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THE TERMS AND SHAPES OF TNE FUNCTIONS NECESSARY FOR CALCULATION OF THE DISSIPATED POWER IN THE PROBLEM OF INDENTATION OF A PUNCH INTO A TWO-PLY MATERIAL

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The paper presents a method of calculation of the majorant of the limit load for the example of a punch into a semispace consisting of an isotropic laminar material. Proving the validity of the suggested theorems and lemma for a selected class of strain types, the calculation of the minimum dissipated power has been significantly simplified. The problem of finding the minimum of the functional has been reduced to the search for the minimum of a function of a single variable.

1. Introduction

The general formula for calculation of the dissipated power at the point M (Fig. 1) is in the form [6, 8, 9, 13, 19]

$$dP(M)=k(M)\left(\frac{\partial v_\alpha}{\partial s_\beta}+\frac{\partial v_\beta}{\partial s_\alpha}+\frac{v_\alpha}{R}+\frac{v_\beta}{S}\right)(ds_\alpha\, ds_\beta), \tag{1.1}$$

where v_α and v_β — coordinates of the velocity of the point M on $\mathbf{M}_\alpha$ and $\mathbf{M}_\beta$. The total dissipated power is equal to

$$P=\int dP(M). \tag{1.2}$$

In the case when a line of velocity discontinuity occurs, the expressions (1.1) and (1.2) are true in the meaning of the Stieltjes integral.

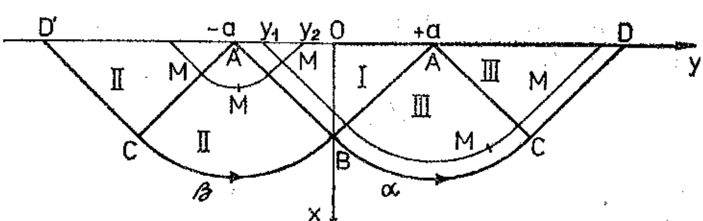

Fig. 1.

In order to use the kinematic method for the determination of the limit load, it is necessary to find the admissible strain field corresponding to the minimum of the total dissipated power. Since it is ruled out to search the for minimum of the dissipated power in the set of all admissible strain fields, so is it necessary to confine oneself to locking for this minimum within a certain class (of the strain fields).

The admissible strain fields which will be considered contain a slip line net of the Prandtl solution for the case of a homogenous medium (Fig. 1).

The slip velocity under the punch is denoted by $v(y)$ y_1 and y_2 ar denoted as in Fig. 1.

According to Eqs. (1.1) and (1.2), the total dissipated power is equal to

$$(1.3)\qquad P=\int_{\mathrm{I,\,II,\,III}} K(M)\left|\frac{\partial v_\alpha}{\partial s_\beta}+\frac{\partial v_\beta}{\partial s_\alpha}+\frac{v_\alpha}{R}+\frac{v_\beta}{S}\right|(ds_\alpha\, ds_\beta)+ \\ +\int_{\mathrm{D'\,CBA}} K(M)|v(a)-U|\,ds+\int_{\mathrm{A'\,BCD}} K(M)\,|U+v(-a)|\,ds+ \\ +\int_{-v}^{a} m(M)\,K(M)\,|v(y)|\,dy,$$

where $m(M)$ and $K(M)$ — the coefficient of roughness and the yield point in pure shearing, respectively (both values determined at the point M).

The integrals in the expression (1.3) are Stieltjes integrals. Substituting $v(-a^-)= =-U$ and $v(a^+)=U$ into Eqs. (1.3) yields.

$$(1.4)\qquad P=\int K(M)\left(\frac{\partial v_\alpha}{\partial s_\beta}+\frac{\partial v_\beta}{\partial s_\alpha}+\frac{v_\alpha}{R}+\frac{v_\beta}{S}\right)(ds_\alpha ds_\beta)+\int_{-a}^{a} m(y)\,K(0,y)\,|v(y)|\,dy$$

if

$$(1.5)\qquad P=P'+\int_{-a}^{a} m(y)\,K(0,y)\,|v(y)|\,dy,$$

where

$$(1.6)\qquad P'=\int K(M)\left(\frac{\partial v_\alpha}{\partial s_\beta}+\frac{\partial v_\beta}{\partial s_\alpha}+\frac{v_\alpha}{R}+\frac{v_\beta}{S}\right)(ds_\alpha\, ds_\beta).$$

