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APPLICATION OF TORSIONAL AND LONGITUDINAL ELASTIC WAVES IN MECHANICAL SYSTEMS

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One-dimensional elastic waves are used for the investigation of mechanical systems consisting of an arbitrary number of elastic elements and rigid bodies. These systems may be loaded by nonperiodic forces and their elements may collide with each other. Elastic elements have finite lengths and are assumed to deform longitudinally or torsionally. Analytical solutions of the systems are given in the form of recurrence formulae. These enable to determine velocities, strains and displacements in an arbitrary cross-section of elastic elements at an arbitrary time instant, taking into account equivalent damping. As an example, the model of a crankshaft of a one-cylinder engine is considered.

1. Introduction

The paper concerns the chosen group of mechanical systems, namely such ones which can be modelled by means of an arbitrary number of homogeneous elastic elements, deformed torsionally or longitudinally, and rigid bodies. The systems can be loaded by nonperiodic forces, and collisions between elements can occur. Such systems comprise colliding machine elements, drive systems, crankshafts of engines and others, in which flexural strains are very small.

In the paper the motion of every elastic element is described by means of the classical wave equation within the restrictions imposed by dimensions of the elements as given in [1], for example. Real damping appearing in the system is replaced by equivalent damping of the viscous type which can be taken into account in boundary conditions or in arbitrary cross-sections of elements.

The solution of the classical wave equation with different initial and boundary conditions can be sought by means of the method of separate variables [3], transformations [5], the method of finite differences along characteristics [2], the Donnell's method [4, 6], and the d'Alembert's method [3, 7]. From the cited literature it follows that these methods were widely exploited for one rod only.

The paper presented concerns a system with an arbitrary number of homogeneous rods deformed torsionally or longitudinally, connected in an appropriate manner. In order to obtain the solution of such systems, the d'Alembert's method was extended. In the extended method every rod has its own functions describing waves travelling only in this rod. Hence it could be possible to notice some regularities in solutions, and to give the solutions in the form of recurrence formulae

It must be emphasized that no regularities for unknown functions were noticed when only one rod was investigated [2-7]. Moreover, the solutions given in the form of recurrence formulae enable the use of one-dimensional torsional and longitudinal waves for many complex mechanical systems in order to determine velocities, strains and displacements in an arbitrary cross-section of elastic elements at an arbitrary time instant, taking into account equivalent damping.

2. Formulation of the problem

We shall consider a mechanical system consisting of M homogeneous elastic elements of different lengths and a certain number of rigid bodies. For convenience, we divide M elastic elements into $N \geqslant M$ elements of equal length l. These elements can deform torsionally or longitudinally. For $t \leqslant 0$ the i-th element has the constant velocity V_{i0}. At instant $t=0$ collisions or nonperiodic forces can occur. At this time instant displacements of all points of the system are the same and are equal to U_0. Forces, acting in the cross-sections $x=(i-1)\,l$, are described by means of the functions

$$(2.1)\qquad F_i(t)=\sum_{k=1}^{m_i} H(t-t_{ik})\left[a_{ik}+b_{ik}(t-t_{ik})+c_{ik}\sin\left(d_{ik}(t-t_{ik})+e_{ik}\right)\right], \qquad i=1, 2, \ldots, N+1$$

that allow to approximate arbitrary forces appearing in real systems where a_{ik}, b_{ik}, c_{ik}, d_{ik}, e_{ik} are appropriate constants, and t_{ik} are fixed time instants. In particular cases, the formula (2.1) can describe a force which is either piece-wisely constant, or piece-wisely linear or sinusoidal.

The real damping is taken into account by introducing an equivalent damping of the viscous type in chosen cross-sections of elastic elements.

Under these assumptions, the determination of displacements, velocities and strains of cross-sections of elastic elements of the considered system for $t \geqslant 0$ is reduced to solving N equations:

$$(2.2)\qquad \frac{\partial^2 U_i(x,t)}{\partial t^2}-c^2\frac{\partial^2 U_i(x,t)}{\partial x^2}=0, \qquad i=1, 2, \ldots, N$$

with the initial conditions,

$$(2.3)\qquad U_i(x,t)=U_0, \qquad \frac{\partial U_i(x,t)}{\partial t}=V_{i0} \quad \text{for} \quad t=0, \quad i=1, 2, \ldots, N$$

and the boundary conditions

$$(2.4)\qquad \begin{gathered} F_1(t)+a_1\frac{\partial^2 U_1(x,t)}{\partial t^2}+b_1\frac{\partial U_1(x,t)}{\partial x}+d_1\left(\frac{\partial U_1(x,t)}{\partial t}-V_{10}\right)=0 \quad \text{for} \quad x=0,\\ U_{i-1}(x,t)=U_i(x,t) \quad \text{for} \quad x=(i-1)\,l, \quad i=2, 3, \ldots, N, \end{gathered}$$

(2.4) [cont.]

$$F_i(t)+a_i\frac{\partial^2 U_{i-1}(x,t)}{\partial t^2}+b_i\frac{\partial U_{i-1}(x,t)}{\partial x}+c_i\frac{\partial U_i(x,t)}{\partial x}+$$

$$+d_i\left(\frac{\partial U_{i-1}(x,t)}{\partial t}-V_{i-1,0}\right)=0 \quad \text{for} \quad x=(i-1)\,l, \quad i=2,3,\ldots,N,$$

$$F_{N+1}(t)+a_{N+1}\frac{\partial^2 U_N(x,t)}{\partial t^2}+b_{N+1}\frac{\partial U_N(x,t)}{\partial x}+d_{N+1}\left(\frac{\partial U_N(x,t)}{\partial t}+\right.$$

$$\left.-V_{N0}\right)=0 \quad \text{for} \quad x=Nl,$$

where $U_i(x,t)$ is the displacement of the i-th elastic element of the system, c is the wave velocity. The boundary conditions (2.4) are the conditions for forces and displacements in the limit cross-sections of elastic elements. The constants a_i, b_i, c_i are determined by parameters of the system, while the constants d_i are related with the equivalent viscous damping in selected cross-sections of the system elements.

The boundary conditions (2.4) cover a number of particular cases of mechanical systems that follow from the fact that the corresponding constants a_i, b_i, c_i, d_i and forces $F_i(t)$ can be equal to zero. For example, if only b_i are different from zero or only c_i are different from zero, then the right-hand end of the i-1-th element or the left-hand end of the i-th element is free. Next, if $F_i(t)=0$ and $a_i=0$, then in the cross-section $x=(i-1)\,l$ neither a rigid body is fixed nor an external force acts, etc. In particular, the conditions (2.4) include the boundary conditions of systems considered in the works [3, 4, 6, 7].

