ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 32, 3, 439—447, 1984
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

ON PLASTIC DESIGN OF ROTATING DISCS OF FINITE THICKNESS

W. ZOWCZAK (WARSZAWA)

A new method of plastic design of axially-symmetric rotating discs is proposed. The method of characteristics, known from the mechanics of plastic flow, has been used to construct statically admissible stress fields and thus to find safe profiles of designed discs. The equations of characteristics derived for the case under consideration differ only slightly from the ones for the standard axially-symmetric case. The derived equations have been used to calculate examples of safe profiles of rotating discs for different values of rotational speed.

1. Introduction

Problems of strength and design of axially-symmetric rotating elements have been studied since the middle of the nineteenth century [17]. The distributions of stresses in elastic cylinders and discs of constant and varying thickness were found and the problem of design of the disc of uniform strength was solved (see for example [11, 16]). In the latter case the assumptions were made that the disc was thin and that the variation of thickness along the radius was small. The formula describing the varying thickness of the rotating disc without a hole was found to be

$$y=y_0 \exp\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right), \tag{1.1}$$

where y_0 is the thickness of the disc in its centre, ρ — mass density of the material of the disc, ω — angular speed, r — radius, σ — permissible stress.

Although elastic discs were discussed in the works quoted above, the formula (1.1) resulted from purely static considerations (without use of any constitutive relations).

The plastic analysis of rotating elements was started by Nadai and Donell [7, 8] in 1929. Since then a number of papers has been published concerning this subject. Various effects have been taken into account, including nonhomogeneity [19] and anisotropy [3, 4]; moreover, the dynamical behaviour of rotating discs or cylinders after reaching the critical angular speed has been analysed [12]. In general, two kinds of rotating elements have been considered: thin discs (of uniform or varying thickness) and long cylinders or tubes, corresponding approximately to the conditions of plane stress and plane strain respectively.

Fewer papers have been devoted to the problem of uniform strength design of rotating discs. Kapkowski and Łukasiewicz have studied the influence of nonuniform temperature distribution on the shape of thin discs of uniform strength [6]. KAPKOWSKI has solved the problem of uniform strength design for a thin rotating disc with a hole, loaded additionally by torque applied by means of shear stresses distributed on the inner and outer edge [5]. In both cases the elastic material is assumed and both solutions reduce to the formula (1.1) while the additional factors vanish. Plastic design of thin rotating discs has been discussed by DRUCKER and SHIELD [1]. Minimum weight designs have been found for discs with a hole, made of a rigid-plastic material obeying the Tresca yield criterion.

In the present paper a new method of plastic design of rotating discs is proposed. The method of characteristics, well-known from the theory of plastic flow has been used as a tool for construction of the statically admissible stress fields which, according to the limit design theorems (see [1, 2, 14]), give safe estimations of the shapes of designed elements. Two-dimensional analysis taking into account variation of stresses across the thickness of the disc (instead of one-dimensional, as in all papers quoted above) enables us to design discs without limitations on their thickness (and on its gradient along the radius). Applications of the method of characteristics presented below are analogous to those proposed in [18] and to examples of design of slopes in the mechanics of granular materials [10, 15].

2. The metod of characteristics in the presence of centrifugal body forces

The method of characteristics is used as the standard method of solving axially symmetric problems in the theory of plastic flow [9, 13] and in the mechanics of soils [15]. In the latter case the own-weight of the medium must be taken into account. In mechanical engineering the own-weight of the material can be usually neglected. While designing moving elements, however, the inertia forces must in some cases be taken into account. This is the case when dealing with rotating elements.

Consider the axi-symmetrical element rotating around its axis with angular speed ω. The equations of equilibrium take the form

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_z}{r}=-\rho\omega^2 r, \\ &\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0, \end{aligned} \tag{2.1}$$

where ρ denotes mass density and r the radius. The material is assumed to be rigid-perfectly plastic and to obey the Tresca yield condition

$$(\sigma_z-\sigma_r)^2+4\tau_{rz}^2=4k^2, \tag{2.2}$$

where k is the maximum shearing stress. It is assumed also that the Haar–Kármán postulate holds, i.e. that the circumferential stress is equal to one of the other principal stresses:

$$\sigma_\theta=\sigma_3=\sigma_1, \tag{2.3}$$

or

$$\sigma_\theta = \sigma_3 = \sigma_2 . \tag{2.4}$$

The above equations enable us to find all components of the stress tensor without considering strains and velocities. The method of solution is exactly the same as in the standard case [9, 13]. Two new unknown functions χ and φ are introduced, the first one being proportional to the sum of principal stresses σ_1 and σ_2

$$\chi = \frac{1}{2k}(\sigma_1 + \sigma_2).$$

while the other is the angle between the direction of the larger principal stress ($\sigma_1 > \sigma_2$) and the r — coordinate axis.