If $v(y)$ is a limited and not decreasing variable in the interval $(-a^-, aa^+)$, then the component under the $|\ |$ sign in the expression for P' (1.6) is always positive and P' is linear in the interval $(v(y), U)$. The present considerations are confined to looking for the minimum of the dissipated power only within the class of strain types defined by Fig. 1.

2. General results

The considered strain types can be connected with the classical incomplete solution of Prandtl for the case of a smooth punch, acting on a homogeneous half--space [20].

The theorem of symmetry

If $k(M)$ and $m(M)$ are symmetrical (even) functions with respect to the axis $OX\{k(X,y)=k(X-y),\ m(y)=m(-y)\}$, then the minimum of the dissipated power is obtained for the symmetrical type of strain.

Proof

If an arbitrary type of deformation is considered, there is always a symmetrical kind of deformation which corresponds to a dissipated power being not higher. Let us assume that a kind of deformation is given, which corresponds to the function $v(y)$ and to the total dissipated power

$$P[v(y)]=P'[v(y)]+\int_{-a}^{a} m(y)\,k(0,y)\,|v(y)|\,dy \tag{2.1}$$

as well as a kind of deformation being symmetrical to the above, which corresponds to the function $\omega(y)=-v(-y)$.

The dissipated power which corresponds to this kind of deformation is—by the symmetry of the quantities m and k — equal to the previous one:

$$P[\omega(y)]=P'[\omega(y)]+\int_{-a}^{a} m(y)\,k(0,y)\,|\omega(y)|\,dy=P[v(y)]. \tag{2.2}$$

Thus, if a symmetrical kind of deformation is taken into account, to which the function $v(y)=\frac{1}{2}[v(y)+\omega(y)]$ corresponding, then, considering the linearity of the function P', we can write

$$P[v(y)]=\frac{P'[v(y)]+P'[\omega(y)]}{2}+\frac{1}{2}\int_{-a}^{a} m(y)\,k(0,y)\,|v(y)+\omega(y)|\,dy. \tag{2.3}$$

Hence

$$P[v(y)]\leqslant P'[v(y)]+\int_{-a}^{a} m(y)\,k(0,y)\,|v(y)|\,dy=P[v(y)]. \tag{2.4}$$

If the values $k(M)$ and $m(M)$ are symmetrical and if only the symmetrical kinds of deformation are considered, then the function $P[v(y)]$ is a linear one, in the interval $(-U, U)$. Consequently, the function $P'[v(y)]$ is linear and the function $v(y)$ is necessarily positive on $(0, a)$, and from this it follows that the expression

$$\int_{-a}^{a} k(0,y)\,m(y)\,|v(y)|\,dy=2\int_{0}^{a} k(0,y)\,m(y)\,|v(y)|\,dy \tag{2.5}$$

is linear.

2.1. *The general theorem*

The hypothesis of symmetry is assumed and the symmetrical kinds of deformation are considered. Let E_u denote the set of functions $v(y)$, satisfying the hypotheses A:

$$\left.\begin{array}{ll}\text{determinate on } -a^-,\ +a^+ & A_1\\ \text{with bounded variations (variability)} & A_2\\ \text{not decreasing} & A_3\\ \text{symmetric} & A_4\\ v(-a^-)=-U,\ v(a^+)=U & A_5\end{array}\right\}A,$$

E'_u denotes a subset of the set E_u, formed by functions, satisfying the hypotheses A and containing two symmetrical jumps of amplitude U:

$$\nu(y) = -U + UH(y+z) + UH(y-z), \quad y \in (-a, a^+), \quad z \in (0, a).$$

The function $P[\nu(y)]$, expressing the total dissipated power, is a functional in the set E_u.