3. Solution

We introduce the nondimensional coordinates $\bar{x}=x/l$, $\tau=ct/l$ and suitable nondimensional displacements $\bar{U}_i(\bar{x},\tau)$ and forces $\bar{F}_i(\tau)$. Then the relations (2.1)–(2.4) take the form

$$\bar{F}_i(\tau)=\sum_{k=1}^{m_i} H(\tau-\tau_{ik})\left[A_{ik}+B_{ik}(\tau-\tau_{ik})+C_{ik}\sin\left(D_{ik}(\tau-\tau_{ik})+E_{ik}\right)\right], \tag{3.1}$$

$$i=1,2,\ldots,N+1,$$

$$\frac{\partial^2 \bar{U}_i(\bar{x},\tau)}{\partial \tau^2}-\frac{\partial^2 \bar{U}_i(\bar{x},\tau)}{\partial \bar{x}^2}=0, \quad i=1,2,\ldots,N, \tag{3.2}$$

$$\bar{U}_i(\bar{x},\tau)=\bar{U}_0, \quad \frac{\partial \bar{U}_i(\bar{x},\tau)}{\partial \tau}=\bar{V}_{i0}, \quad \text{for} \quad \tau=0, \quad i=1,2,\ldots,N,$$

(3.3)

$$E_1\bar{F}_1(\tau)+A_1\frac{\partial^2 \bar{U}_1(\bar{x},\tau)}{\partial \tau^2}+B_1\frac{\partial \bar{U}_1(\bar{x},\tau)}{\partial \bar{x}}+D_1\left(\frac{\partial \bar{U}_1(\bar{x},\tau)}{\partial \tau}-\bar{V}_{10}\right)=0$$

$$\text{for } \bar{x}=0,$$

$$\bar{U}_{i-1}(\bar{x},\tau)=\bar{U}_i(\bar{x},\tau) \quad \text{for} \quad \bar{x}=i-l, \quad i=2,3,\ldots,N,$$

$$(3.4) \quad E_i \bar{F}_i(\tau)+A_i \frac{\partial^2 \bar{U}_{i-1}(\bar{x},\tau)}{\partial\tau^2}+B_i \frac{\partial \bar{U}_{i-1}(\bar{x},\tau)}{\partial\bar{x}}+C_i \frac{\partial \bar{U}_i(\bar{x},\tau)}{\partial\bar{x}}+ \\ +D_i\left(\frac{\partial \bar{U}_{i-1}(\bar{x},\tau)}{\partial\tau}-\bar{V}_{i-1,0}\right)=0 \quad \text{for} \quad \bar{x}=i-1, \quad i=2,3,\ldots,N,$$

$$E_{N+1}\bar{F}_{N+1}(\tau)+A_{N+1}\frac{\partial^2 \bar{U}_N(\bar{x},\tau)}{\partial\tau^2}+B_{N+1}\frac{\partial \bar{U}_N(\bar{x},\tau)}{\partial\bar{x}}+D_{N+1}\cdot \\ \left(\frac{\partial \bar{U}_N(\bar{x},\tau)}{\partial\tau}-\bar{V}_{N0}\right)=0 \quad \text{for} \quad \bar{x}=N.$$

Equations (3.2) and the boundary conditions (3.4) are linear. Thus we postulate solutions of the problem (3.2)–(3.4) in the following form:

$$(3.5) \quad \bar{U}_i(\bar{x},\tau)=U_{0i}(\bar{x},\tau)|_{\bar{V}_{k0}\neq\bar{V}_{k-1,0}}+U_{1i}(\bar{x},\tau)|_{\bar{F}_1\neq 0}+U_{2i}(\bar{x},\tau)|_{\bar{F}_2\neq 0}+ \\ +\ldots+U_{N+1,i}(\bar{x},\tau)|_{\bar{F}_{N+1}\neq 0}+\bar{V}_{i0}\tau+\bar{U}_0, \quad i=1,2,\ldots,N,$$

where the first term is the solution following from collision of the elements at instant $\tau=0$ (that is, with different initial velocities of one, at least, pair of neighbouring elements of the system); the term $U_{si}(\bar{x},\tau)$ is the nondimensional displacement of the i-th element related with the s-th force (that is, $\bar{F}_s(\tau)\neq 0$, disregarding the remaining forces) where $s=1,2,\ldots,N+1$.

We substitute Eq. (3.5) into the boundary conditions (3.4) and obtain boundary conditions successively for $U_{si}(\bar{x},\tau)$ $s=0,1,\ldots,N+1$. For example, for $U_{0i}(\bar{x},\tau)$ they are the following:

$$A_1\frac{\partial^2 U_{01}}{\partial\tau^2}+B_1\frac{\partial U_{01}}{\partial\bar{x}}+D_1\frac{\partial U_{01}}{\partial\tau}=0 \quad \text{for} \quad \bar{x}=0,$$

$$U_{0,i-1}+\bar{V}_{i-1,0}\tau=U_{0i}+\bar{V}_{i0}\tau \quad \text{for} \quad \bar{x}=i-1, \quad i=2,3,\ldots,N,$$

$$(3.6) \quad A_i\frac{\partial^2 U_{0,i-1}}{\partial\tau^2}+B_i\frac{\partial U_{0,i-1}}{\partial\bar{x}}+C_i\frac{\partial U_{0i}}{\partial\bar{x}}+D_i\frac{\partial U_{0,i-1}}{\partial\tau}=0 \\ \text{for} \quad \bar{x}=i-1, \quad i=2,3,\ldots,N,$$

$$A_{N+1}\frac{\partial^2 U_{0N}}{\partial\tau^2}+B_{N+1}\frac{\partial U_{0N}}{\partial\bar{x}}+D_{N+1}\frac{\partial U_{0N}}{\partial\tau}=0 \quad \text{for} \quad \bar{x}=N,$$

but for $U_{si}(\bar{x},\tau)$ with $s=2,3,\ldots,N$, they take the form

$$A_1\frac{\partial^2 U_{s1}}{\partial\tau^2}+B_1\frac{\partial U_{s1}}{\partial\bar{x}}+D_1\frac{\partial U_{s1}}{\partial\tau}=0 \quad \text{for} \quad \bar{x}=0,$$

$$(3.7) \quad A_i\frac{\partial^2 U_{s,i-1}}{\partial\tau^2}+B_i\frac{\partial U_{s,i-1}}{\partial\bar{x}}+C_i\frac{\partial U_{si}}{\partial\bar{x}}+D_i\frac{\partial U_{s,i-1}}{\partial\tau}=0 \\ \text{for} \quad \bar{x}=i-1, \quad i=2,3,\ldots,s-1,s+1,\ldots,N,$$

(3.7) [cont.]
$$E_s \bar{F}_s(\tau) + A_s \frac{\partial^2 U_{s,s-1}}{\partial \tau^2} + B_s \frac{\partial U_{s,s-1}}{\partial \bar{x}} + C_s \frac{\partial U_{ss}}{\partial \bar{x}} + D_s \frac{\partial U_{s,s-1}}{\partial \tau} = 0 \quad \text{for} \quad \bar{x} = s-1,$$

$$U_{s,i-1} = U_{si} \quad \text{for} \quad \bar{x} = i-1, \quad i = 2, 3, \ldots, N,$$

$$A_{N+1} \frac{\partial^2 U_{sN}}{\partial \tau^2} + B_{N+1} \frac{\partial U_{sN}}{\partial \bar{x}} + D_{N+1} \frac{\partial U_{sN}}{\partial \tau} = 0 \quad \text{for} \quad \bar{x} = N.$$

The boundary conditions for $U_{si}(\bar{x}, \tau)$ with $s=1$ and $s=N+1$ are of a similar form.

