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THE BENDING VIBRATIONS OF AN ANISOTROPIC FREE CIRCULAR PLATE OF REGULAR SYMMETRY

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The elasticity moduli determination method of the regular symmetry ingle crystals by bending vibrations excitation in circular plate samples with the middle planes perpendicular to the crystallographic [100] and [110] directions is given in the work. The adaptation of the method for any other pair of directions presents no difficulties. Resonance frequencies of the free edge circular anisotropic plate were determined, too. In order to solve Eqs. (3.1) and (3.3), the consecutive approximation method was used. The method seemed to be the most simple and proper in this case. It enabled to give evident dependence between the resonance frequency of the sample and the elasticity moduli and the Poisson constant. Measurements were taken for Si and Ge single crystals.

1. Introduction

Kirchhoff in his work [1] has given an expression for the bending vibrations frequency of thin circular plates with a free edge. A similar expression for thick plates taking into account shear stresses and rotational inertia was given by Martinček [3]. Both expressions mentioned above are valid for isotropic solids only. The purpose of this work is to determine the frequency of the first resonance of a circular plate of a free edge and cubic anisotropy. This symmetry has three perpendicular elasticity symmetry planes, like the orthotropic one, but elasticity moduli and Poisson's constants are equal in the main directions.

2. Bending vibrations equation of a thin anisotropic circular plate

The general bending vibrations equation of a thin circular anisotropic plate [4, 5] (Fig. 1) can be expressed in the form of the following partial differential equation:

$$D_{11}\frac{\partial^4 W}{\partial x^4}+4D_{16}\frac{\partial^4 W}{\partial x^3\,\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 W}{\partial x^2\,\partial y^2}+4D_{26}\frac{\partial^4 W}{\partial x\partial y^3}+ \\ +D_{22}\frac{\partial^4 W}{\partial y^4}+\varrho h\frac{\partial^2 W}{\partial t^2}=0, \tag{2.1}$$

where $W=W(x,y,t)$ — bending function of the plate. Coefficients of each partial derivative can be obtained from the relation

$$D_{ij} = B_{ij} \frac{h^3}{12}, \tag{2.2}$$

where

$$B_{11} = \frac{1}{\Delta}(s_{22}s_{66}-s_{26}^2), \qquad B_{26} = \frac{1}{\Delta}(s_{12}s_{16}-s_{11}s_{26}),$$

$$B_{22} = \frac{1}{\Delta}(s_{11}s_{66}-s_{16}^2), \qquad B_{66} = \frac{1}{\Delta}(s_{11}s_{22}-s_{12}^2), \qquad \Delta = \begin{vmatrix} s_{11}, & s_{12}, & s_{16} \\ s_{12}, & s_{22}, & s_{26} \\ s_{16}, & s_{26}, & s_{66} \end{vmatrix} \tag{2.3}$$

$$B_{12} = \frac{1}{\Delta}(s_{16}s_{26}-s_{12}s_{66}),$$

$$B_{16} = \frac{1}{\Delta}(s_{12}s_{26}-s_{22}s_{16}),$$

and s_{ij} are the elasticity coefficients. It is to be note that D_{ij} have the meaning of the anisotropic plate rigidity. Respectively, D_{11}, D_{22} have the meaning of the bending rigidity around the x and y axis (see Fig. 1):D_{66} is the torsion rigidity, and D_{16} and D_{26} are side rigidities and refer to the snips of unit width.

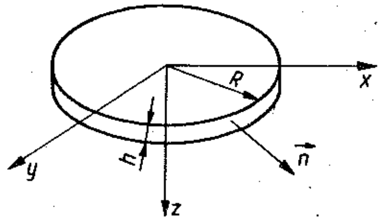

FIG. 1.

