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PLANE VISCOUS FLOW THROUGH THE SYSTEM OF MOVING RODS

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The subject of this paper is a steady flow of an incompressible fluid about the system of two parallel rows of cylindrical rods, moving along their axes. The row of cylinders has been considered as a continuous surface, which allowed for the nonzero normal as well as the tangent velocity component. Both surfaces have been treated as moving walls of a parallel channel. The inner and outer flows have been calculated with the aid of perturbation methods applied to suitable exact solutions of the Navier–Stokes equations. Both solutions have been matched with the use of proper conditions at the wall to obtain a continuous flow description in the whole region. Pressure, velocity and streamlines distributions have been calculated. The error of the adopted approximation has been estimated and discussed.

1. Introduction

The subject of this paper is a steady laminar flow of a viscous incompressible fluid through a system of parallel identical cylindrical rods. Axes of the rods are uniformly distributed in two parallel half-planes being perpendicular to an imper-

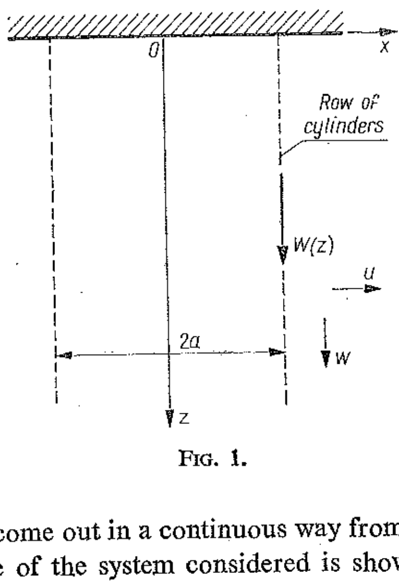

Fig. 1.

meable wall. The rods come out in a continuous way from the wall and move along their axes. The scheme of the system considered is shown in Fig. 1. We use the Cartesian coordinate system in which the z-axis coincides with the axis of symmetry and the x-axis is posed at the lateral wall. The velocity components in the x- and z-directions are denoted by u and w. The distance between the half-planes is equal to $2a$. All the rods move with the same velocity $W(z)$. The motion of the rods induces flow of the fluid surroundings. This is the subject of theoretical investigation in this paper.

The problem formulated here originates from the field of man-made fibres manufacturing. Moving cylindrical rods correspond to the spun fibres, extruded from the spinneret fixed at $z=0$, through a large number of orifices. However, the present work can be considered merely as the first approximation of real processes of fibre formation since it takes into account only hydrodynamic effects, neglecting effects of rheological and thermal nature.

The flow past a row of cylinders has been considered in [1, 2, 3]. According to the results of these papers, a row of discrete cylindrical rods may be treated as a continuous surface which allows for the nonzero normal as well as the tangent velocity component of the fluid at the surface itself. Details of this approximation will be given in the next section. It was first applied to the description of man-made fibre formation in [4]; however, the system considered there differed in geometrical and kinematical properties from that of ours so we could not make use of those results here.

The rows of cylinders treated as continuous surfaces may be thought of as walls of a parallel channel of a width $2a$. In this respect a flow between the half-planes may be considered as a flow in a channel. At the external sides of the walls we expect a flow of a boundary layer type. On the wall itself, which is an interface between the two types of flow, the suitable matching conditions [3] will allow to obtain a continuous flow description at both sides of the wall.

The problem is then very complex since it involves three types of flow:

i) channel flow,
ii) boundary layer flow,
iii) flow over rows of cylindrical rods.

All the three flows mutually interact on each other, what must be taken into account to obtain a realistic description of the problem.

In the present work we did not search for the solution by the direct integration of the Navier–Stokes equations what would involve much numerical computation. Instead we tried to obtain solutions in a relatively simple analytical form by making use of the existing solutions for the component flows: the Poiseuille solution for the inner flow [5, 6], the Schlichting solution for the outer flow [7] and the solutions for the flow over a row of cylinders in the Stokes approximation [2, 3]. The adoption of these solutions to our problem, not exactly corresponding to the conditions for which the solutions have been found, generates errors estimated and discussed in Sect. 6.

2. Flow past a row of cylinders

The starting point of our analysis is a flow past a row of cylinders since in a system given in Fig. 1 it is the element which drives flows in both the inner and outer region. The laminar flow past a row consisting of cylindrical rods, which from now we shall call for brevity a screen, was the subject of very few papers. In the literature we could find only the works of TAMADA and FUJIKAWA [1] and MIYAGI [2] devoted to the flow normal to the screen, that is in the x-direction, and a paper of SZANIAWSKI [3], who considered a flow parallel to the screen in two directions: perpendicular and longitudinal to the cylinders axes, that is in the y- and z-directions (Fig. 2). In the present paper we shall make use of the results given in the above papers so the brief summary of their results will now be given.