The components of the solution (3.5) are sought in the form of a sum of the following functions with the correspondingly selected arguments:

(3.8)
$$U_{si}(\bar{x}, \tau) = f_{si}(\tau - \tau^0_{si} - \bar{x} + \bar{x}^0_{si}) + g_{si}(\tau - \tau^0_{si} + \bar{x} - \bar{x}^0_{si}),$$
$$i = 1, 2, \ldots, N, \quad s = 0, 1, \ldots, N+1,$$

where τ^0_{si} is an instant, and $\bar{x}^0_{si}$ is a limit cross-section of the i-th element in which a disturbance has been caused either by the collision or the s-th force. We assume that for negative arguments the functions f_{si}, g_{si} are equal to zero.

Since in all the relations $(3.6)_{1,3,4}$ and $(3.7)_{1,2,3,5}$ derivatives of appropriate functions appear rather than the functions themselves, we differentiate Eqs. $(3.6)_2$, and $(3.7)_4$ with respect to time for convenience. Next we substitute the solutions (3.8) into the corresponding boundary conditions and thus obtain ordinary differential equations of the first and the second orders for the functions f_{si} and g_{si}. For every case of a mechanical system we have different constants τ^0_{si}, $\bar{x}^0_{si}$ and different ordinary differential equations.

We shall restrict our investigation to obtaining a solution for the displacement $U_{si}(\bar{x}, \tau)$ under action of the s-th force $\bar{F}_s(\tau)$ with $2 \leqslant s \leqslant N$. Then, for $i<s$ the first perturbation arrives at the i-th element to the cross-section $\bar{x}^0_{si} = i$ at instant $\tau^0_{si} = s-i-1$, while for $i \geqslant s$ we have $\bar{x}^0_{si} = i-1$ and $\tau^0_{si} = i-s$. Then the functions (3.8) take the form

(3.9)
$$U_{si}(\bar{x}, \tau) = f_{si}(\tau - \bar{x} + 2i - s + 1) + g_{si}(\tau + \bar{x} - s + 1) \quad \text{for} \quad i = 1, 2, \ldots, s-1,$$
$$U_{si}(\bar{x}, \tau) = f_{si}(\tau - \bar{x} + s - 1) + g_{si}(\tau + \bar{x} + s - 2i + 1) \quad \text{for} \quad i = s, s+1, \ldots, N,$$
$$s = 2, 3, \ldots, N.$$

It follows from the conditions (3.7) that the number of equations of the second order is dependent on the number of constants A_i different from zero. We shall consider the case in which all the constants A_i are different from zero. Then we substitute the relations (3.9) into the boundary conditions (3.7) and obtain the following set of $2N$ ordinary differential equations for the functions $f_{si}(z)$, $g_{si}(z)$:

(3.10)
$$f''_{s1}(z) + r_1 f'_{s1}(z) = -g''_{s1}(z-2) + h_1 g'_{s1}(z-2),$$
$$f'_{si}(z) = f'_{s,i-1}(z-2) + g'_{s,i-1}(z-2) - g'_{si}(z-2), \quad i = 2, 3, \ldots, s-1,$$
$$g'_{si}(z) = f'_{s,i+1}(z-2) + g'_{s,i+1}(z-2) - f'_{si}(z-2), \quad i = s, s+1, \ldots, N-1,$$

(3.10) [cont.]

$$g''_{sN}(z)+r_{N+1}g'_{sN}(z)=-f''_{sN}(z-2)+h_{N+1}f'_{sN}(z-2),$$

$$g''_{s,s-1}(z)+r_s g'_{s,s-1}(z)=-\frac{E_s}{A_s}\bar{F}_s(z)-f''_{s,s-1}(z)+h_s f'_{s,s-1}(z)+e_s g'_{ss}(z),$$

$$f'_{ss}(z)=f'_{s,s-1}(z)+g'_{s,s-1}(z)-g'_{ss}(z),$$

$$g''_{si}(z)+r_{i+1}g'_{si}(z)=-f''_{si}(z)+h_{i+1}f'_{si}(z)+e_{i+1}g'_{s,i+1}(z),$$

$$i=s-2,\ s-3,\ \dots,\ 1,$$

$$f''_{si}(z)+r_i f'_{si}(z)=-g''_{si}(z)+h_i g'_{si}(z)+e_i f'_{s,i-1}(z), \qquad i=s+1,\ s+2,\ \dots,\ N,$$

where

(3.11)

$$r_1=(D_1-B_1)/A_1, \qquad r_{N+1}=(B_{N+1}+D_{N+1})/A_{N+1},$$
$$r_i=(B_i-C_i+D_i)/A_i \quad \text{for} \quad i=2,3,\dots,N,$$
$$h_1=-(B_1+D_1)/A_1, \qquad h_{N+1}=(B_{N+1}-D_{N+1})/A_{N+1},$$
$$h_i=(B_i+C_i-D_i)/A_i \quad \text{for} \quad i=2,3,\dots,s,$$
$$h_i=-(B_i+C_i+D_i)/A_i \quad \text{for} \quad i=s+1,s+2,\dots,N,$$
$$e_i=-2C_i/A_i \quad \text{for} \quad i=2,3,\dots,s,$$
$$e_i=2B_i/A_i \quad \text{for} \quad i=s+1,s+2,\dots,N.$$

The arguments of the right-hand side functions of the first N equations of the set (3.10) are shifted by 2. This fact indicates that Eqs. (3.10) should be solved in the successive intervals of argument z that begin from even numbers. The functions f_{si}, g_{si} are assumed to be equal to zero for negative arguments; then the right-hand sides of these equations are always known as Eqs. (3.10) and are solved in the given succession.

The set of equations (3.10) for the derivatives of the functions f_{si}, g_{si} consists of $N-1$ algebraic equations and $N+1$ ordinary first-order equations with constant coefficients. The latter have the form of the following equation:

(3.12)
$$y'+Ay=P(x)$$

the solution of which for $x \geqslant x_0$ is as follows:

(3.13)
$$y(x)=e^{-A(x-x_0)}\left[\int_{x_0}^{x} P(z)\,e^{A(z-x_0)}\,dz+y(x_0)\right].$$