It has been assumed that $G[\nu(y)]$ is a linear functional $\nu(y)$, for $\nu(y)$ satisfying the hypotheses A_1 through A_4. An arbitrary positive constant is denoted by α and a set E_α is determined of the functions $\nu(y)$ satisfying the hypothese B:

$$\left.\begin{array}{ll} B_1 \text{ through } B_4 \text{ are identical with } A_1 \text{ through} & A_4 \\ \nu(-a^-) = -\alpha,\ \nu(a^+) = \alpha, & B_5 \end{array}\right\} B.$$

Finally E_α, a subset of E_α, has been determined, formed from functions satisfying the hypotheses B and containing two symmetrical jumps with the amplitude α:

$$\nu_y(y) = -\alpha + \alpha H(y+z) + \alpha H(y-z), \quad y \in (-a^-, a^+),$$

where z — the abscissa of the jump on the right hand side (a function parameter), $z \in (0, a)$, H — Heaviside's function.

LEMMA 1

If for a given value α, the functional $G[\nu(y)]$ reaches minimum in the set E_α for $\nu_{z0}(y)$, then the functional $G[\nu(y)]$ reaches minimum in the set E'_α for each positive value α and this minimum corresponds to the same value z_0 from z, thus it does not depend on the value α. This means that many values z can occur here, corresponding to the minimum of the functional $G[\nu(y)]$ in the set E'_α and the result is still valid.

The proof is evident because the functional $G[\nu(y)]$ is linear and the functions E'_{α_1} and E'_{α_2} are homothetic in connection with the positive value of α_2/α_1.

LEMMA 2

If the minimum of the functional $G[\nu(y)]$ in the set E'_α is attainable for the functions $\nu_{z_1}(y)$ and $\nu_{z_2}(y)$, $z_1 \neq z_2$, then the whole function $\nu(y)$ from the set E_α in the form:

$$\nu(y) = \lambda \nu_{z_1}(y) + (1-\lambda)\nu_{z_2}(y), \quad 0 \leqslant \lambda \leqslant 1,$$

is such that

$$G[\nu(y)] = \min_{E_\alpha} G[\nu(y)].$$

This result springs from th linearity of the functional $G[\nu(y)]$ and it can be generalized for more than two functions.

NOTE. The verbal nomenclature of the introduced notions is omitted in further considerations, in order to obtain a more perspicuous notation. Only the corresponding symbols are used.

THEOREM

a) *If G has a minimum in* E'_u, *attainable for only one function* $v_z(y)$, *then this function realizes also the rigorous minimum G in* E_u.

b) *If G has a minimum in* E'_u, *attainable for two functions* $v_{z_1}(y)$ *and* $v_{z_2}(y)$, *then the minimum in* E_u *is equal to the former one and it is attainable only for the functions*

$$v(y)=\lambda v_{z_1}(y)+(1-\lambda)\, v_{z_2}(y), \quad 0\leqslant\lambda\leqslant 1.$$

c) *If G is constant in* E'_u, *then it is constant in* E_u.

Proof

a) Let us have $v(y)\in E_u$. The differential of v on the abscissa z is denoted by $dv(z)$ for $z\in(0, a)$, i.e.

$dv(z)=\frac{dv}{dy}(z)\, dy$, where $\frac{dv}{dy}(z)$ is taken in the meaning of a distribution; thus $dv(z)$ can be a finite quantity (in the case when $v(z)$ has a jump on the abscissa z, $dv(z)$ is positive).

