

## UNSTEADY TRANSONIC CASCADE WITH A SUBSONIC LEADING-EDGE LOCUS

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This paper devises an inviscid, linearised analysis of an unsteady transonic cascade with a subsonic leading-edge locus. First, the flow in the pre-interference zone (the one upstream of the cascade) is determined. This is then used as an initial value to determine the flow in the region between the blades downstream of the pre-interference zone. In deriving the solution for the pre-interference zone, the problem is first cast as an initial-value problem. The periodicity requirement due to the cascade arrangement enables one to eliminate the unknown initial conditions and thus lead to the integral equations for the velocity potentials. The integral equations are solved by successive approximations.

### 1. INTRODUCTION

The oscillatory supersonic flow past a two-dimensional cascade having a nonzero, stagger and arbitrary interblade phase-lag angle and a supersonic leading-edge locus was considered by LANE [1]. The approximation of plane flow was made, i.e. the blade row appears as an infinite lattice of two-dimensional airfoils. This, of course, implies that the actual air passage is annular and that the radius ratio of the blades is not large.

However, the transonic-flow regime is of relevance in the design of gas turbines since the compressor stages often operate in this speed range and the flow in them is basically unsteady. Further, the transonic flow is usually found to be the one most susceptible to flutter and other dynamic instability phenomena. Motivated by the problem of unstable oscillations of cascaded airfoils in transonic flows, the purpose of this paper is to devise an inviscid, linearised analysis for an unsteady transonic cascade with a subsonic leading-edge locus.

With reference to Fig. 1, first, the flow in the pre-interference ① or the one upstream of the cascade is determined. This is then used as an initial value to determine the flow in the region downstream of the pre-interference zone. In deriving the solution for the pre-interference zone, the problem is first cast as initial-value problem. The periodicity requirement due to the cascade arrangement enables one to eliminate the unknown initial conditions and thus lead to the integral equations for the velocity potentials. The integral equations are solved by successive approximations.

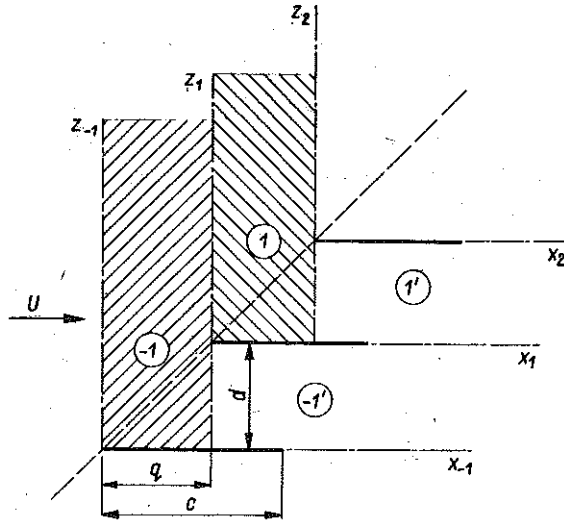


FIG. 1.

## 2. LINEARISED THEORY FOR AN UNSTEADY TRANSONIC CASCADE

Consider an oscillating transonic cascade with a subsonic leading edge locus. Assume an inviscid, two-dimensional flow over thin and slightly-cambered airfoils. Far upstream, the flow is steady and parallel to the airfoil chord. The blades execute small-amplitude oscillations which are harmonic in time and such that the mode and phase lag between any two adjacent airfoils are equal. The assumption of identical blade modes and amplitudes but motion with equal phase lag between adjacent blades is adequate for the flutter problem, since one quantity of interest in the blade-row critical flutter condition is the interblade phase-lag angle.

One has for low-frequency transonic flows past a thin blade,

$$(2.1) \quad (\gamma + 1) \Phi_x \Phi_{xx} - \Phi_{zz} + 2\Phi_{xt} = 0,$$

$$(2.2) \quad z = 0: \quad \Phi_z = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) F(x, t),$$

where  $\Phi$  denotes the velocity potential,  $\gamma$  the ratio of specific heats of the fluid,  $z = F(x, t)$  describes the blade surface, and all quantities have been nondimensionalised using the free-stream velocity  $V$  and the blade chord  $C$ .

