to\infty$

$$\bar{K}(\tau,\tau')=\bar{K}(\tau-\tau')=f_2^*(\tau)\,g_2^*(\tau'), \tag{4.46}$$

where

$$f_2^*(\tau)=E_0\,e^{-\gamma\tau},\qquad g_2^*(\tau')=C_0\,\gamma e^{\gamma\tau'}. \tag{4.47}$$

The integral constitutive equation (4.2) reduced after substitution of Eq. (4.46) and single differentiating to the differential equation of the first order:

$$\partial_\tau t_{ij}(\rho,\tau)+\gamma\left(1+C_0\,\bar{E}(1)\right)t_{ij}(\rho,\tau)=\bar{E}(1)\,\partial_\tau\varepsilon_{ij}(\rho,\tau)+\bar{E}(1)\,\gamma\varepsilon_{ij}(\rho,\tau). \tag{4.48}$$

The displacement function $\bar{u}_{RC}^{(i)}(\tau)$ needs to be determined according to the algorithm of solution given earlier. The solution of Eq. (4.48) (written in the circumferential direction for $\rho=\rho^{(i)}$) takes the form

$$\begin{aligned}\bar{u}_{RC}^{(i)}(\tau)=&\,\bar{u}_{RC}^{(i)}(1)\,e^{-L^{(i)}(\tau-1)}+\gamma\left(1+C_0\,\bar{E}(1)\right)\bar{a}_{21}^{(i)}\frac{M^{(i)}}{L^{(i)}}\left(1-e^{-L^{(i)}(\tau-1)}\right)+\\&+M^{(i)}\,\bar{a}_{22}^{(i)}\,\bar{u}_R^{(i)}(1)\left(\frac{\bar{u}_R^{(i)}(\tau)}{\bar{u}_R^{(i)}(1)}-e^{-L^{(i)}(\tau-1)}\right)+M^{(i)}\,\bar{a}_{22}^{(i)}\,e^{-L^{(i)}\tau}\{\gamma(1+\\&+C_0\,\bar{E}(1))-L^{(i)}\}\int_1^\tau\bar{u}_R^{(i)}(\tau)\,e^{L^{(i)}\tau}\,d\tau,\end{aligned} \tag{4.49}$$

where

$$L^{(i)}=\gamma\,\frac{\bar{a}_{22}^{(i)}\left(1+C_0\,\bar{E}(1)\right)A^{(i)}-\dfrac{\bar{E}(1)}{\rho^{(i)}}}{\bar{a}_{22}^{(i)}\,A^{(i)}-\dfrac{\bar{E}(1)}{\rho^{(i)}}}, \tag{4.50}$$

$$M^{(i)}=-\frac{A^{(i)}}{\bar{a}_{22}^{(i)}\,A^{(i)}-\dfrac{\bar{E}(1)}{\rho^{(i)}}}. \tag{4.51}$$

Appearing in the relations (4.49)–(4.51), the magnitudes $u_{RC}^{(i)}(1)$ and $A^{(i)}$ should be accepted in accordance with the dependences (4.17) and (4.18). The coefficients of prestressing distribution $\bar{a}_{kl}^{i}$ retain the identical form as well, Eqs. (4.26).

The solution of the initial problem (4.48) and (4.10) with respect to the displacement function $\bar{u}^{(i)}(\rho, \tau)$ may be shown after transformations in the uniform, analytical form in the total disc domain:

$$\begin{aligned}(4.52)\quad \bar{u}^{(i)}(\rho,\tau)=&\bar{u}^{(i)}(\rho,1)\,e^{-\gamma(\tau-1)}+\frac{K_1^{(i)}(\rho)}{\gamma-L^{(i)}}\left(e^{-L^{(i)}(\tau-1)}-e^{-\gamma(\tau-1)}\right)+\\&+\frac{K_5^{(i)}(\rho)}{\gamma}\left(1-e^{-\gamma(\tau-1)}\right)+K_3^{(i)}(\rho)\,\bar{u}_R^{(i)}(1)\left[\frac{\bar{u}_R^{(i)}(\tau)}{\bar{u}_R^{(i)}(1)}-e^{-\gamma(\tau-1)}\right]+\\&+\left[K_2^{(i)}(\rho)-\gamma K_3^{(i)}(\rho)\right]e^{-\gamma\tau}\int_1^{\tau}\bar{u}_R^{(i)}(\tau)\,e^{\gamma\tau}\,d\tau+K_4^{(i)}(\rho)\,e^{-\gamma\tau}\int_1^{\tau}e^{(\gamma-L^{(i)})\tau}\times\\&\times\left[\int_1^{\tau}\bar{u}_R^{(i)}(\tau)\,eL^{(1)}\,d\tau\right]d\tau,\end{aligned}$$