Considering the function $E'\, dv(z)$,

$$d\tilde{v}_z(y)=-dv(z)+dv(z)\, H(y+z)+dv(z)\, H(y-z) \tag{2.6}$$

it can be easily proved that

$$v(y)=\int_{z=0}^{z=a+} d\tilde{v}_z(y), \quad \forall y\in(-a, a),$$

therefore

$$G[v(y)]=G\left[\int_{z=0}^{z=a+} d\tilde{v}_z(y)\right] \tag{2.7}$$

and from the linearity of G.

$$G[v(y)]=\int_{z=0}^{z=a} G[d\tilde{v}_z(y)]. \tag{2.8}$$

Let z_0 be thé abscissa, corresponding to the minimum of G in E'_α $(\forall\alpha>0)$, then

$$G[d\tilde{v}_z(y)\geqslant\frac{d\tilde{v}_z(z)}{U}\{\min G \text{ in } E'_u\} \tag{2.8}$$

what is satisfied only for $z=z_0$.

From Eqs. (2.8) and (2.9) it follows that

$$G[v(y)]>\min G \text{ in } E'_u,$$

except for the case when $v(y)=-U+U\cdot H(y+z_0)+UH(y-z_0)$, i.e. the function which realizes the minimum of G in E'_u.

The proof for b) and c) is analogous.

NOTE. If the problem from Sect. 2 is investigated by discretizing it, i.e. by confining it to the staircase function $v(y)$, from the jumps v_i to the stationary abscissae y_i, satisfying the hypotheses A, then we arrive to the problem of linear programming. The basic theorem of linear programming indicates that the minimum of $G[v(y)]$ is attained for the function $v(y)$ which has two symmetrical jumps.

The general theorem proved here can be extended onto the case where all functions satisfying the hypothesis A are considered.

In order to find the minimum of the dissipated power in a determined class of deformation kinds Paragraph 1, assuming the hypothesis of symmetry, it is possible according to Paragraph 2, to confine oneself to the symmetrical kinds of deformation.

Applying the proved theorem, it is possible to confine oneself to such kinds of deformations for which $v(y)$ consists of two symmetrical jumps with the amplitude U.

Thus the search for minimum of the dissipated power is reduced to looking for the minimum of $G[v(y)]$ on E_u'. On E_u', $G[v_z(y)]$ is a function of z: $U\,G(z)$, $z \in (0, a)$. Investigation of this function (calculation of the minimum) is thus much simpler than the original problem.

3. Application of the obtained results to the case of a two-ply material

For the case of a two-ply material the problem is reduced to searching for the best majorant of the limit load, obtained in the class of kinematic solutions, the deformation kinds of which can be linked two the incomplete solution of Prandtl. All the hypotheses from Sect. 2 are satisfied in this case. The final results gained in this Sect. 2 can be applied in this case. The final results gained in this Sect 2 can be applied. The yield point in pure shearing for the lower layer is denoted by K and for the upper layer—by k.

3.1. Calculation of the dissipated power

The expression for the dissipated power is changing with the value h/a. The following cases are considered:

1. The case $h \geqslant a\sqrt{2}$

$$\frac{P'[v(y)]}{2} = k(\pi+2)\,dU,$$

$$\frac{P[v(y)]}{2} = \frac{P'[v(y)]}{2} + mk\int_0^a v(y)\,dy.$$

2. The case $a \leqslant h \leqslant \sqrt{2}a$

$$\frac{P'[v(y)]}{2} = k(\pi+2)\,aU + (K-k)\left\{\int_h^{a+\sqrt{2}} 2r \arccos\frac{h}{r}\,\frac{dv}{dy}\,dr + \right.$$

$$\left. + \int_h^{a+\sqrt{2}} \sqrt{2}\arccos\frac{h}{r}\,[U - v(y)]\,dr\right\},$$

where $r=(a+y)\frac{\sqrt{2}}{2}$

$$\frac{P[\nu(y)]}{2}=\frac{P'[\nu(y)]}{2}+mk\int_0^a \nu(y)\,dy,$$

3. The case $0\leqslant h\leqslant a$

$$\frac{P'[\nu(y)]}{2}=k(\pi+2)aU+(K-k)\left\{\int_h^{h\sqrt{2}} 2r\arccos\frac{h}{r}\,\frac{d\nu}{dy}\,dr+\right.$$
$$+\int_h^{h\sqrt{2}}\sqrt{2}\arccos\frac{h}{r}\,[U-\nu(y)]\,dr+\int_{h\sqrt{2}}^{a+\sqrt{2}}\left[r\frac{\pi}{2}+\right.$$
$$\left.\left.+2(r-h\sqrt{2})\right]\frac{d\nu}{dy}\,dr+\int_{h\sqrt{2}}^{a+\sqrt{2}}\frac{\pi}{4}\sqrt{2}\,[U-\nu(y)]\,dr\right\},$$