Let

$$(2.3) \quad F(x, t) = G(x) + H(x) e^{ikt}, \quad k \ll 1$$

so that one may seek a solution of the form

$$(2.4) \quad \Phi(x, z, t) = \Phi(x, z) + \phi(x, z) e^{ikt},$$

Upon neglecting the higher harmonics, one then obtains

$$(2.5) \quad \varphi_{zz} - (\gamma + 1) \varphi_x \varphi_{xx} = 0,$$

$$(2.6) \quad z = 0: \quad \varphi_z = G(x)$$

and

$$(2.7) \quad \varphi_{zz} - 2ik\varphi_x = (\gamma + 1) (\varphi_x \varphi_{xx})_x,$$

$$(2.8) \quad z = 0: \quad \varphi_z = H_x \equiv -V(x).$$

As shown by Shivamoggi [2], Eq. (2.7) can be well approximated by

$$(2.9) \quad \varphi_{zz} - (K + 2ik) \varphi_x = 0,$$

where

$$(2.10) \quad \begin{aligned} K &= (\gamma + 1) \varphi_{xx}|_{x=x^*}, \\ x &= x^*: \quad \varphi_x = 0. \end{aligned}$$

Consider the pre-interference zone  $\textcircled{1}$ , and let

$$x = 0: \quad \varphi = f(z).$$

Using Eq. (2.11), and Laplace transforming equation (2.9) one obtains

$$(2.12) \quad \bar{\varphi}_{zz} - \alpha^2 \bar{\varphi} = -h(z),$$

where

$$(2.13) \quad \alpha^2 \equiv (K + 2ik) \left( s + \frac{K + 2ik}{4} \right),$$

$$h(z) = (K + 2ik) f(z).$$

Using Eq. (2.8), Eq. (2.12) can be solved to give

$$(2.14) \quad \bar{\varphi} = \frac{1}{\alpha} \bar{V}(s) e^{-\alpha z} + \frac{1}{2\alpha} \left[ \int_z^\infty e^{-\alpha(t-z)} h(t) dt + \int_0^z e^{\alpha(t-z)} h(t) dt + \int_0^\infty e^{-\alpha(t+z)} h(t) dt \right].$$

Upon inverting, one obtains

$$(2.15) \quad \begin{aligned} \varphi = \int_0^x \frac{e^{-\frac{K+2ik}{4} \left[ (x-\xi) + \frac{z^2}{(x-\xi)} \right]}}{\sqrt{\pi (K+2ik)} \sqrt{(x-\xi)}} V(\xi) d\xi + \\ + \frac{1}{2} \sqrt{(K+2ik)} \int_0^\infty \frac{e^{-\frac{(K+2ik)}{4} x}}{\sqrt{\pi x}} f(t) \times \\ \times \left[ e^{-\frac{K+2ik}{4} \frac{(t+z)^2}{x}} + e^{-\frac{(K+2ik)}{4} \frac{(t-z)^2}{x}} \right] dt. \end{aligned}$$

Now, invoking the periodicity requirement due to the cascade arrangement, one has

$$(2.16) \quad \phi_{-n}(x_{-n}, z_{-n}) = e^{-in\sigma} \phi_1(x_1 + nq, z_1 + nd)$$

so that

$$(2.17) \quad f(z_1) = e^{-i\sigma} \phi(q, z_1 + d).$$

Using Eq. (2.17), Eq. (2.15) gives an integral equation for  $\phi$ ,

$$(2.18) \quad \phi(x_1, z_1) = \int_0^{x_1} d\xi \frac{e^{-\frac{(K+2ik)}{4} \left[ (x_1-\xi) + \frac{z_1^2}{(x_1-\xi)} \right]}}{\sqrt{\pi(K+2ik)} \sqrt{(x_1-\xi)}} V(\xi) + \\ + \frac{1}{2} (K+2ik) \int_0^\infty \frac{e^{-\frac{(K+2ik)}{4} x_1}}{\sqrt{\pi x_1}} e^{-i\sigma} \phi(q, t+d) \times \\ \times \left[ e^{\frac{(K+2ik)(t+z_1)^2}{4x_1}} + e^{\frac{(K+2ik)(t-z_1)^2}{4x_1}} \right] dt.$$

By successive approximations, this gives

$$(2.19) \quad \phi(x_1, z_1) = \int_0^{x_1} \frac{e^{-\frac{(K+2ik)}{4} \left[ (x_1-\xi) + \frac{z_1^2}{(x_1-\xi)} \right]}}{\sqrt{\pi(K+2ik)} \sqrt{(x_1-\xi)}} V(\xi) d\xi + \\ + \frac{1}{2} \frac{e^{-i\sigma}}{1-e^{-i\sigma}} \int_0^\infty \frac{e^{-\frac{(K+2ik)}{4} x_1}}{\sqrt{\pi x_1}} \times \\ \times \left[ e^{\frac{(K+2ik)(t+z_1)^2}{4x_1}} + e^{\frac{(K+2ik)(t-z_1)^2}{4x_1}} \right] \times \\ \times \int_0^q \frac{e^{-\frac{(K+2ik)}{4} \left[ (q-\xi) + \frac{(t+d)^2}{(q-\xi)} \right]}}{\sqrt{\pi(q-\xi)}} V(\xi) d\xi.$$

Here, the series

$$e^{-i\sigma} + e^{-2i\sigma} + \dots$$

has been made convergent by assigning a negative imaginary part to  $\sigma$ , and letting  $\text{Im}(\sigma) \Rightarrow 0$ , after the summation has been carried out.

