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DRAG ON A CIRCULAR CYLINDER OSCILLATING IN A DUSTY VISCOUS FLUID

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Rectilinear oscillations of a circular cylinder in an infinite expanse of a viscous, imcompressible fluid with embended small solid spherical particles are studied. Analytical expressions for the velocity of the fluid and the drag experienced by the cylinder due to fluid stresses are given. Explicit expressions for the drag parameters are obtained using the asymptotic expansion of modified Bessel functions, and some limiting cases are discussed.

Nomenclature

u velocity of fluid particles,
v velocity of dust particles,
p pressure,
ϱ density,
$\bar{\Phi}$ body force vector,
t time,
μ coefficient of viscosity,
ν kinetic coefficient of viscosity,
U_0 spatial amplitude of oscillation of the cylinder,
σ frequency of oscillation of the cylinder,
m mass of a dust particle,
k Stokes resistance coefficient,
$\tau = m/k$ particle relaxation time,
N_0 number density of dust particles,
$f = mN_0/\varrho$ mass concentration of dust particle,
g acceleration due to gravity.

1. Introduction

The study of fluids having a uniform distribution of solid spherical particles is an important prelude to understanding phenomena such as blood flow in capillaries, pneumatic conveyance of small grain-like particles, flow in rocket tubes where small carbon or metallic fuel particles are present, environmental pollution, combustion, fluidization, etc. The basic theory of multiphase flows is given in a recent book by Soo [1]. Michael and Miller [2] and Liu [3, 4] have investigated the flow produced by the motion of an infinite plate in a dusty gas. HEALY and YANG [5] have obtained exact solutions of the problem discussed by the authors above using the Laplace transform technique. MICHAEL and NOREY [6] have used the perturbation method to study the laminar flow of a dusty gas between two rotating circular cylinders. Later HEALY and YANG [7] solved the problem of oscillating two-phase flows through a circular channel. Recently INDRASENA and Omar BEN-ZARTY [8] have investigated rotary oscillations of a sphere in a dusty gas and obtained an exact expression for the couple acting on the sphere due to fluid stresses.

In this paper we have studied the rectilinear oscillations of a long circular cylinder along a diameter in an infinite mass of viscous, incompressible fluid having a uniform distribution of solid dust particles. The problem is solved by the method of separation of variables and particular attention is focussed on the drag experienced by the cylinder due to fluid stresses. Explicit expressions for the drag parameters which are introduced in the expressions for the drag are obtained using the asymptotic expansion of modified Bessel functions. Numerical values of these parameters and the drag are tabulated to study their variation with the frequency of oscilation of the cylinder. Some limiting cases are discussed.

2. EQUATIONS OF MOTION

We shall consider the rectilinear oscillations of a circular cylinder of radius a along a diameter with velocity $U_0 \cos \sigma t$ in the infinite expanse of a viscous, incompressible fluid with a uniform distribution of identical spherical dust particles. Taking the spatial amplitude of oscillation to be small in comparison to the radius of the cylinder, i.e. $U_0/a\sigma$ to be small [9], we can get following linearised equations of motion from SAFFMAN [10]:

$$
\begin{aligned}
&\nabla \bar{u} = 0, \\
&\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\varrho} \nabla p + \frac{\Phi}{\varrho} + \nu \nabla^2 \bar{u} + \frac{f}{\tau} (\bar{v} - \bar{u}), \\
&\nabla v = 0, \\
&\tau \frac{\partial \bar{v}}{\partial t} = (\bar{u} - \bar{v}).
\end{aligned}
\tag{2.1}
$$

It is assumed that (i) the interaction between the two phases is in accordance with the Stokes drag law, (ii) there is no radial migration of the particles [11], (iii) there is negligible particle interaction, (iv) sedimentation is negligible, (v) the volume occupied by the particulate phase is negligible and (vi) Browian motion is neglected.

Eliminating $\bar{v}$ between, Eqs. $(2.1)_2$, and $(2.1)_3$, we can get

$$
\left(1 + f + \tau \frac{\partial}{\partial t}\right) \frac{\partial \bar{u}}{\partial t} = \left(1 + \tau \frac{\partial}{\partial t}\right) \left(-\frac{1}{\varrho} \nabla p + \frac{\Phi}{\varrho}\right) + \nu \left(1 + \tau \frac{\partial}{\partial t}\right) \nabla^2 \bar{u}. \tag{2.2}
$$

3. Formation of the boundary-value problem

Choosing the direction of motion of the cylinder along the axis $\theta = 0$ of the cylindrical coordinate system (r, θ, z) and $\bar{e}_r, \bar{e}_\theta, \bar{e}_z$ as base vectors of the system, vector $\bar{u}$ can be written as

$$\bar{u} = u_r(r, \theta)\, e^{i\sigma t}\, \bar{e}_r + u_\theta(r, \theta)\, e^{i\sigma t}\, \bar{e}_\theta. \tag{3.1}$$

Eq. (3.1) and in all subsequent equations only the real parts are to be taken whenever physical quantities are represented by complex quantities.