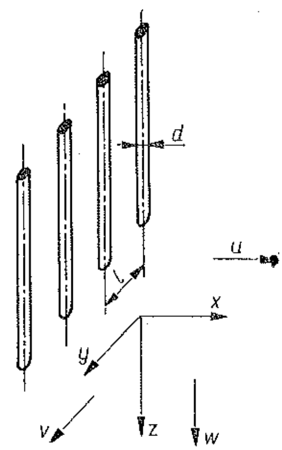

FIG. 2.

The screen considered in [1, 2, 3] consists of an infinite number infinitely long, parallel, identical, equidistant cylindrical rods of a diameter d. The distance between the cylinders is equal to l. It is assumed that the ratio $\delta = d/l$ is small

$$\delta = d/l \ll 1 . \tag{2.1}$$

Considerations of the papers [2 and 3] made in the Stokes approximation of the Navier-Stokes equations allowed to calculate the detailed structure of spatially periodic flow around the cylinders. However, it appeared that a periodic character of the flow is limited to the thin layer of the order l. In this respect the row of cylinders may be treated as a uniform surface of such properties that parameters between both its sides may change in a discontinous way. These parameters are interconnected by relations obtained from conservation laws. The structure of the screen comes into these relations through suitable coefficients.

The relation for a flow normal to the screen reads as follows [1, 2, 3]:

$$U = -lk_x \frac{p_+ - p_-}{\mu}, \tag{2.2}$$

where the permeability coefficient k_x is determined by the relation

$$k_x = \frac{1}{4\pi}\left(\ln\frac{1}{\pi\delta} + \frac{1}{2}\right), \tag{2.3}$$

U is an asymptotic value of a velocity component normal to the screen up-and downstream, p_+ and p_- are asymptotic values of pressure, the subscripts „+" and „−" being in accordance with the direction of the x-axis, μ is the viscosity coefficient.

In the case of a flow parallel to the screen consisting of discrete cylinders, nonslip conditions can be satisfied merely on the surfaces of rigid cylinders. However, according to the approach given in [3] the screen may be considered as a uniform surface on which a fluid moves with a mean uniform slip velocity whose magnitude depends on the asymptotic values of flow parameters at both sides of the screen and on its structure.

The slip velocity in the direction perpendicular to the cylinders' axes is given by the relation

$$\Delta V = lk_y\left(\left.\frac{\partial v}{\partial x}\right|_+ - \left.\frac{\partial v}{\partial x}\right|_-\right), \tag{2.3}$$

and the slip velocity in the direction parallel to the axes reads

$$\Delta W = lk_z\left(\left.\frac{\partial w}{\partial x}\right|_+ - \left.\frac{\partial w}{\partial x}\right|_-\right), \tag{2.4}$$

where the slip coefficients are related to the screen structure by the formulae

$$k_y = \frac{1}{4\pi}\left(\ln\frac{1}{\pi\delta} - \frac{1}{2}\right), \tag{2.5}$$

$$k_z = \frac{1}{2\pi}\ln\frac{1}{\pi\delta}. \tag{2.6}$$

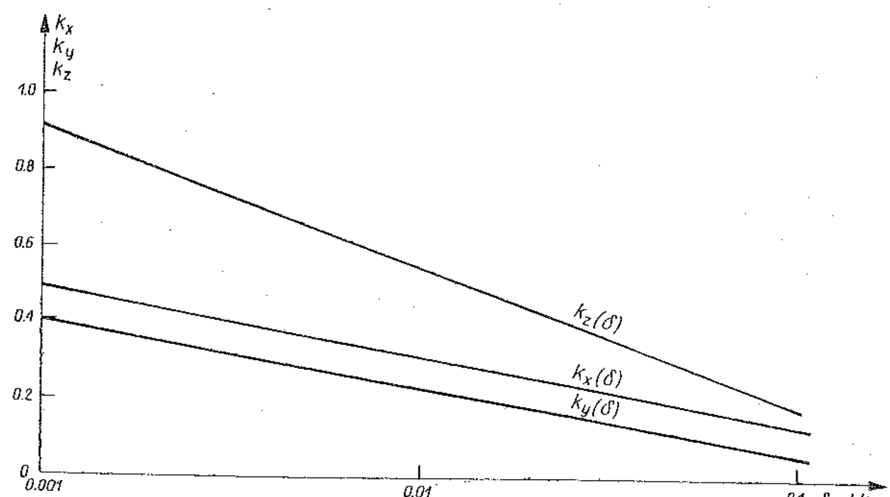

FIG. 3.