Then Eqs. (3.10) can be solved in f'_{si}, g'_{si} in a simple way: namely, we solve them in the given succession in the successive intervals of variability of the argument z beginning from even numbers, making use of the relation (3.13) in equations of the type (3.12). In this way in [7] a fixed rod impacted by a rigid body was solved but the solution was given only for $z \leqslant 8$. It follows from comparison of Eqs. (3.10) and Eq. (3.13) that the solution of Eqs. (3.10) will depend exponentially on the constants r_i. If all the constants r_i are different from each other, the solution for the derivatives of functions f_{si}, g_{si} for the force $\bar{F}_s(z)$ described by the formula (3.1) is the following with $s=2, 3, ..., N$:

$$2n \leqslant z < 2(n+1), \quad n=0, 1, ...,$$

$$(3.14) \quad g'_{sin}(z)=g'_{si,n-1}(z)+\sum_{k=1}^{m_s} H(z_{skn})\Big[\sum_{p=1}^{i} \exp(-r_p z_{skn}) \sum_{j=0}^{n-i+p-1} g_{pj}^{sikn} z_{skn}^j + \sum_{p=i+1}^{s} \exp(-r_p z_{skn}) \sum_{j=0}^{n} g_{pj}^{sikn} z_{skn}^j + \sum_{p=s+1}^{N+1} \exp(-r_p z_{skn}) \times \sum_{j=0}^{n+s-p} g_{pj}^{sikn} z_{skn}^j + g^{sikn} + g_L^{sikn} z_{skn} + g_s^{sikn} \sin(D_{sk} z_{skn}+E_{sk}) + g_c^{sikn} \cos(D_{sk} z_{skn}+E_{sk})\Big], \quad i=1, 2, ..., s-1,$$

$$(3.15) \quad g'_{sin}(z)=g'_{si,n-1}(z)+\sum_{k=1}^{m_s} H(z_{skn})\Big[\sum_{p=1}^{s-1} \exp(-r_p z_{skn}) \sum_{j=0}^{n-s+p-1} g_{pj}^{sikn} z_{skn}^j + \sum_{p=s}^{i+1} \exp(-r_p z_{skn}) \sum_{j=0}^{n-1} g_{pj}^{sikn} z_{skn}^j + \sum_{p=i+2}^{N+1} \exp(-r_p z_{skn}) \times \sum_{j=0}^{n+i-p} g_{pj}^{sikn} z_{skn}^j + g^{sikn} + g_L^{sikn} z_{skn} + g_s^{sikn} \sin(D_{sk} z_{skn}+E_{sk}) + f_c^{sikn} \cos(D_{sk} z_{skn}+E_{sk})\Big], \quad i=s, s+1, ..., N,$$

$$(3.16) \quad f'_{sin}(z)=f'_{si,n-1}(z)+\sum_{k=1}^{m_s} H(z_{skn})\Big[\sum_{p=1}^{i-1} \exp(-r_p z_{skn}) \sum_{j=0}^{n-i+p-1} f_{pj}^{sikn} z_{skn}^j + \sum_{p=i}^{s} \exp(-r_p z_{skn}) \sum_{j=0}^{n-1} f_{pj}^{sikn} z_{skn}^j + \sum_{p=s+1}^{N+1} \exp(-r_p z_{skn}) \times \sum_{j=0}^{n+s-p-1} f_{pj}^{sikn} z_{skn}^j + f^{sikn} + f_L^{sikn} z_{skn} + f_s^{sikn} \sin(D_{sk} z_{skn}+E_{sk}) + f_c^{sikn} \cos(D_{sk} z_{skn}+E_{sk})\Big], \quad i=1, 2, ..., s-1,$$

$$(3.17) \quad f'_{sin}(z)=f'_{si,n-1}(z)+\sum_{k=1}^{m_s} H(z_{skn})\Big[\sum_{p=1}^{s-1} \exp(-r_p z_{skn}) \sum_{j=0}^{n-s+p} f_{pj}^{sikn} z_{skn}^j + \sum_{p=s}^{i} \exp(-r_p z_{skn}) \sum_{j=0}^{n} f_{pj}^{sikn} z_{skn}^j + \sum_{p=i+1}^{N+1} \exp(-r_p z_{skn}) \times \sum_{j=0}^{n+i-p} f_{pj}^{sikn} z_{skn}^j + f^{sikn} + f_L^{sikn} z_{skn} + f_s^{sikn} \sin(D_{sk} z_{skn}+E_{sk}) + f_c^{sikn} \cos(D_{sk} z_{skn}+E_{sk})\Big], \quad i=s, s+1, ..., N,$$

where $f'_{si,n-1}(z)=g'_{si,n-1}(z)=0$ for $n=0$ and $z_{skn}=z-2n-\tau_{sk}$.

The derivatives of the functions $f_{si}(z)$, $g_{si}(z)$ are denoted in Eqs. (3.14)–(3.17) by $f'_{sin}(z)$, $g'_{sin}(z)$, respectively, depending on maximum even number which is smaller than the current argument z. As it follows from the formulae (3.14)–(3.17), the functions $f'_{si}(z)$, $g'_{si}(z)$ depend exponentially on the constants r_i, that is on the constants A_i, B_i, C_i, D_i. The varying coefficients standing at the exponential functions are power series in z_{skn}. The highest powers in these series are connected with a number of reflected waves. All the constants involved in Eqs. (3.14)–(3.17) have four upper indices. The first one provides information on the dependence on the s-th external force, the second — on belonging to the i-th function, the third — on the dependence on the k-th component of the s-th force and the fourth — on the interval to which the current argument z belongs. The constants involved in the power series have two lower indices. The first one is related with the constant r_p and the second one with the j-th power of term z_{skn}. The constants in Eqs. (3.14)–-(3.17) without lower indices are free terms, those with index L stand at the linear term, and those with index s or c stand at the functions sinus or cosinus, respectively.

It follows from the formulae (3.14)–(3.17) that the functions $f'_{si}(z)$, $g'_{si}(z)$ have different forms for $i<s$ and for $i \geqslant s$. Moreover, to determine the derivative of the corresponding function for argument z from a fixed interval, it is necessary to know the derivative of this function in all preceding intervals of argument z. This information follows from the first term of these formulae.

The constants involved in the formulae (3.14)–(3.17) are obtained while solving Eqs. (3.10) successively for $n=0, 1, \ldots$. In the algebraic relations they are algebraic sums of constants of the same type of the corresponding functions. These constants which are obtained using Eq. (3.13) for the functions $f'_{s1n}(z)$, $g'_{s,s-1,n}(z)$, and $f'_{sin}(z)$, $i=s+1, s+2, \ldots, N$, are given in the Appendix.

The constants involved in the solutions for the functions g'_{si}, where $i=N$, $s-2, s-3, \ldots, 1$, are similar to those given in the Appendix. The remaining constants of the formulae (3.15) and (3.16) are found from the algebraic relations of the set of equations (3.10). The above constants are given for $s=2, 3, \ldots, N$. The corresponding constants for $s=1$ or $s=N+1$ are determined in a similar way.

The formulae (3.14)–(3.17) and (3.8) can be utilized to determine the velocities and deformations of arbitrary cross-sections of elastic elements of the considered mechanical system at an arbitrary time instant. These velocities and deformations are caused by the s-th external force $\bar{F}_s(\tau)$ described by Eq. (3.1) for $s=2, 3, \ldots, N$. To determine the velocities and deformations of elements of the system caused by a collision, it is necessary to find the number of the constants A_i being different from zero, to use the boundary conditions (3.6) and to obtain the corresponding solutions for the derivatives of the functions $f_{0i}(z)$, $g_{0i}(z)$. Since the relation (3.13) is employed, it is necessary to take into account the discontinuity of the functions. In turn, after using the solutions (3.5) it is possible to obtain the velocities and deformations of elements of the mechanical system loaded by an arbitrary number of external forces, taking collisions into account. To obtain the displacements, the solutions for the derivatives of the functions $f_{si}(z)$, $g_{si}(z)$ should be integrated.