where $r=(a+y)\frac{\sqrt{2}}{2}$

$$\frac{P[\nu(y)]}{2}=\frac{P'[\nu(y)]}{2}+mk\int_0^a \nu(y)\,dy.$$

3.2. *The functionals G and F*

The functional $F[\nu(y)]$ on E_u, constructed like G in Sect. 2.1 has been determined beginning from:

$$\frac{\frac{1}{2}P'[\nu(y)]-k(\pi+2)aU}{K-k}. \tag{3.1}$$

It has the form:

for $h\geqslant a\sqrt{2}$, $F[\nu(y)]=0$,

for $a\leqslant h\leqslant a\sqrt{2}$

$$F[\nu(y)]=\left\{\int_h^{a+\sqrt{2}} 2r\arccos\frac{h}{r}\,\frac{d\nu}{dy}+\int_h^{a+\sqrt{2}}\sqrt{2}\arccos\frac{h}{r}\,[\nu(a^+)-\nu(y)]\quad r\right\}, \tag{3.2}$$

for $0\leqslant h\leqslant a$

$$F[\nu(y)]=\left\{\int_h^{h\sqrt{2}} 2r\arccos\frac{h}{r}\,\frac{d\nu}{dy}\,dr+\int_h^{h\sqrt{2}}\sqrt{2}\arccos\frac{h}{r}\,[\nu(a^+)-\nu(y)]\,dr+\right. \tag{3.3}$$
$$\left.+\int_{h\sqrt{2}}^{a+\sqrt{2}}\left[\frac{\pi}{2}r+2(r-h\sqrt{2})\right]\frac{d\nu}{dy}\,dr+\int_{h\sqrt{2}}^{a+\sqrt{2}}\sqrt{2}\,\frac{\pi}{4}\,[\nu(a^+)-\nu(y)]\,dr\right\}.$$

The functional $G[\nu(y)]$, corresponding to $P[\nu(y)]$ is in the form

$$\frac{G[\nu(y)]}{2}=k(\pi+2)a\nu(a^+)+mk\int_0^a \nu(y)\,dy+(K-k)F[\nu(y)]. \tag{3.4}$$

3.3. The functions F(z) and G(z)

If the functions $v_z(y)$ from E'_u will be taken for $v(y)$, then G and F will become functions of z: $UF(z)$ and $UG(z)$. According to the relationship (3.4), the following relationship holds between $F(z)$ and $G(z)$:

$$\frac{G(z)}{2} = k(\pi+2)\,a + mk(a-z) + (K-k)\,F(z). \tag{3.5}$$

These functions have different analytical expressions, depending on the parameter h/a and on the variable z. The various expressions for $F(z)$ and for its run are given below.

A. $h \geqslant a\sqrt{2}$,
$F(z) \equiv 0, \quad \forall z \in (0, a)$.

B. $a \leqslant h \leqslant a\sqrt{2}$ (Fig. 2),

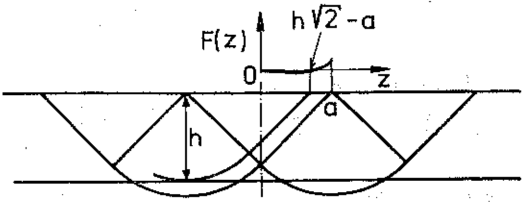

FIG. 2.

B.1. $0 \leqslant z \leqslant h\sqrt{2} - a$,
$F(z) = 0$.