The graphs k_x, $k_y(\delta)$ and $k_z(\delta)$ are plotted in Fig. 3. In this paper we deal with a two-dimensional flow in the plane xz so in next sections we shall use merely Eqs. (2.2) and (2.4). Equation (2.3) for a flow in the y-direction has been included here after [3] for the sake of completeness.

3. Inner and outer flow

In this section we consider the inner flow which develops between parallel moving screens ($0 \leqslant |x| \leqslant a$) and the outer flow on the external sides of the screens ($|x| > a$). For the sake of symmetry we take into account only the half-plane $x \geqslant 0$. Both solutions will be matched with the aid of relations given in Sect. 2.

As it has been mentioned in Sect. 1. the solution of the inner or channel flow will be searched for in the form of a slightly disturbed Poiseuille flow [5, 6]. The velocity components may be then presented as follows:

$$w_i = w_p + \tilde{w}, \qquad u_i = \tilde{u}, \tag{3.1}$$

where

$$w_p = w_1 - \frac{1}{2} \frac{dP}{dz} a^2 (1 - \xi^2), \qquad 0 \leqslant \xi \leqslant 1 \tag{3.2}$$

is the basic Poiseuille solution, and $\tilde{w}$, $\tilde{u}$ are its small perturbations. Quantities which appeared in Eq. (3.2) have the following meaning:

$$P = \frac{p_i - p_0}{\mu}, \qquad p_0 = \text{const}, \tag{3.3}$$

is a kinematic pressure function,

$$\xi = x/a \tag{3.4}$$

is a nondimensional coordinate and w_1 is a fluid velocity at the wall of the channel. The subscripts „i" and „0" refer to the inner and outer flows, respectively, while the subscript „1" refers to the wall. The velocity w_1 differs from the velocity of cylindrical rods W by the component ΔW which is a slip velocity discussed in the previous section

$$w_1 = W + \Delta W. \tag{3.5}$$

The basic solution used for the description of the outer flow is the solution given by Schlichting [7]. In application to our case this solution corresponds to a boundary layer flow over a continuous permeable flat wall moving at a constant velocity w_1, with a uniform suction velocity $u_1 < 0$. We present this solution in the form

$$w_0 = w_1 \exp\left[\frac{u_1 a}{\nu}(\xi - 1)\right], \qquad \xi \geqslant 1. \tag{3.6}$$

Equations (3.2) and (3.6) are the exact solutions of the Navier-Stokes equations, provided w_1, u_1, and dP/dz are constant. In this paper we allow that all the three quantities may vary in the z-direction $w_1=w_1(z)$, $u_1=u_1(z)$, $P'=P'(z)$ what makes that Eqs. (3.2) and (3.6) may be merely the approximate solutions of our problem. The error of this approximation will be examined in Sect. 6.

From the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{3.7}$$

we can calculate a normal velocity distribution for the inner and outer flow, with the boundary conditions $u(0)=0$ and $u(1)=u_1$, respectively.

$$\begin{aligned} u_i &= \xi\left[\frac{P''a^2}{6}(3-\xi^2)-w_1'a\right], \qquad 0\leqslant\xi\leqslant 1, \\ u_0 &= u_1+\frac{\nu}{u_1}\left(w_1'-w_1\frac{u_1'}{u_1}\right)\left\{1-\exp\left[\frac{u_1 a}{\gamma}(\xi-1)\right]\right\}- \\ &\qquad -\frac{w_1}{u_1}au_1'(\xi-1)\exp\left[\frac{u_1 a}{\nu}(\xi-1)\right], \end{aligned} \tag{3.8}$$

where the primes (′) denote differentiation with respect to z.

The normal velocity $u_1(z)$ on the base of Eqs. (2.2) and (3.3) reads

$$u_1=lk_x P(z). \tag{3.9}$$

Having calculated velocity distribution on both sides of the wall we can obtain the relation for the velocity at the wall itself $w_1(z)$ from Eq. (3.5). Differentiating Eqs. (3.6) and (3.2) with respect to x and inserting calculated derivatives at $\xi=1$ into Eqs. (2.4) and (3.5), we arrive at the relation

$$w_1=\frac{W-\alpha k_z Pa^2}{1-\alpha k_z\dfrac{u_1 a}{\nu}}, \tag{3.10}$$

where

$$\alpha=l/a. \tag{3.11}$$

Application of the model presented in the previous section to this problem is justified if the distance l between the rods in the screen is small as compared to the characteristic dimension of the system. Hence we have

$$\alpha\ll 1. \tag{3.12}$$

Assuming that k_z and $u_1 a/\nu$ are at most of the order unity and making use of Eq. (3.12), we obtain

$$\alpha k_z\frac{u_1 a}{\nu}\ll 1. \tag{3.13}$$

This condition allows to express w_1 (3.10) in a simplified form

$$w_1 \approx W - ak_z P' a^2 . \tag{3.14}$$

The knowledge of the velocity field enables calculations of the streamlines $Q=$ const from the equation

$$dQ = wdx - udz , \tag{3.15}$$

where Q denotes the volume flow rate.