If the plates edge is free, then in order to solve Eq. (2.3) the following boundary conditions are set

$$\begin{aligned} M_n &= 0, \\ N_n + \frac{\partial H_{tn}}{\partial s} &= 0, \end{aligned} \tag{2.4}$$

where M_n — bending moment, H_{tn} — torsional moment, N_n — shear forces, $\mathbf{n}$ — external normal to the plate's edge surface and $\partial/\partial s$ — plate's edge arc derivative.

From Eq. (2.1) a following timeless equation is easily obtained:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} - \varrho h \omega^2 w = 0. \tag{2.5}$$

The elasticity coefficients, which appear in D_{ij} where the main directions overlap the coordinates axes, are functions of the angles between the axes and main directions.

3. The equations of the problem and their solutions

Two cases are considered. In the first one the middle plane of the plate is parallel to the crystallographic plane (100); x, y, z axes of the coordinates are identical with the main elasticity directions (see Fig. 2). Then Eq. (2.5) modifies to

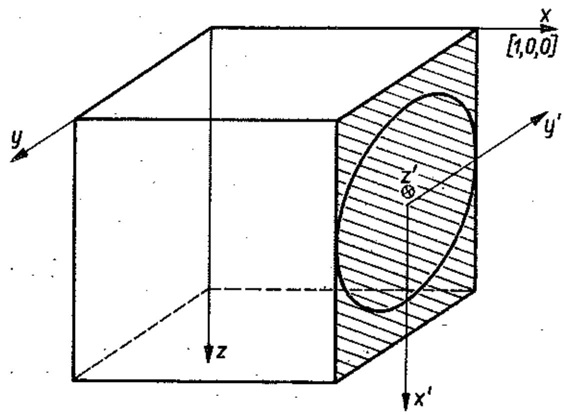

Fig. 2.

(3.1) $$\frac{\partial^4 w}{\partial x^4} + 2\frac{D_3}{D}\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{\varrho h \omega^2}{D} w = 0,$$

where

$$D = \frac{Eh^2}{12(1-\nu^2)},$$

$$D_3 = \frac{Eh^3 \nu}{12(1-\nu^2)} + \frac{Gh^3}{6}$$

and E — the main direction's Young modulus, G — the main direction's rigidity modulus and ν — the main direction's Poisson constant.

Here advantage was taken of the fact that in the regular system all rigidity moduli in main directions are identical.

In the second case the middle plane of the plate is parallel to the crystallographic (110) plane (Fig. 3); the x'-axis direction is a main direction, but y, z are not. Thus the coefficients D_{ij} of Eq. (2.5) are to be reduced to the respective coefficients related to the main axes [6]. The following is obtained:

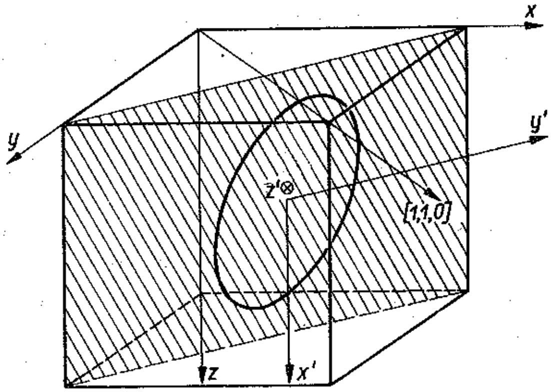

FIG. 3.

$$D'_1 \frac{\partial^4 w}{\partial x^4} + 2D'_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D'_2 \frac{\partial^4 w}{\partial y^4} - \varrho h \omega^2 w = 0, \tag{3.2}$$

because $D'_{16} = D'_{26} = 0$. There $D'_{11} = D'_1$, $D'_{22} = D'_2$ and $D'_{12} + 2D'_{66} = D'_3$.