In terms of these functions, the nonvanishing components of the stress tensor are expressed by the formulae

$$\begin{aligned} \sigma_r &= 2k\chi + k\cos 2\varphi, \\ \sigma_z &= 2k\chi - k\cos 2\varphi, \\ \tau_{rz} &= k\sin 2\varphi, \\ \sigma_\theta &= k(2\chi \mp 1). \end{aligned} \tag{2.5}$$

The plus sign in the last formula corresponds to the equality (2.3), while the minus sign yields from the equality (2.4). The stresses expressed by the above formulae identically satisfy the yield condition and the Haar-Kármán postulate. Substituting (2.5) into the equations (2.1), one obtains a set of two partial differential quasi--linear equations in two unknown functions χ and φ:

$$\frac{\partial \chi}{\partial r} - \sin 2\varphi \frac{\partial \varphi}{\partial r} + \cos 2\varphi \frac{\partial \varphi}{\partial z} = \frac{1}{2r}(-\cos 2\varphi \mp 1) - \frac{\rho \omega^2 r}{2k},$$

$$\frac{\partial \chi}{\partial z} + \cos 2\varphi \frac{\partial \varphi}{\partial r} + \sin 2\varphi \frac{\partial \varphi}{\partial z} = -\frac{1}{2r}\sin 2\varphi,$$

which differs from the standard case by the additional term representing the inertia forces $-\rho\omega^2 r/2k$, in the first equation. This system of equations is of the hyperbolic type, thus it has two families of real characteristics which can be found by a standard procedure (see for example [13]).

The equations of characteristics of the first family (α-lines) take the form

$$\begin{aligned} \frac{dz}{dr} &= \operatorname{tg}\left(\varphi + \frac{\pi}{4}\right), \\ d\chi + d\varphi &= -\frac{1}{2r}(dz \mp dr) - \xi r\, dr, \end{aligned} \tag{2.6}$$

while the equations of the second family (β-lines) are

$$\begin{aligned} \frac{dz}{dr} &= \operatorname{tg}\left(\varphi - \frac{\pi}{4}\right), \\ d\chi - d\varphi &= -\frac{1}{2r}(dz \mp dr) - \xi r\, dr . \end{aligned} \tag{2.7}$$

The upper signs in the equations $(2.6)_2$ and $(2.7)_2$ correspond to the formula (2.4), while the lower ones yield from the equality (2.3). The characteristics form the angles of $\pm\pi/4$ with the principal directions, thus also in the present case they coincide with the slip-lines.

The coefficient ξ is expressed by the formula

$$\xi = \frac{\rho\omega^2}{2k}.$$

Setting $\omega=0$ (and thus $\xi=0$) in the above equations one obtains well-known standard formulae derived for the case of the absence of body forces.

Thus integration of the partial differential equations of equilibrium has been reduced to the integration of the ordinary differential equations of characteristics. This is usually carried out by means of the finite difference method. Discretization of the equations (2.6)–(2.7) and solving of the basic boundary value problems proceed identically as in the standard axi-symmetrical case and will not be described here.

3. Numerical examples

The equations written above have been used to construct statically admissible stress fields and thus to obtain safe estimations of the profiles of the rotating discs. It has been assumed that on the axis of symmetry AA' of the disc (Fig. 1a) the ma-