For future use, let

$$(2.20) \quad q(z_1) \equiv \phi(q, z_1).$$

Consider now domain  $\textcircled{1}$  downstream of the pre-interference zone. Upon putting

$$(2.21) \quad \tilde{x}_1 = x_1 - q, \quad \phi(x_1) = \tilde{\phi}(\tilde{x}_1)$$

and Laplace transforming, Eqs. (2.9) and (2.8) give

$$(2.22) \quad \bar{\bar{\phi}}_{z_1 z_1} - \tilde{\alpha}^2 \bar{\bar{\phi}} = -\tilde{h}(z_1),$$

$$(2.23) \quad z_1 = 0: \quad \bar{\bar{\phi}}_{z_1} = -\bar{V}(\tilde{s}),$$

$$(2.24) \quad z_1 = d: \quad \bar{\bar{\phi}}_{z_1} = -e^{-i\sigma} \bar{V}(\tilde{s}),$$

where

$$(2.25) \quad \tilde{\alpha}^2 = (K+2ik) \left( \tilde{s} + \frac{K+2ik}{4} \right),$$

$$h(z_1) = (K+2ik) q(z_1).$$

Using Eqs. (2.23) and (2.24), and solving Eq. (2.22), one obtains

$$(2.26) \quad \bar{\bar{\phi}} = -\frac{\bar{V}(\tilde{s})}{\tilde{\alpha}} \frac{e^{-i\sigma} \cosh \tilde{\alpha} z_1 - \cosh \tilde{\alpha} (z_1 - d)}{\sinh \tilde{\alpha} d} +$$

$$+ \frac{1}{2\tilde{\alpha}} \left[ \int_{z_1}^d e^{-\tilde{\alpha}(t-z_1)} \tilde{h}(t) dt + \right.$$

$$+ \int_0^{z_1} e^{\tilde{\alpha}(t-z_1)} \tilde{h}(t) dt + \int_0^d e^{-\tilde{\alpha}(t+z_1)} h(t) dt -$$

$$\left. \frac{\int_0^d \{e^{\tilde{\alpha}(t-d)} + e^{-\tilde{\alpha}(t+d)}\} \tilde{h}(t) dt}{\sinh \tilde{\alpha} d} \cosh \tilde{\alpha} z_1 \right].$$

Upon inverting, this gives

$$(2.27) \quad \bar{\bar{\phi}}(\tilde{x}_1, z_1) = -\int_0^{\tilde{x}_1} V(\xi) \left[ \frac{e^{-i\sigma}}{\sqrt{(K+2ik)}} e^{-\frac{(K+2ik)(\tilde{x}_1-\xi)}{4}} \times \right.$$

$$\times \left\{ \frac{1}{d\sqrt{(K+2ik)}} + \frac{2}{d\sqrt{(K+2ik)}} \times \right.$$

$$\times \left. \sum_{n=1}^{\infty} (-1)^n e^{-\frac{n^2\pi^2(\tilde{x}_1-\xi)}{d^2(K+2ik)}} \cos \frac{n\pi z_1}{d} \right\} -$$

$$- \frac{1}{\sqrt{(K+2ik)}} e^{-\frac{(K+2ik)(\tilde{x}_1-\xi)}{4}} \left\{ \frac{1}{d\sqrt{(K+2ik)}} + \right.$$

$$+ \frac{2}{d\sqrt{(K+2ik)}} \sum_{n=1}^{\infty} (-1)^n e^{-\frac{n^2\pi^2(\tilde{x}_1-\xi)}{d^2(K+2ik)}} \times$$