The formulae (3.14)–(3.17) do not regard particular cases of the mechanical system under consideration. They have been obtained with all A_i involved in the conditions (3.7) different from zero and with the constants r_i different from each other, $i=1, 2, ..., N+1$. Each particular case of the constants A_i, r_i requires a separate solution of the problem (3.2)–(3.4). The procedure and the character of the obtained results are the same. The difference in the solutions will consist only in numbers of the exponential functions and the upper limits of summation of the series. Moreover, no Heaviside function will appear in the case of collisions.

4. Example

As an example, we consider a model of a crankshaft of a one-cylinder four-stroke engine, Fig. 1, [8, 9]. The model consists of two elastically deformable main journals and of three rigid bodies having the mass moments of inertia I_1, I_2, I_3 with respect to the axis of rotations. The main journals (1) and (2) are subject only to torsional strains in accordance to the Hook law. Their flexural strains are disregarded because they are small (3%) as compared to torsional ones, [8, 9]. Each of the main journals

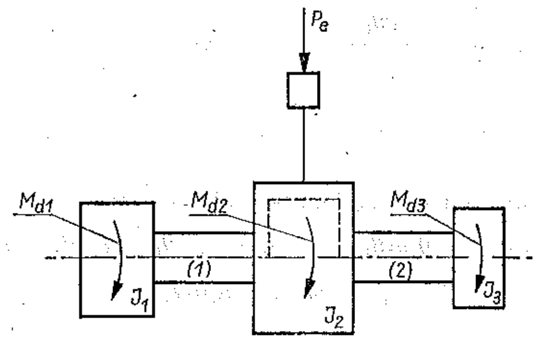

Fig. 1. The model of a crankshaft of a one-cylinder engine.

has length l, mass density ρ, shear modulus G and polar moment of inertia I_0: The rigid body (2) is loaded by the external moment $M_2(t)$ which depends on force P_e, Fig. 1, [8, 9]. In the example this moment is aproximated by the piece-wisely linear function

$$M_2(t)=\sum_{k=1}^{m_2} a_{2k} H(t-t_{2k}), \tag{4.1}$$

where a_{2k} are constants and t_{2k} are fixed time instants. Damping in the system is described by damping moments M_{di} loading rigid bodies. It is also assumed that in $t=0$ displacements and velocities of the cross-sections of the main journals are equal to zero.

Since the model is loaded only by moment $M_2(t)$, displacement U_i in p. 3 is equal to U_{2i} and henceforth is written as $\theta_i(x, t)$. Then the investigation of torsional displacements and strains in the main journals is reduced to solving two equations:

$$\frac{\partial^2 \theta_i}{\partial t^2}-c^2 \frac{\partial^2 \theta_i}{\partial x^2}=0, \quad i=1, 2, \tag{4.2}$$

with the boundary conditions

$$
(4.3)\qquad
\begin{aligned}
&I_1 \frac{\partial^2 \theta_1}{\partial t^2} - GI_0 \frac{\partial \theta_1}{\partial x} + d_1 \frac{\partial \theta_1}{\partial t} = 0 \quad \text{for} \quad x=0,\\
&M_2(t) - I_2 \frac{\partial^2 \theta_1}{\partial t^2} - GI_0 \left(\frac{\partial \theta_1}{\partial x} - \frac{\partial \theta_2}{\partial x}\right) - d_2 \frac{\partial \theta_1}{\partial t} = 0 \quad \text{for} \quad x=l,\\
&\theta_1 = \theta_2 \quad \text{for} \quad x=l,\\
&I_3 \frac{\partial^2 \theta_2}{\partial t^2} + GI_0 \frac{\partial \theta_2}{\partial x} + d_3 \frac{\partial \theta_2}{\partial t} = 0 \quad \text{for} \quad x=2l
\end{aligned}
$$

and initial conditions

$$
(4.4)\qquad \theta_i = \frac{\partial \theta_i}{\partial t} = 0 \quad \text{for} \quad t=0, \quad i=1,2,
$$

where $c^2 = G/\rho$, d_i are coefficients of equivalent damping, and the damping moments M_{di} in (4.3) were assumed:

$$
(4.5)\qquad
\begin{aligned}
&M_{d_0}(t) = -d_0 \frac{\partial \theta_0}{\partial t} \quad \text{for} \quad x=0,\\
&M_{di}(t) = -d_i \frac{\partial \theta_{i-1}}{\partial t} \quad \text{for} \quad x=(i-1)\,l, \quad i=2,3.
\end{aligned}
$$

Upon the introduction of nondimensional quantities

$$
\bar{x} = x/l, \quad \tau = ct/l, \quad \bar{\theta}_i = \theta_i/\theta_0, \quad \bar{M}_2 = M_2\,\theta_0\, l^2/I_1 c^2, \quad D_i = d_i\, l/I_1 c,
$$
$$
K_i = I_0\, \rho l/I_i, \quad E_i = I_1/I_i,
$$

the relations (4.2)–(4.4) take the form

$$
(4.6)\qquad \frac{\partial^2 \bar{\theta}_i}{\partial \tau^2} - \frac{\partial^2 \bar{\theta}_i}{\partial \bar{x}^2} = 0, \quad i=1,2,
$$

$$
(4.7)\qquad
\begin{aligned}
&\frac{\partial^2 \bar{\theta}_1}{\partial \tau^2} - K_1 \frac{\partial \bar{\theta}_1}{\partial \bar{x}} + D_1 \frac{\partial \bar{\theta}_1}{\partial \tau} = 0 \quad \text{for} \quad \bar{x}=0,\\
&E_2 \bar{M}_2(\tau) - \frac{\partial^2 \bar{\theta}_1}{\partial \tau^2} - K_1 E_2 \frac{\partial \bar{\theta}_1}{\partial \bar{x}} + K_2 \frac{\partial \bar{\theta}_2}{\partial \bar{x}} - D_2 E_2 \frac{\partial \bar{\theta}_1}{\partial \tau} = 0 \quad \text{for} \quad \bar{x}=1,\\
&\bar{\theta}_1 = \bar{\theta}_2 \quad \text{for} \quad \bar{x}=1,\\
&\frac{\partial^2 \bar{\theta}_2}{\partial \tau^2} + K_2 \frac{I_2}{I_3} \frac{\partial \bar{\theta}_2}{\partial \bar{x}} + D_3 \frac{I_1}{I_3} \frac{\partial \bar{\theta}_2}{\partial \tau} = 0 \quad \text{for} \quad \bar{x}=2,
\end{aligned}
$$

$$
(4.8)\qquad \bar{\theta}_i = \frac{\partial \bar{\theta}_i}{\partial \tau} = 0 \quad \text{for} \quad \tau=0, \quad i=1,2.
$$