B.2. $h\sqrt{2} - a \leqslant z \leqslant a$,

$$F(z) = 2(a+z)\arccos\frac{h\sqrt{2}}{a+z} - h\sqrt{2}\log\frac{\sqrt{2}}{2}\,\frac{(a+z) + \sqrt{(a+z)^2 - 2h^2}}{h}$$

and it is an increasing function of z.

C. $\frac{a\sqrt{2}}{2} \leqslant h \leqslant a$ (Fig. 3).

C.1. $0 \leqslant z \leqslant h\sqrt{2} - a$.

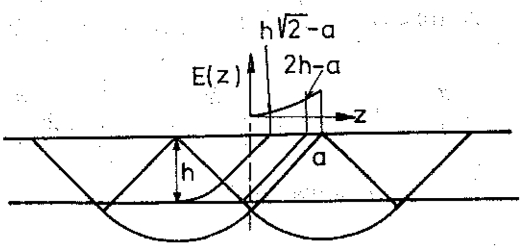

FIG. 3.

C.2. $h\sqrt{2}-a \leqslant z \leqslant 2h-a$,

$$F(z)=2(a+z)\arccos\frac{h\sqrt{2}}{a+z}-h\sqrt{2}\log\frac{\sqrt{2}}{2}\frac{(a+z)+\sqrt{(a+z)^2-2h^2}}{h}$$

and it is an increasing function of z.

C.3. $2h-a \leqslant z \leqslant a$,

$$F(z)=\left(\frac{\pi}{2}+1\right)(a+z)-2h-h\sqrt{2}\log\left(1+\sqrt{2}\right)$$

and it is an increasing function of z.

D. $(2-\sqrt{2})\,a \leqslant h \leqslant a$ (Fig. 4).

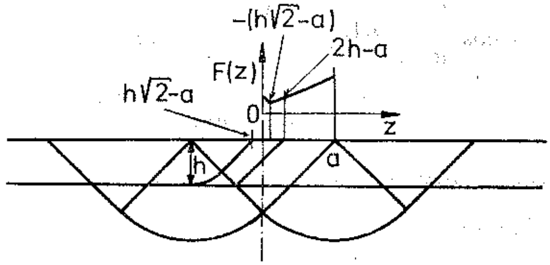

FIG. 4.

D.1. $0 \leqslant z \leqslant -(h\sqrt{2}-a)$,

$$F(z)=2\left[(a-z)\arccos\frac{h\sqrt{2}}{a-z}+(a+z)\arccos\frac{h\sqrt{2}}{a+z}\right]-$$
$$-h\sqrt{2}\log\frac{\left[a+z+\sqrt{(a+z)^2-2h^2}\right]\left[a-z+\sqrt{(a-z)^2-2h^2}\right]}{2h^2}$$

and it is a decreasing function of z.

D.2. $-(h\sqrt{2}-a) \leqslant z \leqslant 2h-a$,

$$F(z)=2(a+z)\arccos\frac{h\sqrt{2}}{a+z}-h\sqrt{2}\log\frac{\sqrt{2}}{2}\frac{(a+z)+\sqrt{(a+z)^2-2h^2}}{h}$$

as in C.2.

D.3. $2h-a \leqslant z \leqslant a$,

$$F(z)=\left(\frac{\pi}{2}+1\right)(a+z)-2h-h\sqrt{2}\log\left(1+\sqrt{2}\right).$$

E. $\frac{a}{2} \leqslant h \leqslant (2-\sqrt{2})\,a$ (Fig. 5).

E.1. $0 \leqslant z \leqslant 2h-a$ as in D.1.

$$F(z)=2\left[(a-z)\arccos\frac{h\sqrt{2}}{a-z}+(a+z)\arccos\frac{h\sqrt{2}}{a+z}\right]-$$
$$-h\sqrt{2}\log\frac{\left[a+z+\sqrt{(a+z)^2-2h^2}\right]\left[a-z+\sqrt{(a-z)^2-2h^2}\right]}{2h^2}.$$

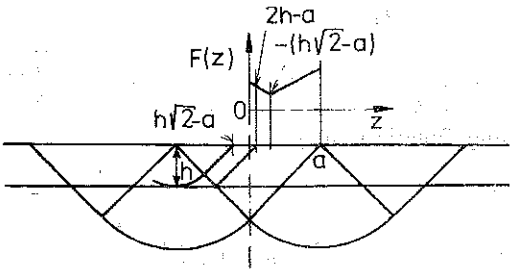

FIG. 5.