Integrating Eq. (3.15) and making use of the relations for the velocity components in the inner and outer flow, we obtain the following streamlines equations:

$$\begin{aligned} \frac{Q}{a} &= w_1 \xi - \frac{P' a^2}{6} \xi (3-\xi^2), \quad 0 \leqslant \xi \leqslant 1 , \\ \frac{Q}{a} &= w_1 \left\{ 1 + \frac{\nu}{u_1 a} \left[1 - \exp\left(\frac{u_1 a}{\nu} (\xi - 1) \right) \right] \right\} - \frac{P' a^2}{3}, \quad \xi > 1 . \end{aligned} \tag{3.16}$$

4. Calculation of the pressure function

The pressure function $P(z)$, unknown as yet, which appeared in all the expressions for the velocity and streamlines distribution may be found from the continuity condition for the transversal velocities $(3.8)_1$, (3.9) at the wall [5, 6]

$$u(1, z) = u_1(z) . \tag{4.1}$$

Putting $\xi=1$ into Eq. $(3.8)_1$ and inserting it together with the relation (3.14) into Eq. (4.1) we obtain, after some rearrangements, the second order differential equation for the pressure function

$$(1+3ak_z) P'' a^3 - 3ak_x Pa = 3W' a . \tag{4.2}$$

This equation has some resemblance to the Reynolds equation in the hydrodynamic theory of lubrication [8] for pressure in a bearing. It is also similar to the analogous equations from the previous papers [5, 6] except from the factor in brackets at the second derivative taking into account the velocity slip at the wall.

Equation (3.18) contains one additional function $W(z)$ not specified as yet, which ought to describe kinematics of the wall. We assume that $W(z)$ increases from 0 to W_∞, according to the exponential relation [6]

$$W(z) = W_\infty (1 - e^{-\beta \cdot z/a}) , \tag{4.3}$$

where β is a nondimensional parameter. We also assume that the coefficients $k_x(\delta)$ and $k_z(\delta)$ remain constant.

Inserting Eq. (4.3) after differentiating into Eq. (4.2) and introducing the nondimensional quantities

$$\Pi(\zeta) = \frac{P}{\bar{P}}, \quad \bar{P} = \frac{W_\infty}{a} \frac{\sqrt{3}}{\sqrt{ak_x(1+3ak_z)}} , \tag{4.4}$$

(4.5)
$$\zeta=\omega\frac{z}{a},\qquad w=\frac{\sqrt{3\alpha k_x}}{\sqrt{1+3\alpha k_z}},$$

(4.6)
$$\gamma=\frac{\beta}{\omega},$$

we transform Eq. (4.2) into a nondimensional form

(4.7)
$$\Pi''-\Pi=\gamma e^{-\gamma\zeta},$$

which contains only one nondimensional parameter γ and is identical with that obtained in [6] although the variables here have not exactly the same meaning.

The boundary conditions are as follows:

(4.8)
$$\Pi'(0)=0,\quad \lim_{\zeta\to\infty}\Pi(\zeta)=0.$$

The first of them provides that $w=0$ at $z=0$ (see Eqs. (3.2), (3.14) and (4.3)) what results from the assumption that the lateral wall ($z=0$) is impermeable. The second condition results from the requirement that the pressure function be finite at infinity.

The solution of Eq. (4.7) with the boundary conditions (4.8) has the following form [6]:

(4.9)
$$\Pi(\zeta)=\begin{cases}-\dfrac{\gamma}{\gamma^2-1}(\gamma e^{-\zeta}-e^{-\gamma\zeta}) & \text{for}\quad \gamma\neq 1,\\[2mm] -\dfrac{1}{2}(1+\zeta)e^{-\zeta} & \text{for}\quad \gamma=1.\end{cases}$$

This function has been plotted in Fig. 4. It is seen that the pressure at the inner side of the wall is always less than the ambient pressure except the asymptotic value in infinity where both pressure equalize.

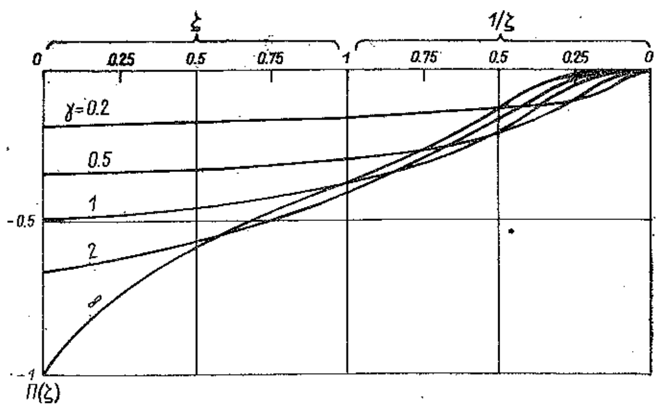

FIG. 4.