According to the relations (3.9), the solutions $\bar{\theta}_i$ are sought in the form

$$
(4.9)\qquad \bar{\theta}_i(\bar{x}, \tau) = f_{2i}(\tau - \bar{x} + 1) + g_{2i}(\tau + \bar{x} - 1), \quad i=1,2
$$

and equations for the unknown functions $f_{2i}(z)$, $g_{2i}(z)$ are

$$
(4.10)\quad
\begin{aligned}
& f_{21}''(z)+r_1 f_{21}'(z)=-g_{21}''(z-2)+h_1 g_{21}'(z-2),\\
& g_{22}''(z)+r_3 g_{22}'(z)=-f_{22}''(z-2)+h_3 f_{22}'(z-2),\\
& g_{21}''(z)+r_2 g_{21}'(z)=E_2 \bar{M}_2(z)-f_{21}''(z)+h_2 f_{21}'(z)-e_2 g_{22}'(z),\\
& f_{22}'(z)=f_{21}'(z)+g_{21}'(z)-g_{22}'(z),
\end{aligned}
$$

where

$$
r_1=K_1+D_1, \quad h_1=K_1-D_1, \quad r_2=K_2+(K_1+D_2)E_2, \quad h_2=(K_1-D_2)E_2-K_2,
$$
$$
e_2=2K_2, \quad r_3=(K_2/E_2+D_3)E_3, \quad h_3=(K_2/E_2-D_3)E_3.
$$

The set of equations (4.10) can be obtained from Eqs. (3.10) for $N=2$ and $s=2$.

We start solving Eqs. (4.10) with the interval $0\leqslant z<2$ using Eq. (3.13) and for $r_1\neq r_2\neq r_3$. Then

$$
\begin{aligned}
& f_{210}'(z)=g_{220}'(z)=0,\\
& g_{210}'(z)=E_2\sum_{k=1}^{m_2} H(z-\tau_{2k})\left[g^{21k0}+g_{20}^{21k0}\exp\left(-r_2(z-\tau_{2k})\right)\right],\\
& f_{220}'(z)=E_2\sum_{k=1}^{m_2} H(z-\tau_{2k})\left[f^{22k0}+f_{20}^{22k0}\exp\left(-r_2(z-\tau_{2k})\right)\right],
\end{aligned}
$$

where

$$
g^{21k0}=\bar{a}_{2k}/r_2, \quad g_{20}^{21k0}=-g^{21k0}, \quad f^{22k0}=g^{21k0}, \quad f_{20}^{22k0}=g_{20}^{21k0}.
$$

For the next interval $2\leqslant z<4$, we get

$$
\begin{aligned}
f_{211}'(z)=E_2\sum_{k=1}^{m_2} H(z-2-\tau_{2k})\Big[f^{21k1}+f_{10}^{21k1}\exp\left(-r_1(z-2-\tau_{2k})\right)+\\
+f_{20}^{21k1}\exp\left(-r_2(z-2-\tau_{2k})\right)\Big],\\
g_{211}'(z)=E_2\sum_{k=1}^{m_2} H(z-2-\tau_{2k})\Big[g^{22k1}+g_{20}^{22k1}\exp\left(-r_2(z-2-\tau_{2k})\right)+\\
+g_{30}^{22k1}\exp\left(-r_3(z-2-\tau_{2k})\right)\Big],\\
g_{211}'(z)=g_{210}'(z)+E_2\sum_{k=1}^{m_2} H(z-2-\tau_{2k})\Big[g^{21k1}+g_{10}^{21k1}\exp\left(-r_1(z-2-\tau_{2k})\right)+\\
+\exp\left(-r_2(z-2-\tau_{2k})\right)\sum_{j=0}^{1} g_{2j}^{21k1}(z-2-\tau_{2k})^j+g_{30}^{21k1}\exp\left(-r_3(z-2-\tau_{2k})\right)\Big],
\end{aligned}
$$

where $f^{21k1}=h_1 g^{21k0}/r_1$, $f_{20}^{21k1}=(r_2+h_1)g_{20}^{21k0}/(r_1-r_2)$, $f_{10}^{21k1}=$
$=-f^{21k1}-f_{20}^{21k1}$,

$g^{22k1}=h_3 f^{22k0}/r_3$, $g_{20}^{22k1}=-(r_2+h_3)f_{20}^{22k0}/(r_2-r_3)$, $g_{30}^{22k1}=-g^{22k1}-g_{20}^{22k1}$,

$$g^{21k1}=(h_2 f^{21k1}+e_2 g^{22k1})/r_2, \quad g_{10}^{21k1}=-(r_1+h_2)f_{10}^{21k1}/(r_1-r_2),$$
$$g_{21}^{21k1}=(r_2+h_2)f_{20}^{21k1}+e_2 g_{20}^{22k1}, \quad g_{30}^{21k1}=e_2 g_{30}^{22k1}/(r_2-r_3),$$
$$g_{20}^{21k1}=-g^{21k1}-g_{10}^{21k1}-g_{30}^{21k1},$$

while function $f'_{221}(z)$ is the algebraic sum of $f'_{211}(z)$, $g'_{211}(z)$ and $g'_{221}(z)$.

After analogous transformations for $z \geqslant 4$ one notes that the analytical solutions for the functions $g'_{2in}(z)$ and $f'_{2in}(z)$ may be finally represented by the formulae (3.14)–(3.17) for $N=2$, $s=2$ and with $g_L^{2ikn}=g_s^{2ikn}=g_c^{2ikn}=f_L^{2ikn}=f_s^{2ikn}=f_c^{2ikn}=0$. All remaining constants in the solutions can te deduced from the relations given above and from the formulae presented in the Appendix.

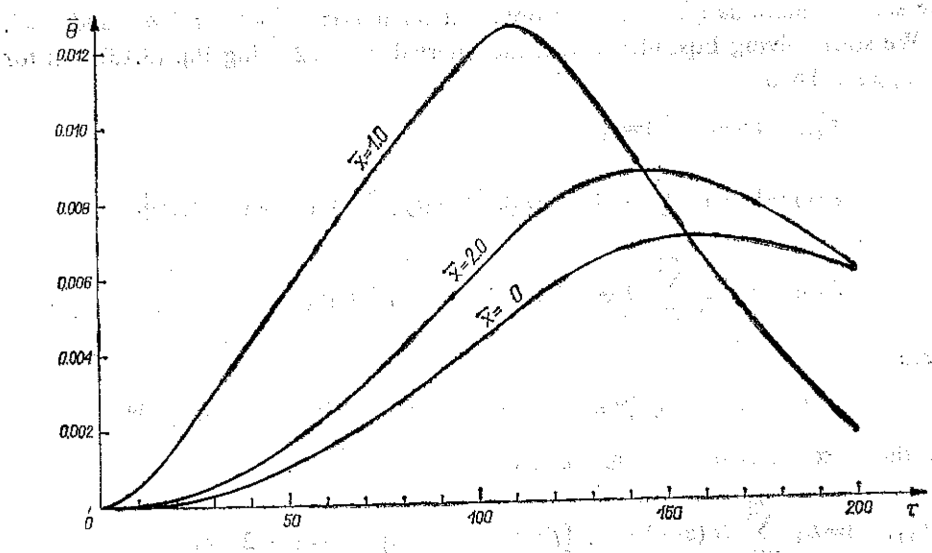

FIG. 2. The displacements of the cross-sections $\bar{x}=0, 1, 2$ of the model of a crankshaft of a one-cylinder engine.