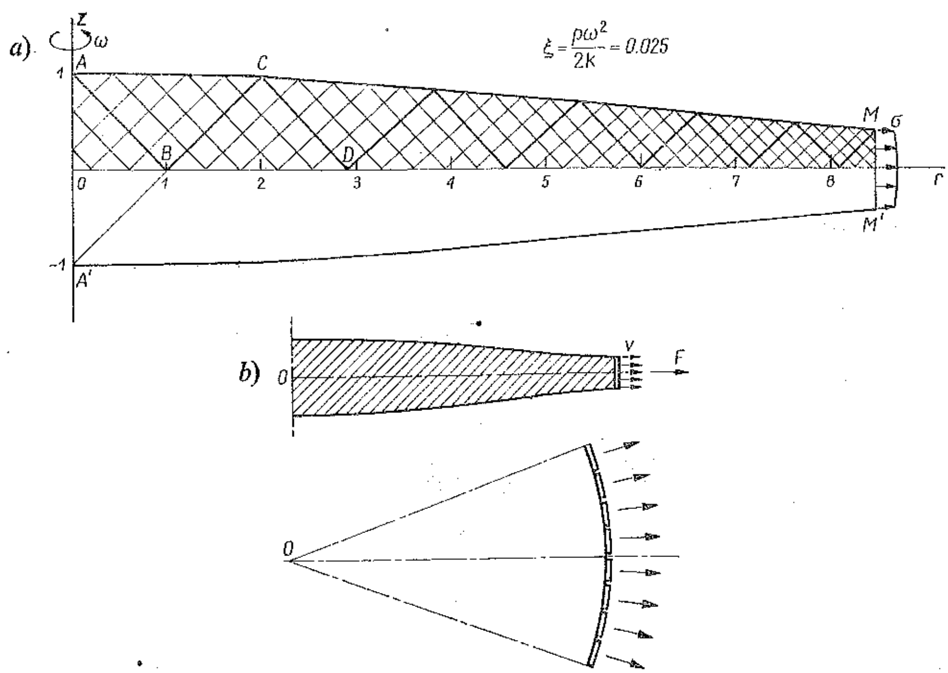

Fig. 1.

terial is in the state of biaxial uniform tension $\sigma_r=\sigma_\theta=2k$, ($\sigma_z=0$), i.e. equal to the yield stress (a similar assumption has been made in the classical solution of uniform strength design of rotating discs [11, 16]). It has been assumed also that circumferential stresses in the disc have maximum (positive) values, i.e. that the equality (2.3) holds.

The construction of the stress field starts from the axis AA' on which the boundary conditions ($\chi=0.5$, $\varphi=0$) are prescribed. In the region $AA'B$, stresses may be found in an elementary way. It is easy to notice that for the reasons of symmetry the principal directions in the whole region $AA'B$ are constant ($\varphi=0$). Thus the characteristics form a rectilinear mesh. However, the value of χ is not constant throughout this region but, due to the presence of body forces, depends on the distance from the axis

$$\chi=\frac{1-\xi r^2}{2}.$$

Having found the characteristic AB (which belongs to the β family) one can solve the inverse boundary value Cauchy problem in the region ABC. This problem consists in determining the state of stress in the whole region together with the shape of the free boundary AC on the basis of given values of χ and φ along the characteristic AB. Here the mesh of characteristics has been found numerically. The shape of the boundary AC has been determined from the condition:

$$\frac{dz}{dr}=\operatorname{tg}\varphi,$$

and that at every point of it we have $\chi=1/2$.

The position of the segment AA' has been chosen to be symmetrical in relation to the r-axis, so the r-axis determines the plane of symmetry of the disc. At every point of this plane we have $\varphi=0$. Thus the stress field may be constructed for one (say upper) half of the disc only.

Having solved the inverse boundary value Cauchy problem in ABC we obtain the characteristic BC (which belongs to the α family). On the basis of the known value $\varphi=0$ along BD and calculated values of χ and φ along BC, the mixed boundary value problem can be solved in the whole region BDC and the characteristic CD determined.

Then the above sequence of boundary value problems is to be repeated as many times as required in order to obtain the safe profile of the disc up to its prescribed outer radius. The stresses σ on the external boundary MM' (Fig. 1a) result from the constructed statically admissible stress field. Their integration along MM' gives us a resultant force F acting on the unit length of the circumference of the disc.

Assume that the conditions on the boundary MM' are of the kinematic type, e.g. in the form of the velocities in the direction r, uniformly distributed along MM'. (This can be realized by means of a set of rigid plates attached on the external contour as it has been shown in Fig. 1b). According to the limit design theorems the disc of the shape so obtained will not collapse under inertia forces caused by its rotation and under the external loading on its outer boundary MM', provided this loading does not exceed the calculated value F.