The greatest sucking effect, which occurs at the initial cross-section of the channel $z=0$, will be the subject of more detailed examination. By means of Eqs. (4.9) and (4.3) the pressure drop at $z=0$ may be presented in the form

$$(4.10)\qquad P(0)\frac{a}{W_\infty}=-\frac{1}{\sqrt{3ak_x(\delta)}}\,\frac{3\beta}{\beta\sqrt{1+3ak_z(\delta)}+\sqrt{3ak_x(\delta)}}$$

which allows to examine the influence of the wall structure (a and δ) and the wall extensibility (β).

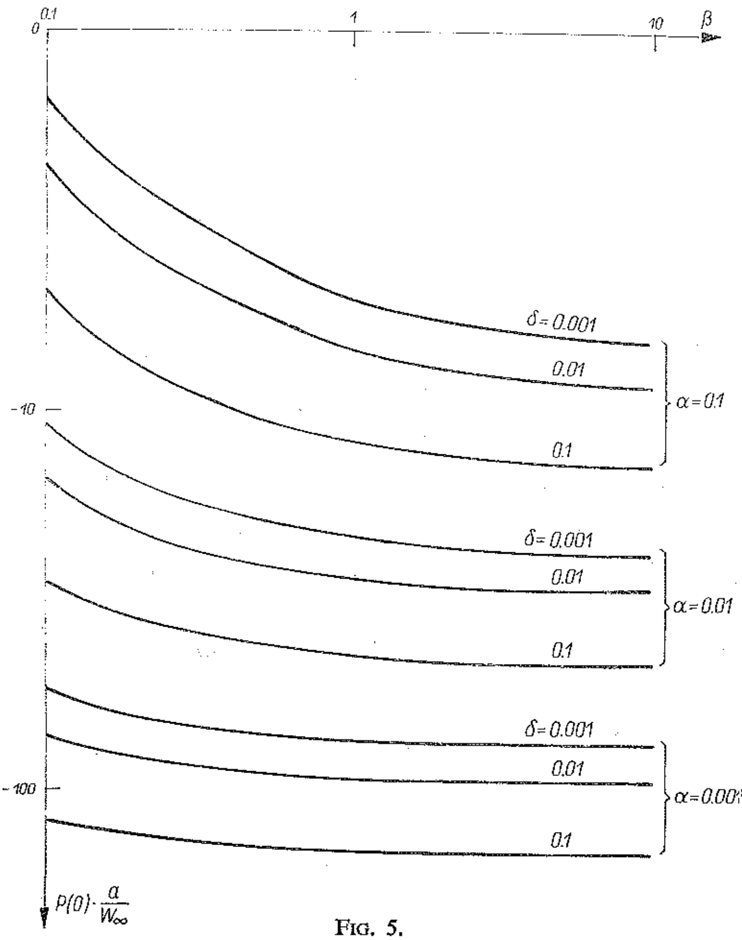

FIG. 5.

The relation (4.10) has been plotted in Fig. 5, for a wide range of the three parameters a, β, δ. We can see that a higher hydrodynamic resistance connected with a more close packing of the rods in the row (small a, high δ) results in a higher pressure drop between the outer and inner side of the wall. The growth of the extensibility factor β also increases the pressure drop, what is more pronounced for walls of lower hydrodynamic resistance.

5. Streamlines distribution

The streamlines equations (3.16) can be transformed with the aid of Eqs. (4.4)–(4.6) into the nondimensional form

(5.1)
$$q_i = \frac{Q_i}{W_\infty a} = \psi\xi - \frac{\xi(3\sigma-\xi^2)}{3\sigma-1}\Pi',$$
$$q_0 = \frac{Q_0}{W_\infty a} = \psi - \Pi' - \left(\psi - \frac{3(\sigma-1)}{3\sigma-1}\Pi'\right)\frac{1-e^{\omega\Pi \mathrm{Re}(\xi-1)}}{\omega\Pi \mathrm{Re}},$$

where

(5.2)
$$\sigma = 1 + 2ak_z(\delta),$$

(5.3)
$$\psi(\zeta) = \frac{W}{W_\infty} = 1 - e^{-\gamma\zeta},$$

(5.4)
$$\mathrm{Re} = \frac{W_\infty a}{\nu}.$$

Equation (5.1) for the inner flow depends on two parameters: γ Eq. (4.6) and the slip factor defined by Eq. (5.2). Equation $(5.1)_2$ depends on three parameters since besides the two mentioned above it contains a product ωRe which we shall treat as the third parameter.