The equation (3.2) can be written in the form [8]

$$\Delta\Delta w + 2a \frac{\partial^4 w}{\partial x^2 \partial y^2} + b \frac{\partial^4 w}{\partial x^4} - \frac{\varrho h \omega^2}{D'_2} w = 0, \tag{3.3}$$

where

$$a = \frac{D'_3}{D'_2} - 1, \quad b = \frac{D'_1}{D'_2} - 1,$$

$$D'_1 = \frac{E^2 + 2EG(1-\nu)}{E + 2G(1 - \nu - 2\nu^2)} \frac{h^3}{12},$$

$$D'_2 = \frac{4EG}{E + 2G(1 - \nu - 2\nu^2)} \frac{h^3}{12},$$

$$D'_3 = h^3 \left[\frac{G}{6} + \frac{1}{12} \frac{E^2 v + 2EGv(1-v)}{E + 2G(1 - v - 2v^2)} \right].$$

To calculate the first resonance of the bending vibrations the subsequent approximations method was used with Eqs. (3.2) and (3.3), introducing to Eq. (3.2) a parameter $\varepsilon = D_3/D - 1$ [7] and to Eq. (3.3) $\varepsilon = \max(|a|, |b|)$ [8]. The following results were obtained: for a plate with the (100) plane

$$\varrho h \omega_{100}^2 = D \left(\frac{16 x_1^4}{R^4} - \frac{2 \iint w_0 \frac{\partial^4 w_0}{\partial x^2 \partial y^2} dx dy}{\iint w_0^2 \, dx dy} \right) + D_3 \frac{2 \iint w_0 \frac{\partial^4 w_0}{\partial x^2 \partial y^2} dx dy}{\iint w_0^2 \, dx dy}; \tag{3.4}$$

for a plate with the (110) plane

(3.5) $$\varrho h\omega_{110}^2 = D_1' \frac{\iint w_0 \frac{\partial^4 w_0}{\partial x^4}\,dxdy}{\iint w_0^2\,dxdy} + D_3' \frac{2\iint w_0 \frac{\partial^4 w_0}{\partial x^2 \partial y^2}\,dxdy}{\iint w_0^2\,dxdy} + D_2'\left(\frac{16x_1^4}{R^4} - \frac{2\iint w_0 \frac{\partial^4 w_0}{\partial x^2 \partial y^2}\,dxdy}{\iint w_0^2\,dxdy} - \frac{\iint w_0 \frac{\partial^4 w_0}{\partial x^4}\,dxdy}{\iint w_0^2\,dxdy}\right),$$

x_n^2 is a coefficient dependent on the resonance number and Poisson's constant ν, given in the work [2]. w_0 — in Eqs. (3.4) and (3.5) is the bending function for free vibrations of an isotropic circular plate [9]. For vibrations the n-th component of this function is in the following form:

(3.6) $$w_0(r, \varphi) = [A_n J_n(kr) + B_n I_n(kr)]\,[\sin n\varphi + \cos n\varphi],$$

where A_n, B_n — constants, $k^4 = \varrho h\omega^2/D$ and J_n, I_n — the Bessel function of the first order and the modified first-order Bessel function, respectively.

For a free edge circular plate we have two boundary conditions:

(3.7) $$\frac{\partial^2 w_0}{\partial r^2} + \frac{\nu}{r}\frac{\partial w_0}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w_0}{\partial \varphi^2} = 0, \quad r = R,$$
$$\frac{\partial^3 w_0}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w_0}{\partial r^2} - \frac{1}{r^2}\frac{\partial w_0}{\partial r} + \frac{2-\nu}{r^2}\frac{\partial^3 w_0}{\partial r\partial \varphi^2} - \frac{3-\nu}{r^3}\frac{\partial^2 w_0}{\partial \varphi^2} = 0, \quad r = R.$$

The first resonance frequencies are to be determined; this means frequencies of the plate with two nodal diameters. In this case the order of the Bessel function is the same as the number of nodal diameters present. This leads to the solution for w_0 in the form

(3.8) $$w_0(r, \varphi) = [AJ_2(kr) + BI_2(kr)]\,[\sin 2\varphi + \cos 2\varphi].$$

In further considerations we use the Bessel functions $J_2(kr)$ and $I_2(kr)$ in series expansion; hence