$$\left. \left. \times \cos \frac{n\pi(z_1-d)}{d} \right\} \right] d\xi +$$

$$\begin{aligned}
& + \int_0^{x_1} \frac{e^{-\frac{(K+2ik)}{4} \left[ (\bar{x}_1 - \xi) + \frac{z_1^2}{(\bar{x}_1 - \xi)} \right]}}{\sqrt{\pi(K+2ik)} \sqrt{(\bar{x}_1 - \xi)}} V(\xi) d\xi + \\
& + \frac{1}{2} \sqrt{(K+2ik)} \int_0^d \frac{e^{-\frac{(K+2ik)}{4} \bar{x}_1}}{\sqrt{\pi \bar{x}_1}} q(t) \times \\
& \times \left[ e^{\frac{(K+2ik)(t+z_1)^2}{4 \bar{x}_1}} + e^{\frac{(K+2ik)(t-z_1)^2}{4 \bar{x}_1}} \right] dt - \\
& - \frac{1}{2} \sqrt{(K+2ik)} \left[ \int_0^{x_1} e^{-\frac{(K+2ik)\xi}{4}} \times \left\{ \frac{1}{d \sqrt{(K+2ik)}} + \right. \right. \\
& + \left. \left. \frac{2}{d \sqrt{(K+2ik)}} \sum_{n=1}^{\infty} (-1)^n e^{\frac{n^2 \pi^2 \xi}{d^2 (K+2ik)}} \cos \frac{n \pi z_1}{d} \right\} \times \right. \\
& \times \int_0^d e^{\frac{(K+2ik)(\bar{x}_1 - \xi)}{4}} \left\{ \frac{\sqrt{(K+2ik)}(t-d)}{2 \sqrt{\pi} (\bar{x}_1 - \xi)^3} e^{-\frac{(K+2ik)(t-d)}{4(\bar{x}_1 - \xi)}} + \right. \\
& \left. \left. + \frac{\sqrt{(K+2ik)}(t+d)}{2 \sqrt{\pi} (\bar{x}_1 - \xi)^3} e^{-\frac{(K+2ik)(t+d)}{4(\bar{x}_1 - \xi)}} \right\} q(t) dt d\xi.
\end{aligned}$$

The pressure jump across blade 1 is then given by

$$\begin{aligned}
(2.28) \quad p(x_1, 0^-, t) - p(x_1, 0^+, t) &= \\
&= e^{ikt} [e^{-i\sigma} p(x_1, d^-) - p(x_1, 0^+)] = \\
&= 2\gamma(x_1) e^{ikt},
\end{aligned}$$

where

$$(2.29) \quad \gamma(x_1) = -\frac{1}{2} \left( \frac{\partial}{\partial x_1} + ik \right) \left[ e^{-i\sigma} \left\{ \begin{array}{l} \varphi(x_1, d^-) \\ \tilde{\varphi}(x_1, d^-) \end{array} \right\} - \left\{ \begin{array}{l} \varphi(x_1, 0^+) \\ \tilde{\varphi}(x_1, 0^+) \end{array} \right\} \right], \quad x_1 \begin{array}{l} < \\ > \end{array} q.$$

Note that for the case  $q=0$ , one obtains the results corresponding to the sonic leading-edge locus. This is to be expected because of the lack of any pre-interference in the sonic leading-edge limit.

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## REFERENCES

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2. B. K. SHIVAMOGGI, Arch. Mech. [Submitted for publication, 1981].

## STRESZCZENIE

### NIEUSTALONA PALISADA OKOŁODŹWIĘKOWA Z PODDŹWIĘKOWĄ KRAWĘDZIĄ NATARCIA

W pracy przedstawiono sposób zlinearyzowanej analizy nieustalonej palisady okołodźwiękowej z poddźwiękową krawędzią natarcia. Określa się najpierw obszar przedinterferencyjny (w górę palisady), który służy następnie do ustalenia warunków początkowych dla przepływu w obszarze między łopatkami w dół palisady. Przy określaniu rozwiązania dla obszaru przedinterferencyjnego zagadnienie stawia się wstępnie jako problem początkowo-brzegowy. Warunek periodyczności umożliwia wyeliminowanie nieznanych warunków początkowych i prowadzi do równań całkowych dla potencjału prędkości. Równania te rozwiązuje się metodą kolejnych przybliżeń.

## РЕЗЮМЕ

### НЕУСТАНОВИВШАЯСЯ ОКОЛОЗВУКОВАЯ ПАЛИСАДА С ДОЗВУКОВОЙ ГРАНЕЙ АТАКИ

В работе представлен способ линеаризованного анализа неустановившейся околозвуковой палисады с дозвуковой граней атаки. Сначала определяется прединтерференционная область (вверх палисады), которая служит затем для установления начальных условий для течения между лопастями вниз палисады. При определении решения для прединтерференционной области задача ставится предварительно как начально-краевая задача. Условие периодичности дает возможность исключения неизвестных начальных условий и приводит к интегральным уравнениям для потенциала скорости. Эти уравнения решаются методом последовательных приближений.

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