Equation (2.2), using Eq. (3.1) can be resolved into the following equations:

$$\begin{aligned} \left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}(u_r e^{i\sigma t}) &= \left(1+\tau\frac{\partial}{\partial t}\right)\left(-\frac{1}{\varrho}\frac{\partial p'}{\partial r}\right) - \\ &\quad -\nu\left(1+\tau\frac{\partial}{\partial t}\right)\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right)e^{i\sigma t}, \\ \left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}(u_\theta e^{i\sigma t}) &= \left(1+\tau\frac{\partial}{\partial t}\right)\left(-\frac{1}{\varrho}\frac{\partial p'}{r\partial \theta}\right) + \\ &\quad +\nu\left(1+\tau\frac{\partial}{\partial t}\right)\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)e^{i\sigma t}, \end{aligned} \tag{3.2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and

$$p' = p + \varrho g r \sin\theta + \text{constant}.$$

In view of the equation of continuity, u_r and u_θ can be expressed by a stream function $\psi(r, \theta)$ as

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = \frac{\partial \psi}{\partial r}. \tag{3.3}$$

Substituting Eqs. (3.3) and (3.2) respectively simplifies to

$$\begin{aligned} \left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}e^{i\sigma t}\right) &= \left(1+\tau\frac{\partial}{\partial t}\right)\left(-\frac{1}{\varrho}\frac{\partial p'}{\partial r}\right) - \\ &\quad -\nu\left(1+\tau\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial \nabla^2}{\partial \theta}\psi\right)e^{i\sigma t}, \\ \left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial r}e^{i\sigma t}\right) &= \left(1+\tau\frac{\partial}{\partial t}\right)\left(-\frac{1}{\varrho}\frac{\partial p'}{r\partial \theta}\right) + \\ &\quad +\nu\left(1+\tau\frac{\partial}{\partial t}\right)\left(\frac{\partial \nabla^2}{\partial r}\psi\right)e^{i\sigma t}. \end{aligned} \tag{3.4}$$

Elimination of p' from the above equations leads to

$$\left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}(\nabla^2\psi)\,e^{i\sigma t}=\nu\left(1+\tau\frac{\partial}{\partial t}\right)(\nabla^4\psi)\,e^{i\sigma t}. \tag{3.5}$$

4. Solution of the problem

Now we have to solve the differential equation (3.5) to find ψ subject to the adherence condition

$$\begin{aligned} u_r(a,\theta) &= U_0 \operatorname{Cos}\theta,\\ u_\theta(a,\theta) &= -U_0 \operatorname{Sin}\theta \end{aligned} \tag{4.1}$$

and at infinity we have the usual requirement of finiteness for all the physical quantities of the problem. The boundary conditions (4.1) and (3.3) suggest expressing ψ as

$$\psi(r,\theta)=F(r)\operatorname{Sin}\theta \tag{4.2}$$

which, on substitution in Eq. (3.5), leads to the following differential equation:

$$D^2F(r)-n^2DF(r)=0, \tag{4.3}$$

where

$$n^2=\frac{i\sigma}{\nu}\,\frac{1+f+\tau i\sigma}{1+\tau i\sigma}$$

and

$$D=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}.$$

The boundary conditions (4.1) can be expressed in terms of F in the following form:

$$F(a)=-aU_0,\quad F'(a)=-U_0. \tag{4.4}$$

Integrating Eq. (3.4), we get the differential equation

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}-\left(n^2+\frac{1}{r^2}\right)F=C_1r+\frac{C_2}{r}, \tag{4.5}$$

where C_1 and C_2 are constants of integration.

The solution of Eq. (4.5) is given by

$$F(r)=AI_1(nr)+BK_1(nr)-\frac{C_1}{n^2}r-\frac{C_2}{n^2}\frac{1}{r}, \tag{4.6}$$

where A and B are constants, I_1 and K_1 are modified Bessel functions of the first order [12].

In view of the requirement of finiteness at infinity for all physical quantities of the problem we have to take the solution as

$$F(r) = BK_1(nr) + \frac{C}{r}, \tag{4.7}$$

where $C = -C_2/n^2$.