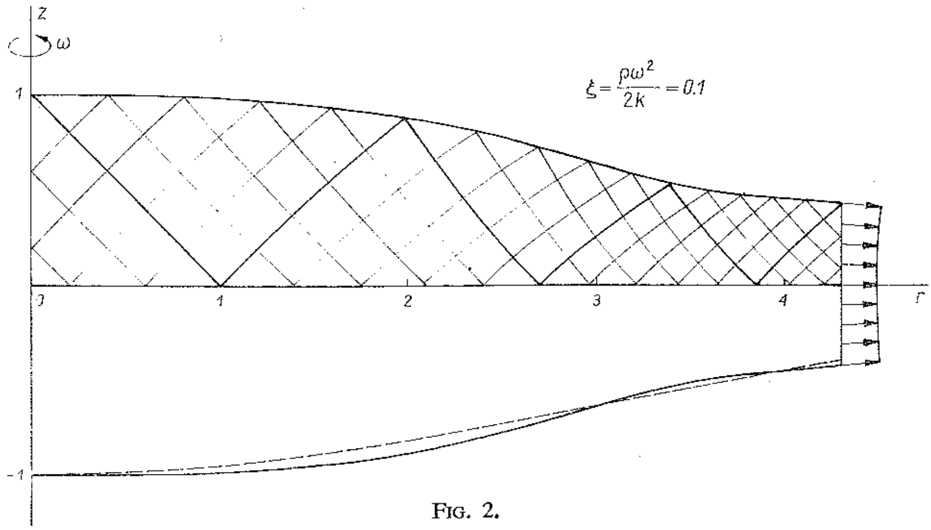

FIG. 2.

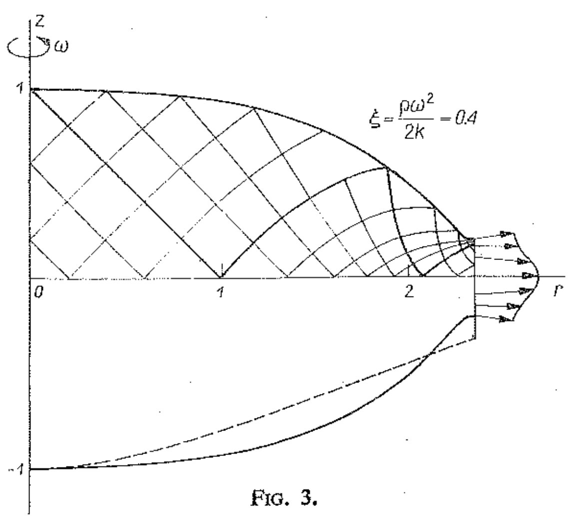

FIG. 3.

The example presented in Fig. 1a has been calculated for a particular value of the characteristic parameter ξ, namely $\xi=0.025$. This roughly corresponds to the disc of the thickness in the center (for $r=0$) 0.2 m, made of the steel whose yield stress is $2k=310$ MPa rotating with rotational speed of 3000 revolutions per minute ($\omega=100\pi\ \mathrm{s}^{-1}$). The obtained solution differs only slightly from the classical solution (1.1).

In the next figures analogous examples are presented. The values of the characteristic parameter have been chosen to be $\xi=0.1$ (Fig. 2) and $\xi=0.4$ (Fig. 3), which correspond to rotational speeds of 6000 and 12000 rpm respectively (thickness in the center of the discs and the material of which they are made being the same). For these relatively larger rotational speeds the differences between the classical solutions for the thin discs (shown by the dashed line) and the present solutions are clearly visible. Particularly in the last example, where thickness of the disc changes rapidly in the radial direction, the use of the solution (1.1) neglecting stresses in the z-direction can give only rough approximation of the required shape.

4. Final remarks

The presented examples should be treated as illustrations rather of the new approach to the design of rotating discs, than the solutions recommended for practical use. The boundary condition in the form of biaxial tension ($\chi=0.5$) uniformly distributed along AA' (Figs. 1–3), has been assumed in order to make the present problem compatible with the classical problem of uniform strength design of a rotating disc. It should be noted, however, that this assumption is somewhat artificial and causes that, for greater value of ξ, the curve determining the profile of the disc is not monotonic but shows a tendecy to oscillate. It seems that the slight decreasing of χ along AC would be advantageous, but the choice of a particular distribution of χ for an assumed value of the parameter ξ remains an open question.

There is also the possibility of solving an analogous problem, the boundary conditions being prescribed on the outer boundary of the disc. In this case the construction of the statically admissible stress field proceeds in the opposite direction, towards the centre. However, the difficulties arise while computing the stresses in the vicinity of the axis. The special stress field should be constructed in this region because for the arbitrarily assumed conditions on the outer boundary the material in the centre will not be generally in the plastic state. Thus the method of characteristics cannot be used there.

The method presented above may also be applied to the plastic design of other axially-symmetric rotating machine parts (tubes and shafts of varying diameter etc.).

Acknowledgement

The author expresses his gratitude to Mr. J. Świszulski M. Sc., and Mr. J. Olszewski M. Sc., from the Computer Centre of the Institute of Building Mechanization for their help in performing the calculations.