where the functions $K_j^{(i)}(\rho)$ $(j=1,2,\ldots,5)$ have been defined separately for the regions:

(a)

$$(4.53)\qquad K_j^{(i)}(\rho)=B_j^{(i)}\,\frac{\rho}{\bar{E}(1)}\,\frac{1-\mu+(1+\mu)\dfrac{1}{\rho^2}}{1-\left(\dfrac{1}{\rho^{(i)}}\right)^2}$$

and (b)

$$(4.54)\qquad K_j^{(i)}(\rho)=-B_j^{(i)}\,\frac{\rho}{\bar{E}(1)}\,\frac{\bar{a}_{12}^{(i)}}{\bar{a}_{22}^{(i)}}\,\frac{(1-\mu)\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2+(1+\mu)\left(\dfrac{\rho^{(i)}}{\rho}\right)^2}{1-\left(\dfrac{\rho^{(i)}}{\rho_2}\right)^2}.$$

The coefficients $B_j^{(i)}$ (for $j=1,2,\ldots,5$) accepted in the relations (4.53) and (4.54) are given by the formulas

$$(4.55)\qquad\begin{aligned}B_1^{(i)}=&-\bar{a}_{22}^{(i)}\left[\bar{u}_{RC}^{(i)}(1)-\bar{u}_R^{(i)}(1)\,\bar{a}_{22}^{(i)}\,M^{(i)}-\frac{M^{(i)}}{L^{(i)}}\,\bar{a}_{21}^{(i)}\,\gamma\left(1+C_0\bar{E}(1)\right)\right]\left[L^{(i)}-\right.\\&\left.-\gamma\left(1+C_0\bar{E}(1)\right)\right],\\B_2^{(i)}=&-\bar{a}_{22}^{(i)}\left[M^{(i)}\,\bar{a}_{22}^{(i)}\,L^{(i)}-\left(1+C_0\bar{E}(1)\right)\left(1+2M^{(i)}\,\bar{a}_{22}^{(i)}\right)\right],\\B_3^{(i)}=&\,\bar{a}_{22}^{(i)}\left(M^{(i)}\,\bar{a}_{22}^{(i)}+1\right),\\B_4^{(i)}=&\,M^{(i)}\left(\bar{a}_{22}^{(i)}\right)^2\left[L^{(i)}-\gamma\left(1+C_0\bar{E}(1)\right)\right],\\B_5^{(i)}=&\,\bar{a}_{21}^{(i)}\,\gamma\left(1+C_0\bar{E}(1)\right)\left[1+\frac{M^{(i)}}{L^{(i)}}\,\bar{a}_{22}^{(i)}\,\gamma\left(1+C_0\bar{E}(1)\right)\right].\end{aligned}$$

The initial conditions for the displacement function $\bar{u}^{(i)}(\rho, 1)$ are presented by the relations given earlier: (4.33) in the region (a) and (4.36) in the region (b) of the disc.

The iterative method shown above, which makes it possible to take into account the influence of interaction of prestressing cables, does not undergo changes. However, the present solution simplifies to a numerical test of the functions with an application of numerical methods of calculation for the integrals appearing in the formulas (4.49) and (4.52).

Below, in the numerical analysis of creep process of the disc, the constants describing the properties of concrete will be taken as follows:

$$C_0=3.6\cdot 10^{-3}, \quad E_0=6.25\cdot 10^{2}, \quad A=6.89\cdot 10^{-4}, \quad \alpha=1.40, \quad \beta=0.6, \quad \gamma=0.728. \tag{4.56}$$

These data correspond to a description of creep course for concretes out of Portland cement. The constants s_0 and t_0 introduced for dimensional purposes are taken in calculations as the ultimate strength of the concrete $s_0=40$ MPa and the load instant $t_0=28$ days.