The slip factor σ may be estimated from Eq. (5.2) assuming that realistic values which can be taken for α and δ are about 0.1 and 10^{-4}, respectively. We thus get $\sigma \approx 1.25$ and take it as a limiting value.

The streamline $q=0$ has been chosen to coincide with the axes of the coordinate system xz; however, the analysis of Eq. $(5.1)_1$ displays the possibility of a supplementary line $q=0$ which extends from the axis ξ to the axis ζ thus bypassing the origin of the coordinate system. The form of this streamline is obtained from Eq. $(5.1)_1$ putting $q=0$

(5.5)
$$\xi^2 = 3\sigma - \frac{\psi(\zeta)}{\Pi'(\zeta)}(3\sigma-1).$$

The points in which the line $q=0$ intersects the coordinate axes, $(\xi_0, 0)$ and $(0, \xi)$, have been found from Eq. (5.5) and the results are plotted in Fig. 6. The supplementary streamline appears if the following condition is satisfied:

(5.6)
$$\gamma > 3\sigma - 1,$$

which results from Eq. (5.5) for $\xi^2 > 0$. We can see from Fig. 6 and Eq. (5.6) that the slip effect has a significant influence on the generation of the suplementary streamline and all the same on the inner flow.

The influence of the slip effect on the outer flow can be detected in Fig. 7, where the difference $q_\infty - q_1$ multiplied by ωRe is plotted against ζ for several values of γ, $q_\infty = \lim_{\xi\to\infty} q_i(\xi, \zeta)$ is the flow rate outside the boundary layer and $q_1 = q(1, \zeta)$

is the flow rate at the wall. It can be seen that the slip effect has a negligible influence on the flow rate distribution on the outer side of the wall.

As an example of a flow through the system considered we present the streamlines distribution calculated for the case of $\gamma=5$, to have a flow pattern with the

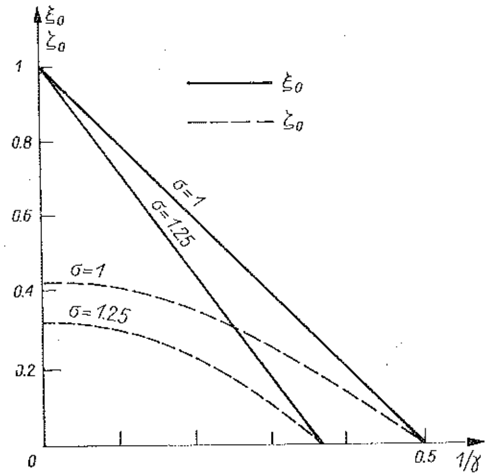

FIG. 6.

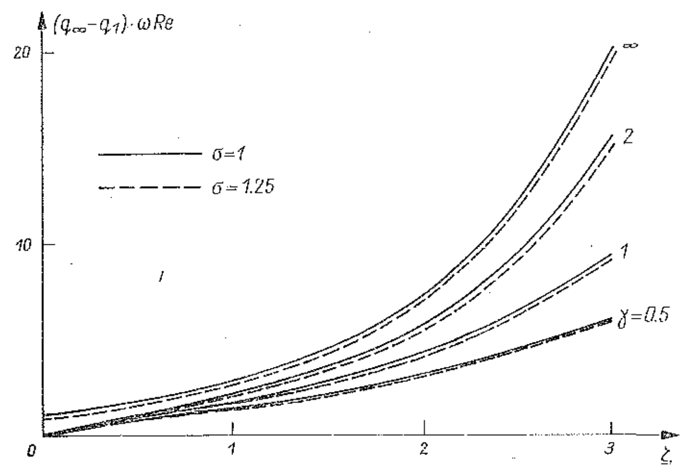

FIG. 7.

supplementary streamline $q=0$ (Fig. 8). For the inner flow we take into account the parameter σ while for the outer flow we retained only the parameter ωRe having neglected σ as less important in this region. It is seen that below the streamline $q=0$ the secondary current circulates with a negative rate flow forming a kind of a separation bubble. In the outer flow we can observe the significant influence of the Reynolds number on the boundary layer thickness.

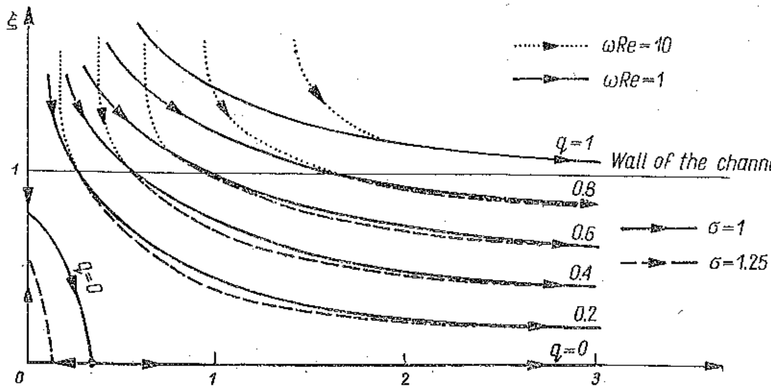

FIG. 8.