References

1. D. C. Drucker, R. T. Shield, *Design for minimum weight*, Proceedings of the IXth International Congress of Applied Mechanics, vol. 5, Bruxelles 1957.
2. D. C. Drucker, R. T. Shield, *Bounds on minimum weight design*, Quart. Appl. Math., **15**, 3, 269–271, 1957.
3. D. Durban, V. Birman, *Elasto-plastic analysis of an anisotropic rotating disc*, Acta Mech., **49**, 1, 1–10, 1983.
4. W. W. Feng, *Finite deformation of anisotropic plastic rotating discs*, AIAA J., **16**, 1205–1207 1978.
5. J. Kapkowski, *Discs of uniform strength subjected to axially symmetric loads* [in Polish], Arch. Bud. Masz., **12**, 2, 249–260, 1965.
6. J. Kapkowski, S. Łukasiewicz, *The influence of temperature on the shape of rotating discs of uniform strength*, Bull. Acad. Polon. Sci., Série Sci. tech., **9**, 1, 7–16, 1965.
7. A. Nadai, L. H. Donnell, *Stress distribution in rotating discs of ductile material after the yield point has been reached*, ASME Trans., APM — 51–16, 173–181, 1929.
8. A. Nadai, *Theory of flow and fracture of solids*, McGraw-Hill, 1950.
9. R. T. Shield, *On the plastic flow of metals under conditions of axial symmetry*, Proc. Roy. Soc., **233A**, 267–287, 1955.
10. V. V. Sokolovsky, *Statics of granular medium* [in Russian], Fizmatgizdat, Moscow 1960.
11. A. Stodola, *Die Dampfturbinen*, Springer, Berlin 1904.
12. W. Szczepiński, *Dynamic expansion of a rotating solid cylinder of mild steel*, Arch. Mech., **9**, 1, 75–87, 1967.
13. W. Szczepiński, *Introduction to the mechanics of the plastic forming of metals*, PWN, Noordhoff and Siithoff, 1979.
14. W. Szczepiński, *Limit analysis and plastic design of structural elements of complex shape*, Progress in Aerospace Sciences, Vol. 12, Pergamon Press, Oxford 1972.
15. W. Szczepiński, *Limit states and kinematics of granular media* [in Polish], PWN, Warszawa 1974
16. S. P. Timoshenko, *Problems of strength in steam turbines* [in Russian] Вестник общества технологов, **19**, 7, стр. 266–277, С. Петербург 1912 (reprinted in: С. П. Тимошенко, *Прочность и колебания элементов конструкции*, Наука, Москва 1975).
17. S. P. Timoshenko, *History of the strength of materials*, McGraw-Hill, 1953.
18. W. Zowczak, *Design of complex machine parts by means of statically admissible continuous stress fields*, ZAMM, 64, 4–5, 1984.
19. W. Życzkowski, *Limit state of nonhomogeneous rotating discs* [in Polish], Rozpr. Inż. 5, 49–96, 1957.

Streszczenie

O PLASTYCZNYM PROJEKTOWANIU WIRUJĄCYCH TARCZ SKOŃCZONEJ GRUBOŚCI

Zaproponowano nową metodę plastycznego projektowania osiowosymetrycznych tarcz wirujących. Do budowy statycznie dopuszczalnych pól naprężeń wyznaczających bezpieczny profil tarczy, zastosowano znaną z mechaniki plastycznego płynięcia metodę charakterystyk. Wyprowadzone równania charakterystyk dla przypadku występowania odśrodkowych sił bezwłasności różnią się nieznacznie od równań dla przypadku niewystępowania sił masowych. Posługując się wyprowadzonymi równaniami wyznaczono przykłady bezpiecznych profili tarcz wirujących dla różnych prędkości obrotowych.

Резюме

О ПЛАСТИЧЕСКОМ ПРОЕКТИРОВАНИИ ВРАЩАЮЩИХСЯ ДИСКОВ КОНЕЧНОЙ ТОЛЩИНЫ

Предлагается новый метод пластического проектирования осесимметричных вращающихся дисков. При конструировании статически допускаемых полей напряжений использован, известный в теории пластического течения, метод характеристик. Выведенные уравнения характеристик, в случае выступления центробежных сил инерции, незначительно отличаются от уравнений в случае невыступления массовых сил. Используя выведенные уравнения, вычислены примеры безопасных профилей вращающихся дисков для разных скоростей вращения.

INSTITUTE OF BUILDING MECHANIZATION

Received March 9, 1984.