Using the boundary conditions (4.4), the constants B and C can be determined as

$$\begin{aligned} B &= \frac{-2U_0 a}{K_1(na) + anK_1'(na)}, \\ C &= U_0 a^2 \frac{K_1(na) - anK_1'(na)}{K_1(na) + anK_1'(na)}. \end{aligned} \tag{4.8}$$

$$\begin{aligned} u_r(r,\theta)\, e^{i\sigma t} &= U_0\left[\left(\frac{a}{r}\right)^2 + \frac{2a}{r^2}\,\frac{aK_1(na) - rK_1(nr)}{K_1(na) + ank_1'(na)}\right] e^{i\sigma t}\operatorname{Cos}\theta, \\ u_\theta(r,\theta)\, e^{i\sigma t} &= U_0\left[\left(\frac{a}{r}\right)^2 - \frac{2a}{r^2}\,\frac{aK_1(na) + r^2 nK_1'(nr)}{K_1(na) + anK_1'(na)}\right] e^{i\sigma t}\operatorname{Sin}\theta. \end{aligned} \tag{4.9}$$

Using Eq. (4.2) and (4.4) in Eqs. (3.4) and integrating the resulting equations, we can determine the pressure p′ in the following form:

$$p' = -C\frac{\mu n^2}{r}\operatorname{Cos}\theta\cdot e^{i\sigma t} + A_1 e^{-t/\tau} + \text{Const}, \tag{4.10}$$

where A_1 is a constant of integration.

5. Drag

The drag $D(t)$ due to the fluid stresses τ_{ij} on the oscillating cylinder of length L is given by

$$D(t) = aL\int_0^{2\pi}(\tau_{rr}\operatorname{Cos}\theta - \tau_{r\theta}\operatorname{Sin}\theta)_{r=a}\,d\theta, \tag{5.1}$$

where

$$(\tau_{rr})_{r=a} = C\frac{\mu n^2}{a}\operatorname{Cos}\theta\cdot e^{i\sigma t} = A_1 e^{-t/\tau} + \varrho\, gr\sin\theta + \text{const}$$

$$(\tau_{r\theta})_{r=a} = -\mu n^2\left(U_0 a + \frac{C}{a}\right)\operatorname{Sin}\theta\cdot e^{i\sigma t}.$$

Substituting these in Eq. (5.1) and performing integration, we find that

$$D(t) = -M' U_0 \nu n^2\left[1 - \frac{4K_1(na)}{K_1(na) + anK_1'(na)}\right] e^{i\sigma t}, \tag{5.2}$$

where $M' = \pi a^2 LP$ is the mass of the fluid displaced by a length L of the cylinder.

It is not possible to separate the Bessel functions K_1 and K_1' into real and imaginary parts because they are of integral order and of complex arguments. Therefore it is not possible to write explicit expressions for K and K'. We can write explicit expressions which are useful for large values of $|na|$ if we use an asymptotic expansion for K_1 and K_1'. For large values of $|z|$ we can express K_1 and K_1' asymptotically as

$$K_1(z) \approx \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(1+\frac{3}{8z}+\dots\right),$$

$$K_1'(z) \approx -\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(1+\frac{7}{8z}+\dots\right)$$

provided $|\arg z| < 3\pi/2$ [12].

Using these expressions in Eq. (5.2) and separating the real part, we can get

$$D(t) \approx M' U_0 \sigma (K \operatorname{Sin} \sigma t - K' \operatorname{Cos} \sigma t), \tag{5.3}$$

where

$$K = \alpha^2 \sin 2\varphi + \frac{4}{a}\sqrt{\frac{\nu}{\sigma}}\, \alpha \operatorname{Sin} \varphi, \tag{5.4}$$

$$K' = \alpha^2 \operatorname{Cos} 2\varphi + \frac{4}{a}\, \alpha \sqrt{\frac{\nu}{\sigma}} \operatorname{Cos} \varphi + \frac{2\nu}{\sigma a^2},$$

$$|n|^2 = \frac{\sigma}{\nu}\alpha^2; \tag{5.5}$$

$$\alpha^2 = \left[\frac{(1+f)^2 + \tau^2\sigma^2}{1+\tau^2\sigma^2}\right]^{\frac{1}{2}} \tag{5.6}$$

and

$$2\varphi = \frac{\pi}{2} + \tan^{-1}\frac{\tau\sigma}{1+f} - \tan^{-1}\tau\sigma.$$

For the sake of convenience K and K' introduced in the expression (5.3) may be defined as drag parameters.