After integration of the appropriate formulae (3.14)–(3.17) and using the relations (4.9), numerical calculations were performed for $K_1=0.005$, $K_2=0.00035$, $E_2=K_2/K_1$, $E_3=0.137/K_1$, $D_1=0.5$, $D_2=1.0$, $D_3=0.3$ and for the external moment $\bar{M}_2(\tau)=0.00065\,[0.27H(\tau)-0.48H(\tau-104)]$. The plots of the displacements for the cross-sections $\bar{x}=0$, 1 and 2 of the main crankshaft journals for $0 \leqslant \tau \leqslant 200$ are drawn in Fig. 2. It follows from Fig. 2 that the maximum displacements occur in the cross-section $\bar{x}=1$.

5. CONCLUSIONS

The paper presents a method to investigate a chosen group of mechanical systems, the elastic elements of which deform longitudinally or torsionally. It uses one-dimensional waves and is based on a proper selection of the arguments fo functions (3.8) and special construction of the recurrence formulae developed by the author.

The equations of motion and boundary conditions are given for a system consisting of an arbitrary number of elastic elements interconnected by rigid bodies. A solution is obtained for such a mechanical system for which Eqs. (3.10) consist of $N-1$ algebraic equations and $N+1$ differential equations of the type (3.12) with different constants r_i. Similar solutions are obtainable for other numerous mechanical systems described by the relations (3.2)–(3.4). Differences concern only the number of exponential functions and upper limits of the series.

Analytical solutions for mechanical systems described by the relations (3.2)–(3.4) are obtained in the form of recurrence formulae. Another method of obtaining solutions also in the form of recurrence formulae has been proposed by Donnell [6]. However, this method is not very efficient because it can be employed only for very simple mechanical systems [4] within a very short time interval. On the other hand, the method proposed here is applicable for many complex mechanical systems over an arbitrary interval of time. The obtained recurrence formulae allow to determine velocities, strains and displacements at arbitrary cross-sections of elastic elements and at an arbitrary time instant, taking into account equivalent damping.

As an example of the application of the method, the model of a crankshaft of a one-cylinder engine is considered.

Appendix

The constants appearing in the solutions (3.14), (3.16) and (3.17) for the functions $f'_{sin}(z)$, $g'_{s,s-1,n}(z)$ and $f'_{sin}(z)$, $i=s+1, s+2, \ldots, N$, are:

$$f_L^{s1kn}=h_1\, H(n-1)\, g_L^{s1k,n-1}/r_1,$$

$$f^{s1kn}=(h_1\, g^{s1k,n-1}-f_L^{s1kn}-g_L^{s1k,n-1})\, H(n-1)/r_1,$$

$$w_s^{1n}=(h_1\, g_s^{s1k,n-1}+D_{sk}\, g_c^{s1k,n-1})\, H(n-1),$$

$$w_c^{1n}=(h_1\, g_c^{s1k,n-1}-D_{sk}\, g_s^{s1k,n-1})\, H(n-1),$$

$$f_s^{s1kn}=(r_1\, w_s^{1n}+D_{sk}\, w_c^{1n})/(r_1^2+D_{sk}^2),$$

$$f_c^{s1kn}=(r_1\, w_c^{1n}-D_{sk}\, w_s^{1n})/(r_1^2+D_{sk}^2),$$

$$f_{1j}^{s1kn}=-H(n-2-j)\, g_{1j}^{s1k,n-1}+(r_1+h_1)\, g_{1,j-1}^{s1k,n-1}/j, \quad j=n-1, n-2, \ldots, 1,$$

$$f_{pj}^{s1kn}=[-(j+1)\, H(n-2-j)\,(g_{p,j+1}^{s1k,n-1}+f_{p,j+1}^{s1kn})+(r_p+h_1)\, g_{pj}^{s1k,n-1}]/(r_1-r_p),$$
$$p=2, 3, \ldots, s, \quad j=n-1, n-2, \ldots, 0,$$

$$f_{pj}^{s1kn}=[-(j+1)\, H(n+s-p-2-j)(g_{p,j+1}^{s1k,n-1}+f_{p,j+1}^{s1kn})+(r_p+h_1)\, g_{pj}^{s1k,n-1}]/(r_1-r_p),$$
$$p=s+1, s+2, \ldots, N+1, \quad j=n+s-p-1, n+s-p-2, \ldots, 0,$$

$$f_{10}^{s1kn} = -f^{s1kn} - H(n-1)\sum_{p=2}^{s} f_{p0}^{s1kn} - \sum_{p=s+1}^{N+1} H(n+s-p-1) f_{p0}^{s1kn} -$$
$$- f_s^{s1kn}\sin E_{sk} - f_c^{s1kn}\cos E_{sk},$$

$$g_L^{s,s-1,kn} = -E_s B_{sk}/A_s r_s \quad \text{for} \quad n=0,$$
$$g^{s,s-1,kn} = -(E_s A_{sk}/A_s + g_L^{s,s-1,kn})/r_s \quad \text{for} \quad n=0,$$
$$g_s^{s,s-1,kn} = -r_s E_s C_{sk}/A_s (r_s^2 + D_{sk}^2) \quad \text{for} \quad n=0,$$
$$g_c^{s,s-1,kn} = E_s D_{sk} C_{sk}/A_s (r_s^2 + D_{sk}^2) \quad \text{for} \quad n=0,$$
$$g_L^{s,s-1,kn} = (h_s f_L^{s,s-1,kn} + e_s g_L^{sskn})/r_s \quad \text{for} \quad n \geqslant 1,$$
$$g^{s,s-1,kn} = (h_s f^{s,s-1,kn} + e_s g^{sskn} - f_L^{s,s-1,kn} - g_L^{s,s-1,kn})/r_s \quad \text{for} \quad n \geqslant 1,$$
$$v_s^{s-1,n} = h_s f_s^{s,s-1,kn} + D_{sk} f_c^{s,s-1,kn} + e_s g_s^{sskn} \quad \text{for} \quad n \geqslant 1,$$
$$v_c^{s-1,n} = h_s f_c^{s,s-1,kn} - D_{sk} f_s^{s,s-1,kn} + e_s g_c^{sskn} \quad \text{for} \quad n \geqslant 1,$$
$$g_s^{s,s-1,kn} = (r_s v_s^{s-1,n} + D_{sk} v_c^{s-1,n})/(r_s^2 + D_{sk}^2) \quad \text{for} \quad n \geqslant 1,$$
$$g_c^{s,s-1,kn} = (r_s v_c^{s-1,n} - D_{sk} v_s^{s-1,n})/(r_s^2 + D_{sk}^2) \quad \text{for} \quad n \geqslant 1,$$