The displacement thickness δ^* is the case of a moving wall can be expressed in the form

$$\delta^* = \frac{1}{w_1} \int_a^\infty w dx . \tag{5.7}$$

Inserting Eq. (3.6) into Eq. (5.7) we obtain

$$\frac{\delta^*}{a} = -\frac{1}{\omega \mathrm{Re}\, \Pi(\zeta)}, \tag{5.8}$$

what means that in the case considered the displacement thickness is simply proportional to the reciprocal of the nondimensional pressure function (4.9).

6. THE VALIDITY OF THE MODEL

The application of the Poiseuille solution to the channel flow needs the velocity perturbations $\tilde{u}$ and $\tilde{w}$ to be small as compared with the Poiseuille velocity profile

$$\frac{\tilde{u}}{w_p} \ll 1, \quad \frac{\tilde{w}}{w_p} \ll 1. \tag{6.1}$$

The first conditions is generally satisfied except for the neighbourhood of the lateral wall ($z=0$) where it locally fails, what may be seen in Fig. 8. The second condition needs more detailed examination. The axial velocity perturbation, so far neglected, may be calculated from the linearized Navier-Stokes equations in a similar way as it was carried out in [5, 6]. In this respect we give here the final formula for $\tilde{w}$, without repeating the whole procedure

$$\tilde{w} = \frac{W' a^2}{\nu} [W W_1(\xi) - \tfrac{1}{2} P' a^2 W_2(\xi)] + \frac{a k_x P a}{\nu} [W W_3(\xi) - \tfrac{1}{2} P' a^2 W_4(\xi)], \tag{6.2}$$

where

$$W_1(\xi)=\frac{(1-\xi^2)^2}{4(3\sigma-1)},$$

$$W_2(\xi)=\frac{(1-\xi^2)^2}{60(3\sigma-1)}[15\sigma+7-(15\sigma-7)\xi^2+2\xi^4],$$

$$W_3(\xi)=\frac{1-\xi^2}{4(3\sigma-1)}[6\sigma-(1+\xi^2)],$$

$$W_4(\xi)=\frac{1-\xi^2}{30(3\sigma-1)}(45\sigma^2+1+\xi^2+\xi^4).$$

The condition $(6.1)_2$ is always satisfied at infinity ($\zeta\to\infty$) and at the wall ($\xi=1$), what resulted from the adopted boundary conditions [6]; however, in the interior of the channel, including the initial cross-section, $z=0$, the fulfillment of this condition is not obvious and needs examination. Since the strongest deviation from the Poiseuille flow occurs at the plane $z=0$, we shall confine ourselves to the analysis of the condition $(6.1)_2$ in the origin of the coordinate system (0, 0).

Let us introduce the ratio

$$\varepsilon_i=\frac{\tilde{w}(0,0)}{W(0,0)}. \tag{6.3}$$

It is not exactly the condition $(6.1)_2$ since as a reference velocity, for the sake of simplicity, W instead w_p has been introduced. Inserting Eqs. (6.2) and (5.3) into Eq. (6.3) we obtain

$$\varepsilon_i=\beta\,\mathrm{Re}\,|F(\gamma,\sigma)|\ll 1, \tag{6.4}$$

where

$$F(\gamma,\sigma)=W_1(0)-AW_2(0)-\frac{1}{\gamma+1}[W_3(0)-AW_4(0)],$$

$$A=\frac{3}{3\sigma-1}\,\frac{\gamma}{\gamma+1}.$$

The ratio $\varepsilon_i/(\beta\,\mathrm{Re})$ has been plotted against γ for a wide range of the slip factor $\sigma=1$ and Eq. (1.25) (Fig. 9).

For the outer flow we shall estimate the error of approximation inserting the velocity components (3.6) and $(3.8)_2$ and their derivatives into the boundary layer equation

$$w\frac{\partial w}{\partial z}+u\frac{\partial w}{\partial x}=\nu\frac{\partial^2 w}{\partial x^2}. \tag{6.5}$$

Having carried out the calculations and limiting ourselves to the plane $\xi=1$ we notice that all the terms balance each other except one

$$\Delta=WW', \tag{6.6}$$

which remains as the nonzero remainder. Taking the viscous term of Eq. (6.5) as a reference quantity we define the error of the approximation as

$$\varepsilon_0 = \frac{\Delta}{\nu \dfrac{\partial^2 w}{\partial x^2}} = \frac{W' \nu}{u_1^2}. \tag{6.7}$$