As an example the solution for the disc prestressed four times with $\rho_2=3$ will be shown. The cables lie along with circles having radii: $\rho^{(1)}=1.5$, $\rho^{(2)}=2$, $\rho^{(3)}=2.5$, $\rho^{(4)}=3$. For the useful load $\bar{p}_0=0.25$ the accepted prestressing is characterized by the following magnitudes:

$$\bar{a}_1^{(1)}=8.75\cdot 10^{-2}, \quad \bar{a}_1^{(2)}=6.25\cdot 10^{-2}, \quad \bar{a}_1^{(3)}=5\cdot 10^{-2}, \quad \bar{a}_1^{(4)}=3.75\cdot 10^{-2},$$
$$\bar{a}_2^{(1)}=40, \quad \bar{a}_2^{(2)}=30, \quad \bar{a}_2^{(3)}=24, \quad \bar{a}_2^{(4)}=20,$$

The solution for the radial and circumferential stress functions and displacement functions is shown in Figs. 4, 5 and S. Solid lines illustrate the solution in which interaction of cables is taken into account, on the contrary to dashed lines where this effect is disregarded ($\bar{u}_R^{(i)}=0$). At the same time light lines indicate the functions at the instant load ($\tau=1$) while heavy lines show their shape at the instant $\tau=5$ when the creep process finishes. Additionally, the circumferential stress function is shown in Fig. 5 with a centre line (for $\tau=5$). This function has been obtained from the solution, neglecting the ageing effect of concrete (analytical solution).

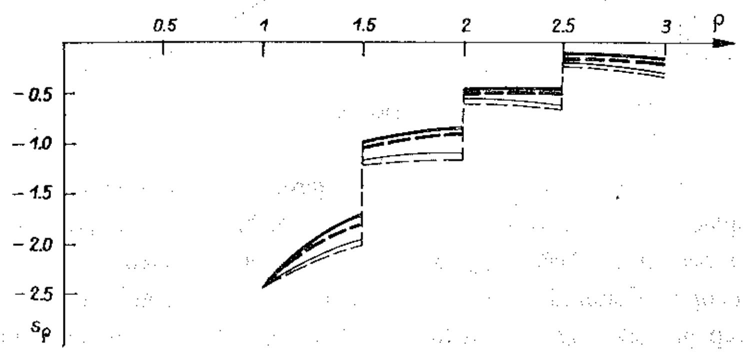

FIG. 4.

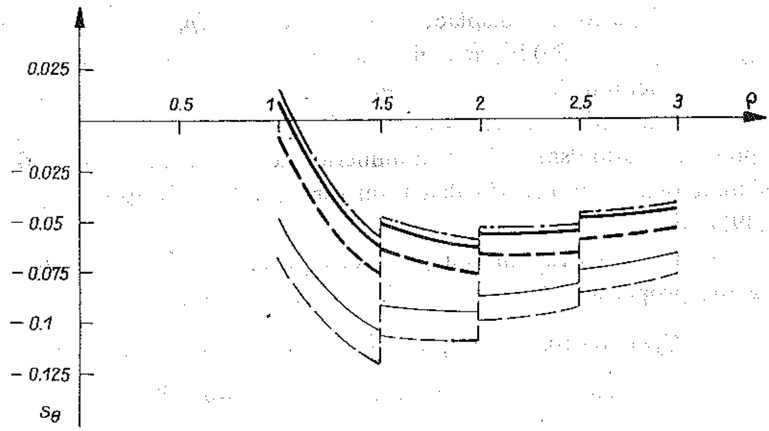

Fig. 5.