$$g_{pj}^{s,s-1,kn} = [(j+1)H(n-s+p-1-j)(f_{p,j+1}^{s,s-1,kn} + g_{p,j+1}^{s,s-1,kn}) - (r_p + h_s)\times$$
$$\times f_{pj}^{s,s-1,kn} - e_s H(n-s+p-1-j) g_{pj}^{sskn}]/(r_p - r_s), \quad p=1,2,\ldots,s-1,$$
$$j = n-s+p, n-s+p-1, \ldots, 0,$$

$$g_{sj}^{s,s-1,kn} = -H(n-1-j) f_{sj}^{s,s-1,kn} + [(r_s + h_s) f_{s,j-1}^{s,s-1,kn} + e_s g_{s,j-1}^{sskn}]/j,$$
$$j = n, n-1, \ldots, 1,$$

$$g_{pj}^{s,s-1,kn} = [-(j+1)(H(n+s-p-2-j) f_{p,j+1}^{s,s-1,kn} + H(n+s-p-1-j) g_{p,j+1}^{s,s-1,kn}) +$$
$$+ (r_p + h_s) H(n+s-p-1-j) f_{pj}^{s,s-1,kn} + e_s g_{pj}^{sskn}]/(r_s - r_p),$$
$$p = s+1, s+2, \ldots, N+1, \quad j = n+s-p, n+s-p-1, \ldots, 0,$$

$$g_{s0}^{s,s-1,kn} = -g^{s,s-1,kn} - \sum_{p=1}^{s-1} H(n-s+p) g_{p0}^{s,s-1,kn} - \sum_{p=s+1}^{N+1} H(n+s-p)\times$$
$$\times g_{p0}^{s,s-1,kn} - g_s^{s,s-1,kn}\sin E_{sk} - g_c^{s,s-1,kn}\cos E_{sk},$$

$$f_L^{sikn} = [h_i H(n-1) g_L^{sikn} + e_i f_L^{s,i-1,kn}]/r_i, \quad i = s+1, s+2, \ldots, N,$$
$$f^{sikn} = [h_i H(n-1) g^{sikn} + e_i f^{s,i-1,kn} - H(n-1) g_L^{sikn} - f_L^{sikn}]/r_i,$$
$$i = s+1, s+2, \ldots, N,$$

$$w_s^{in} = H(n-1)(h_i g_s^{sikn} + D_{sk} g_c^{sikn}) + e_i f_s^{s,i-1,kn}, \quad i = s+1, s+2, \ldots, N,$$
$$w_c^{in} = H(n-1)(h_i g_c^{sikn} - D_{sk} g_s^{sikn}) + e_i f_c^{s,i-1,kn}, \quad i = s+1, s+2, \ldots, N,$$
$$f_s^{sikn} = (r_i w_s^{in} + D_{sk} w_c^{in})/(r_i^2 + D_{sk}^2), \quad i = s+1, s+2, \ldots, N,$$
$$f_c^{sikn} = (r_i w_c^{in} - D_{sk} w_s^{in})/(r_i^2 + D_{sk}^2), \quad i = s+1, s+2, \ldots, N,$$

$$f_{pj}^{sikn} = [(j+1)\big(H(n-s+p-1-j) f_{p,j+1}^{sikn} + H(n-s+p-2-j) g_{p,j+1}^{sikn}\big) - \\ -(r_p+h_i) H(n-s+p-1-j) g_{pj}^{sikn} - e_i f_{pj}^{s,i-1,kn}]/(r_p-r_i),$$

$$i=s+1, s+2, \ldots, N, \quad p=1, 2, \ldots, s, \quad j=n-s+p, n-s+p-1, \ldots, 0,$$

$$f_{pj}^{sikn} = [(j+1)\big(H(n-1-j) f_{p,j+1}^{sikn} + H(n-2-j) g_{p,j+1}^{sikn}\big) - (r_p+h_i) \times \\ \times H(n-1-j) g_{pj}^{sikn} - e_i f_{pj}^{s,i-1,kn}]/(r_p-r_i),$$

$$i=s+1, s+2, \ldots, N, \quad p=s+1, s+2, \ldots, i-1, \quad j=n, n-1, \ldots, 0,$$

$$f_{ij}^{sikn} = -H(n-1-j) g_{ij}^{sikn} + [(r_i+h_i) g_{i,j-1}^{sikn} + e_i f_{i,j-1}^{s,i-1,kn}]/j,$$

$$i=s+1, s+2, \ldots, N, \quad j=n, n-1, \ldots, 1,$$

$$f_{pj}^{sikn} = [-(j+1) H(n+i-p-1-j)(f_{p,j+1}^{sikn} + g_{p,j+1}^{sikn}) + (r_p+h_i) g_{pj}^{sikn} + \\ + e_i H(n+i-p-1-j) f_{pj}^{s,i-1,kn}]/(r_i-r_p), \quad i=s+1, s+2, \ldots, N,$$

$$p=i+1, i+2, \ldots, N+1, \quad j=n+i-p, n+i-p-1, \ldots, 0,$$

$$f_{i0}^{sikn} = -f^{sikn} - \sum_{p=1}^{s} H(n-s+p) f_{p0}^{sikn} - \sum_{p=s+1}^{i-1} f_{p0}^{sikn} - \sum_{p=i+1}^{N+1} H(n+i-p) f_{p0}^{sikn} - \\ -f_s^{sikn} \sin E_{sk} - f_c^{sikn} \cos E_{sk}, \quad i=s+1, s+2, \ldots, N.$$

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Streszczenie

WYKORZYSTANIE SKRĘTNYCH I PODŁUŻNYCH FAL SPRĘŻYSTYCH W UKŁADACH MECHANICZNYCH

W pracy wykorzystano jednowymiarowe fale sprężyste do badania układów mechanicznych złożonych z dowolnej liczby elementów sprężystych i brył sztywnych. Układy te są obciążone siłami nieokresowymi oraz mogą w nich występować zderzenia. Elementy sprężyste tych układów mają skończone długości oraz odkształcają się wzdłużnie lub skrętnie. Rozwiązania analityczne otrzymano

w postaci wzorów rekurencyjnych, które pozwalają wyznaczać prędkości, odkształcenia i przemieszczenia dowolnych przekrojów poprzecznych elementów sprężystych w dowolnej chwili czasu przy uwzględnieniu tłumienia zastępczego. Jako przykład rozpatrzono model wału korbowego silnika jednocylindrowego.

Р е з ю м е

ИСПОЛЬЗОВАНИЕ ТОРСИОННЫХ И ПРОДАЛЬНЫХ УПРУГИХ ВОЛН В МЕХАНИЧЕСКИХ СИСТЕМАХ

В статье используются одмомерные упругие волны для исследования механических систем, состоящих из произвольного числа других элементов и жестких тел. Эти системы нагружени непориодическими силами, кромо того в них могут выступать соударения. Упручие элементы этих систем имеют конечную длину, а продальные деформации кручения. Аналитические решения получены в виде рекурретных формул, которые позволяют определять скорости, деформации и перемещения произвольных поперечных сечей упругих элементов в произвольный момент времени с учетом заменного демпрофирования. В качестве примера рассматривается модель коленчатого вала одноцилиндтого двигателя.

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