E.2. $2h-a \leqslant z \leqslant -(h\sqrt{2}-a)$

$$F(z)=2(a-z)\arc\cos\frac{h\sqrt{2}}{a-z}+\left(\frac{\pi}{2}+1\right)(a+z)-2h-$$

$$-h\sqrt{2}\log\left(1+\frac{\sqrt{2}}{2}\right)\frac{[a-z+\sqrt{(a-z)^2-2h^2}]}{h}$$

and it is a decreasing function of z.

E.3. $(a-h\sqrt{2}) \leqslant z \leqslant a$,

$$F(z)=\left(\frac{\pi}{2}+1\right)(a+z)-2h-h\sqrt{2}\log(1+\sqrt{2}).$$

F. $0 \leqslant h \leqslant \frac{a}{2}$ (Fig. 6).

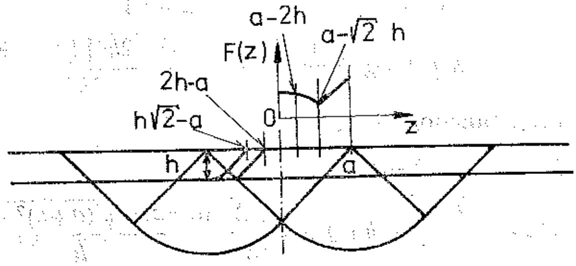

FIG. 6.

F.1. $0 \leqslant z \leqslant a-2h$,

$F(z)$ is const

$F(z)=(\pi+2)a-4h-2h\sqrt{2}\log(1+\sqrt{2})$.

F.2. $(a-2h) \leqslant z \leqslant (a-h\sqrt{2})$ as in E.2.

$$F(z)=2(a-z)\arc\cos\frac{h\sqrt{2}}{a-z}+\left(\frac{\pi}{2}+1\right)(a+z)-2h-$$

$$-h\sqrt{2}\log\left(1+\frac{\sqrt{2}}{2}\right)\frac{[a-z+\sqrt{(a-z)^2-2h^2}]}{h}$$

and it is a decreasing function of z.

F.3. $(a-h\sqrt{2}) \leqslant z \leqslant a,$

$$F(z)=\left(\frac{\pi}{2}+1\right)(a+z)-2h-h\sqrt{2}\log\left(1+\sqrt{2}\right).$$

4. Results obtained for a smooth punch and $k<K<\infty$

In the case of a smooth punch, $G(z)$ is reduced to the form

$$\frac{G(z)}{2}=(\pi+2)\,ak+(K-k)\,F(z). \tag{4.1}$$

Since $k<K$, the minimization of $G(z)$ is reduced to the minimization of $F(z)$ according to Eq. (4.1).

Using the expressions for $F(z)$ and the results of investigations of their behaviour, the minimum of $F(z)$ is attained $\forall h$, for $z=|h\sqrt{2}-a|$ and it assumes the value

if $a \geqslant \frac{a\sqrt{2}}{2}$

$$F\left(|h\sqrt{2}-a|\right)=0;$$

if $(2-\sqrt{2})\,a \leqslant h \leqslant \frac{a\sqrt{2}}{2}$

$$F\left(|h\sqrt{2}-a|\right)=2\left(2a-h\sqrt{2}\right)\arccos\frac{h\sqrt{2}}{2a-h\sqrt{2}}+$$

$$-h\sqrt{2}\log\frac{\sqrt{2}}{2}\,\frac{2a-h\sqrt{2}+2\sqrt{a^2-ah\sqrt{2}}}{h};$$

if $0 \leqslant h \leqslant (2-\sqrt{2})\,a$

$$F(h\sqrt{2}-a|)=\left(\frac{\pi}{2}+1\right)\left(2a-h\sqrt{2}\right)-2h-h\sqrt{2}\log\left(1+\sqrt{2}\right).$$