(3.9) $$w_0 = \left[A\left(\frac{k^2 r^2}{8} - \frac{k^4 r^4}{96} + \frac{k^6 r^6}{3072}\right) + B\left(\frac{k^2 r^2}{8} + \frac{k^4 r^4}{96} + \frac{k^6 r^6}{3072}\right)\right] \times [\sin 2\varphi + \cos 2\varphi].$$

The A and B constants are determined from the boundary conditions (3.7). We then obtain

(3.10) $$\frac{B}{A} = -\frac{384(1-\nu) - 192z + (15+\nu)z^2}{384(1-\nu) + 192z + (15+\nu)z^2},$$

where $z = kR$.

Calculating the integrals in the expressions (3.4) and (3.5), we obtain

$$\iint w_0 \frac{\partial^4 w_0}{\partial x^2 \partial y^2} dxdy = 4\pi A^2\left(1+\frac{B}{A}\right)\frac{z^3}{128R^2}\left[\frac{1}{48}z\left(1+\frac{B}{A}\right)+\right.$$
$$\left.+\frac{1}{768}z^2\left(\frac{B}{A}-1\right)+\frac{1}{30720}z^3\left(1+\frac{B}{A}\right)\right],$$
$$\iint w_0^2\,dxdy = \pi A^2R^2\left[\frac{1}{192}z^2\left(1+\frac{B}{A}\right)^2+\frac{1}{1536}z^3\left(1+\frac{B}{A}\right)\times\right.$$
$$\times\left(\frac{B}{A}-1\right)+\frac{1}{61440}z^4\left(1+\frac{B}{A}\right)^2+\frac{1}{46080}z^4\left(\frac{B}{A}-1\right)^2+$$
$$\left.+\frac{1}{884736}z^5\left(1+\frac{B}{A}\right)\left(\frac{B}{A}-1\right)+\frac{1}{66060288}z^6\left(1+\frac{B}{A}\right)^2\right], \tag{3.11}$$
$$\iint w_0\frac{\partial^4 w_0}{\partial x^4}\,dxdy = \pi A^2\left(1+\frac{B}{A}\right)\frac{z}{512R^2}\left[z^3\left(1+\frac{B}{A}\right)+\right.$$
$$\left.+\frac{1}{16}z^4\left(\frac{B}{A}-1\right)+\frac{1}{640}z^5\left(1+\frac{B}{A}\right)\right].$$

Above integrals one to be calculated in limits $(0, R)$.
The integrals above are to be calculated in the limits $(0, R)$ and $(0, 2\pi)$. The parameter z is a function of the Poisson constant and it is determined from equating to zero of the determinant created by substituting the expression (3.9) with several Poisson onstants to the boundary conditions (3.7).

The values of the integrals of Eq. (3.11) calculated numerically for several Poisson contants are shown in Table 1.

Table 1. The values of integrals of Eq. (3.11).

ν	z	$\frac{R^2}{\pi}\iint w_0 \frac{\partial^4 w_0}{\partial x^2 \partial y^2} dx\,dy$	$\frac{1}{\pi R^2}\iint \omega_0^2\, dx\, dy$	$\frac{R^2}{\pi}\iint w_0 \frac{\partial^4 w_0}{\partial x^4} dx\,dy$
0.0	5.435	0.549	0.116	1.648
0.1	5.283	0.522	0.119	1.566
0.2	5.102	0.486	0.122	1.457
0.3	4.889	0.440	0.123	1.320
0.4	4.637	0.385	0.122	0.155
0.5	4.336	0.321	0.119	0.963

4. Determination of the elasticity moduli of the regular symmetry anisotropic plates

Equations (3.4) and (3.5) give evident dependence of the free edge circular plate's vibration frequencies on the bending and torsion rigidities. Since the

Poisson constant can be determined from the first resonance to the second resonance frequency ratio in a way similar to that shown in MARTINČEK and KIRCHHOFF'S works [1, 3], substituting to Eqs. (3.4) and (3.5) its value for the (100) middle planes orientations and using Table 1, we obtain an equations system allowing to calculate E and G by measuring the respective resonance frequencies. The elasticity moduli and the Poisson constant of Si and Ge were determined by means of the presented method.