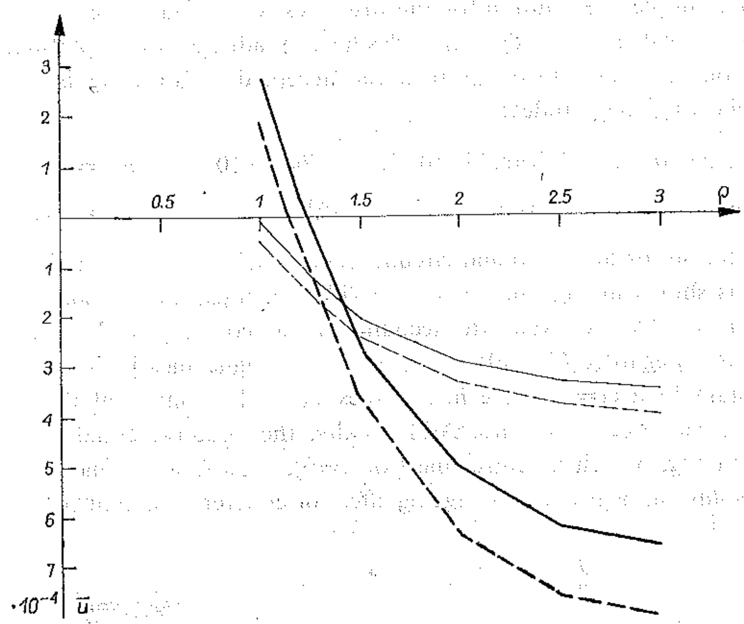

Fig. 6.

The assymptotic behaviour of the circumferential stress functions (Fig. 7) and radial displacement functions (all of cables), Fig. 8, shows stoppage of the creep process in $\tau=5$ time. According to the criterion of restricted prestress stretching stresses in concrete should not be greater than admissible magnitudes after stoppage of the creep process (Fig. 7 solid line). Not taking into account cable interaction, Fig. 7 — dashed line, may be connected with eliminating disadvantageous stress

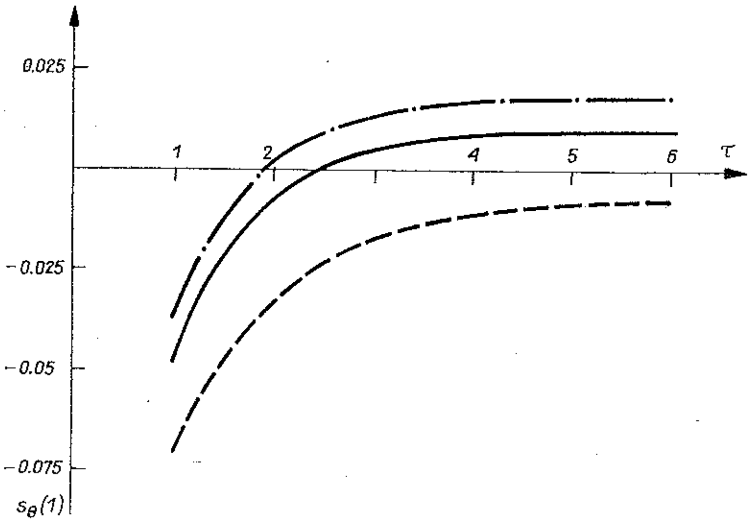

FIG. 7.

redistribution in analysis. Neglecting the ageing effect ($\bar{E}(\tau)$=const, $\bar{\varphi}(\tau)$=const) and keeping the values of the material constants leads to greater loss of prestress caused by greater displacements of prestressed cables [7].

The applied constitutive relation of linear visco-elasticity is valid for stresses smaller than 50 per cent of the ultimate strength of the concrete. In the case of