The minimum value of the dissipated power in the considered case is thus equal to

$$P=2U\left\{(\pi+2)\,ak+(K-k)\,F\left(|h\sqrt{2}-a|\right)\right\}, \tag{4.2}$$

which leads to the expression for the majorant of the limit load:

$$P_0^{(2)}=(\pi+2)\,k+(K-k)\,\frac{F\left(|h\sqrt{2}-a|\right)}{a}. \tag{4.3}$$

The shape of the function $\frac{F\left(|h\sqrt{2}-a|\right)}{a}$ depedning on $\frac{h}{a}$ is shown in Fig. 7.

It should be noted that for $\frac{h}{a} \geqslant \frac{\sqrt{2}}{2}$ the relationship (4.3) yields the value $P_0^{(2)}=(\pi+2)\,K$, which is the exact value of the limit load, and for $\frac{h}{a}=0$, $P_0^{(2)}=$ $=(\pi+2)\,K$ which is also the value of the limit load, Fig. 7.

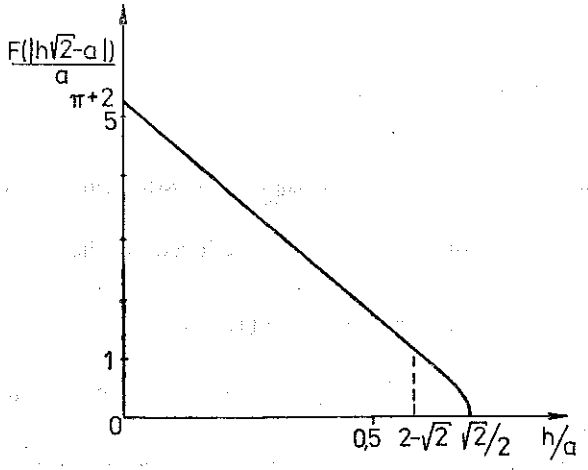

FIG. 7.

4.1. *Results for a rough punch and* $k<K<\infty$

In the case in question, the expression (3.5) for $G(z)$ cannot be reduced. The minimization of $G(z)$ depends clearly on m and k/K and the numerical calculations must be carried out for each considered case. They yield the minimum of the dissipated power for the velocity fields E_u, allowing to determine the majorant of the limit load $P_m^{(2)}$.

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STRESZCZENIE

WYRAŻENIA I PRZEBIEGI FUNKCJI KONIECZNYCH DO OBLICZANIA MOCY ROZPROSZONEJ W PROBLEMIE WCISKANIA STEMPLA W DWUWARSTWĘ

W pracy podano metodę obliczania majoranty obciążenia granicznego dla przypadku wciskania stempla w półpłaszczyznę z materiału izotropowego warstwowego. Wykazując prawdziwość zaproponowanych twierdzeń i lematów dla wybranej klasy sposobów deformacji, znacznie uproszczono obliczanie minimum mocy rozproszonej, sprowadzając problem poszukiwania minimum funkcjonału do poszukiwania minimum funkcji jednej zmiennej.

Резюме

ВЫРАЖЕНИЯ И ХОД ФУНКЦИЙ НЕОБХОДИМЫХ ДЛЯ ВЫЧИСЛЕНИЯ МОЩНОСТИ В ЗАДАЧЕ ВДАВЛИВАНИЯ ШТАМПА В ДВОЙНОЙ РАССЕЯННОЙ СЛОЙ

В работе приведен метод вычисления мажоранты предельной нагрузки для случая вдавливания штампа в полуплоскость из слоистого изотропного материяла. Показывая справедливость предложенных теорем и лемм для избранного класса способов деформации, значительного упрощается вычисление минимума рассеянной мощности, сводя проблему поиска минимума функционала к поиску минимума функции одной пременной.

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