6. Discussion

The second term $-M' U_0 \sigma K' \operatorname{Cos} \sigma t$ in the expression for the force $D(t)$ always opposes the movement of the cylinder and thus a damping force is out of the phase with the acceleration. This damping force causes decay of the oscillations of the cylinder if left free. The quantity KM' is called the virtual mass of the cylinder and depends on the frequency σ and the parameters f and τ in a complicated form. It is seen from Tables 1, 2,

Table 1. $\tau = 0.50$

σ	K	K'	$D'(t) = D(t)/M' U_0$
1	5.08018	8.13861	4.06515
2	3.85386	4.92368	3.01819
3	3.31619	3.71009	−11.54636
4	3.00017	3.04996	17.07841
5	2.78661	2.62768	−16.33377
10	2.26199	1.68182	−25.95659
20	1.89293	1.10181	−43.79758
30	1.72939	0.86824	−51.29337
40	1.63183	0.73577	−40.08223
50	1.56522	0.64826	−7.95996

Table 2. $\tau = 0.20$

σ	K	K'	$D'(t) = D(t)/M' U_0$
1	5.812436	8.12346	4.02840
2	3.690807	4.92994	3.11206
3	3.36335	3.73200	−11.69629
4	3.03686	3.07881	17.26471
5	2.81413	2.65829	−16.51755
10	2.26869	1.70386	−26.18056
20	1.89344	1.11277	−43.91328
30	1.72919	0.87519	−51.28276
40	1.63153	0.74079	−39.96637
50	1.56492	0.65217	−7.78015

Table 3. Clean viscous fluid ($\tau = 0.00$)

σ	K	K'	$D'(t) = D(t)/M' U_0$
1	5.00000	7.99999	3.99245
2	3.82843	4.82842	3.04951
3	3.30940	3.64272	−11.49507
4	3.00000	2.99999	16.94460
5	2.78886	2.58885	−16.14935
10	2.26491	1.66490	−25.82550
20	1.89443	1.09442	−43.75172
30	1.73030	0.86363	−51.32372
40	1.63246	0.73245	−40.17337
50	1.56569	0.64568	−8.08495

and 3 that for any given frequency of oscillation the drag experienced by the cylinder due to a dusty fluid is higher than the drag due to a clean viscous fluid. This is so because of the presence of the dust particles in the fluid. From Eqs. (5.4) and (5.5) it follows that for large values of $\tau\sigma$ the effect of dust particles on the fluid motion is reduced and α, φ, K, K' and D approach their corresponding values for an ordinary viscous fluid. When the masses of the dust are small, their influence on the fluid motion is reduced and ultimately as $m \to 0$ the term $\frac{f}{\tau}(u-v)$ which represents the drag on the fluid due to the dust particles disappears in the equation of motion (2.2) and the drag parameters K and K' simplify to

$$K = (1+2\beta),$$
$$K' = \beta\,(2+\beta),$$

where

$$\beta = \left(\frac{2\nu}{\sigma a^2}\right)^{\frac{1}{2}}.$$

It can be seen that the expression (5.3) with the above values of K, K' gives the drag on the cylinder due to the clean viscous fluid. Numerical values of K, K' are tabulated in Tables 1, 2, and 3 to represent their variation with the frequency for different values of τ.

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Streszczenie

OPÓR DZIAŁAJĄCY NA WALEC KOŁOWY DRGAJĄCY W ZAPYLONYM PŁYNIE LEPKIM

Rozważa się prostoliniowe drgania walca kołowego w nieskończonej objętości lepkiego płynu nieściśliwego z zawiesiną w postaci małych kulek sztywnych. Podano wyrażenia analityczne dla prędkości płynu i oporu napotykanego przez walec. Jawne wyrażenie dla parametrów oporu uzyskano za pomocą rozwinięć asymptotycznych zmodyfikowanych funkcji Bessela. Omówiono również pewne przypadki graniczne.

Резюме

СОПРОТИВЛЕНИЕ ДЕЙСТВУЮЩЕЕ НА КРУГОВОЙ ЦИЛИНДР КОЛЕБЛЮЩИЙСЯ В ЗАПЫЛЕННОЙ ВЯЗКОЙ ЖИДКОСТИ

Рассуждаются прямолинейные колебания кругового цилиндра в бесконечном объеме вязкой несжимаемой жидкости со взвесей в виде малых жестких шариков. Приведены аналитические выражения для скорости жидкости и сопротивления, встречаемого цилиндром. Явные выражения для параметров сопротивления получены при помощи асимптотических разложений модифицированных функций Бесселя; обсуждены тоже некоторые предельные случаи.

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