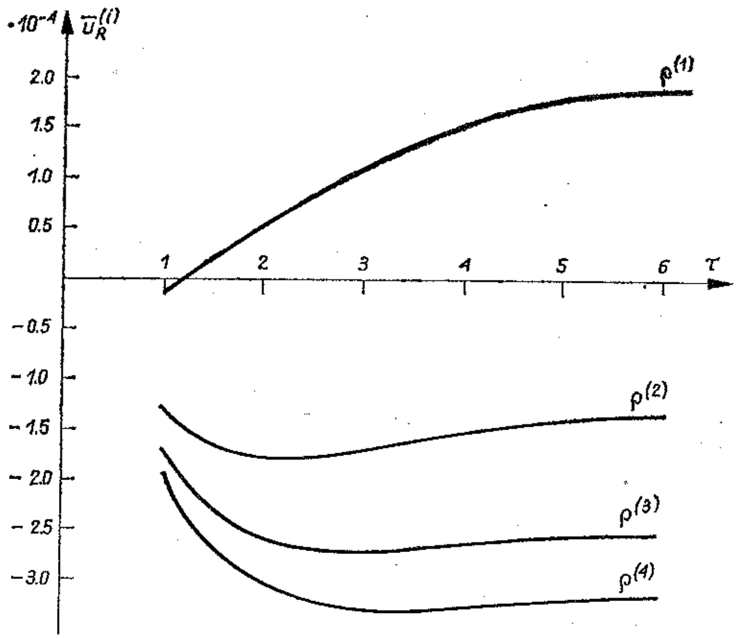

FIG. 8.

discs, constraint is related to a substitute stress calculated according to the Mises hypothesis

$$\bar{s}=(s_\rho^2+s_\theta^2-s_\rho s_\theta)^{1/2}\leqslant 0.5. \tag{4.57}$$

The substitute stress functions for $\tau=1$ (light line) and $\tau=5$ (heavy line) are shown in Fig. 9.

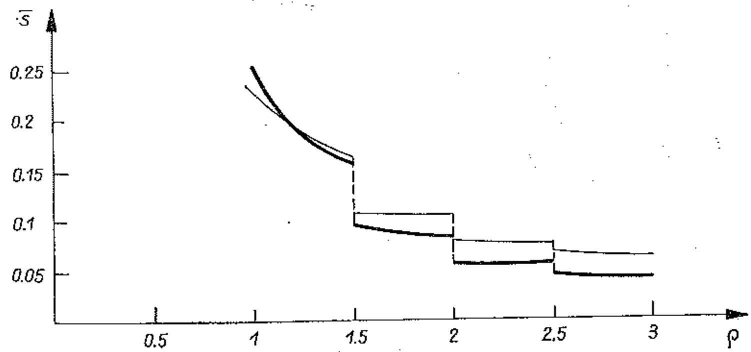

FIG. 9.

5. Final remarks

The applicability of the analysis presented above is connected with frequent cases of the work of prestressed discs as a construction element of high chimneys, television and cooling towers, nuclear reactors. The practical realization of those constructions is connected with the adoption of a suitable prestress program determined by technological conditions. The numerical example shows the solution of the creep process of the disc with initial conditions formulated for simultaneous application of both load and prestressing. The method of the solution presented in this paper enables calculation for arbitrary initial conditions. In this case the realization of the solutions is only connected with the drafting of the program on the digital computer.

Assuming the work of the disc in normal temperatures in formulating the initial--boundary problem, the stress relaxation in cables has been neglected. This behaviour can be regarded if the prestress coefficients $\bar{a}_1^{(i)}$ and $\bar{a}_2^{(i)}$ are taken as functions of time according to the relaxation law for steel. The coefficients of prestress distribution $\bar{a}_{kl}^{(i)}$ will also be the functions of time having influence on the form of equations drafting the initial problem for the function of displacement.

The solution for nonageing concrete leads to greater loss of prestress. This indicates that the nonageing model gives a safer design than the ageing model.

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STRESZCZENIE

REOLOGIA WSTĘPNIE SPRĘŻONYCH TARCZ KOŁOWO-SYMETRYCZNYCH

Przedstawiono rozwiązanie zagadnienia pełzania tarczy pierścieniowej sprężonej układem kabli koncentrycznych. Zakładając równomierny rozkład kabli wyznaczono numerycznie wielkości sił sprężających. Problem wzajemnego oddziaływania kabli rozwiązano sposobem iteracyjnym.

Резюме

РЕОЛОГИЯ ПРЕДВАРИТЕЛЬНО НАПРЯЖЕННЫХ ДИСКОВ С КРУЧОВОЙ СИММЕТРИЕЙ

Представлено решение ползучести кольцевого диска подвертенного предварительному напряжению посредством концентрически расположенных пучков. Преднолагая равномерное распределение арматуры были численно определены напрагающих усилий. Задача о взаимодействи пучков была решена интеративным методом.

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