The measurement method and the equipment used were described in the work [10]. The results obtained are collected in Table 2. Units:

$$\text{a)} \ -10^{10}\frac{\text{N}}{\text{m}^2}, \quad \text{b)} \ -10^{-12}\frac{\text{m}^2}{\text{N}}.$$

Table 2. Results.

	E^a_{100}	G^a_{100}	ν_{100}	s^b_{11}	s^b_{44}	$-s^b_{12}$	c^a_{11}	c^a_{12}	c^a_{44}
n-type Si 15—30 Ω cm	12.981	7.830	0.27	7.704	12.771	2.080	16.226	6.001	7.830
Intrinsic Ge 50 Ω cm	10.412	6.569	0.28	9.600	15.220	2.690	13.316	5.178	6.569

5. CONCLUSIONS

The results obtained fit in sufficiently well with the data given by other authors [11, 12], where different methods were applied to determine the rigidity constants. The considerations presented above show that in order to determine the complete set of elastic constants of the regular symmetry anisotropic solid it is enough to measure two lowest resonance frequencies of the circular plate sample with two orientations, e.g. [100] and [110]. When the main elastic moduli and the Poisson constant are known, it is easy to calculate all elastic constants and moduli in any direction.

The method applied is limited to the first approximation only, because the second one gives a very small contribution to the frequency. Taking only the three-term Bessel function series was due to the condition that changes of the third digit in the elastic constants are sufficient to obtain the assumed error magnitude.

The method can be easily adapted to determine the complete set of moduli with the bending vibrations frequencies of any other pair of normal orientations, e.g. [110] and [111] or [100] and [111].

The errors of determination of the Young modulus and rigidity modulus, calculated for an example for Si, are $0.034\cdot 10^{10}$ N/m^2 and $0.020\cdot 10^{10}$ N/m^2, respectively.

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Streszczenie

DRGANIA GIĘTNE PŁYTY ANIZOTROPOWEJ O BRZEGACH SWOBODNYCH I SYMETRII REGULARNEJ

W pracy podano metodę określania modułów sprężystości monokryształów drogą wymuszania drgań giętnych w płytkach kołowych o powierzchniach środkowych wyciętych prostopadle do osi krystalograficznych [100] i [110]. Zastosowanie metody do dowolnie dobranej innej pary osi nie przedstawia trudności. Określono również częstości rezonansowe płyt anizotropowych o brzegach swobodnych. Do rozwiązania równań (3.1) i (3.3) zastosowano metodę kolejnych przybliżeń, którą uznać można za najbardziej stosowaną w rozważanym przypadku. Metoda ta umożliwia wykazanie zależności między częstością rezonansową próbki a modułami sprężystości i stałą Poissona. Wykonano pomiary na monokryształach i krzemu i germanu.

Резюме

ИЗГИБНЫЕ КОЛЕБАНИЯ АНИЗОТРОПНОЙ ПЛИТЫ СО СВОБОДНЫМИ КРАЯМИ И С РЕГУЛЯРНОЙ СИММЕТРИЕЙ

В работе проведен метод определения модулей упругости монокристаллов путем вынуждения изгибных колебаний в круговых плитках со срединными поверхностями, вырезанными перпендикулярно к кристаллическим осям 100 и 110. Применение метода к произвольно подобранной другой паре осей не представляет трудности. Определены тоже резонансные частоты анизотропных плит со свободными краями. Для решения уравнений [7 и 9] применен метод последовательных приближений, который можно считать наиболее подходящим в рассматриваемом случае. Этот метод дает возможность показать зависимости между резонансной частотой образца и модулями упругости, а также коэффициентом Пуассона. Проведены измерения на монокристаллах кремня и германия.

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