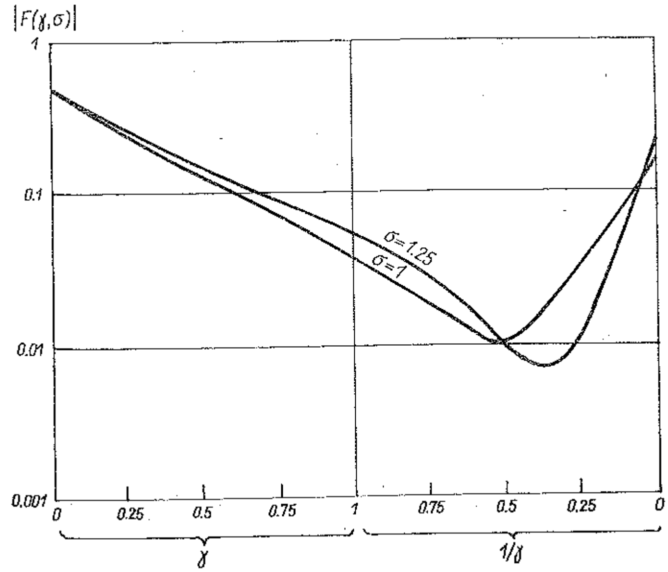

FIG. 9.

With the aid of Eqs. (5.3), (3.9) and (4.4)-(4.6) we transform Eq. (6.7) into the form

$$\varepsilon_0 = \frac{(\gamma^2-1)^2}{\beta \,\mathrm{Re}} \, \frac{e^{-\gamma\zeta}}{(\gamma e^{-\zeta} - e^{-\gamma\zeta})^2}. \tag{6.8}$$

It is seen from Eq. (6.8) that ε_0 tends to zero with $\zeta \to \infty$ provided $\gamma > 2$. In this respect we shall confine here our analysis to the initial plane $\zeta = 0$ where the magnitude of ε_0 is the highest. Hence the condition of the validity of the model reads

$$\varepsilon_0(0) = \frac{(\gamma+1)^2}{\beta \,\mathrm{Re}} \ll 1. \tag{6.9}$$

Comparing the conditions (6.4) and (6.9) we may notice some contradiction since the error in the inner flow increases with the factor β Re while in the outer flow the error decreases. This contradiction resulted from the fact that the inner flow modelling was related to the hydrodynamic theory of lubrication [8] based on the low Reynolds number approximation whereas the outer flow calculations originated from the boundary layer approach being the high Reynolds number approximation.

Nervertheless we can find the common range where both approximations are acceptable. From the combination of Eqs. (6.4) and (6.9) we can eliminate β Re and get the following relation:

$$\varepsilon_i\, \varepsilon_0 = (\gamma+1)^2\, |F(\gamma, \sigma)|. \tag{6.10}$$

Taking 10^{-2} for $|F(\gamma, \sigma)|$ (Fig. 9) and 10 for $(\gamma+1)^2$ and assuming equal errors of approximations for the inner and outer flow we obtain that $\varepsilon = \varepsilon_i = \varepsilon_0$ is of the order $(0.1)^{1/2}$. This value, which is a compromise between contradict requirements, seems to be rather high but we ought to have in mind that it is a local value which decreases with the distance from the plane $\zeta=0$. The factor β Re calculated from Eq. (6.4) or Eq. (6.9) is, in this case, about 30. We ought to remember that in the definition of Re (5.4) a half-width a is used as a characteristic length. However, the Reynolds number based on the streamwise characteristic length, generally used in the boundary layer analysis, can be much higher than Re.

In this paper we tried to obtain a simple analytical solution of the problem by means of approximate methods; in particular, of those based on the perturbation of the Poiseuille flow, repeated after our previous papers [5, 6]. Although we have found the range where the application of the model may be estimated as justified, the approximation is not quite satisactory. A more general model of a wider range of applicability should include in the analysis of the inner flow, inertial terms of the Navier-Stokes equations.

Appendix. Row of cylinders as a limiting case of a porous medium

It can be shown that a row of cylinders may be considered as a limiting case of a porous medium consisting of parallel cylindrical rods. Flow in the channel with porous walls of such a structure has been considered in our previous paper [6] and others, listed in [9].

A filtration velocity in this medium, which we shall call a volume bundle of rods, is determined by the Darcy law. After [6] it may be written in the form

$$U = -\frac{S}{\mu} F_{\perp}(\varphi) \frac{\partial p}{\partial x}, \tag{A.1}$$

where S is the cross-sectional area of the bundle pear one rod, and the nondimensional coefficient $F_{\perp}(\varphi)$ of the filtration normal to the rods is given, after [10] by the formula

$$F_{\perp} = \frac{1}{8\pi}\left(\ln\frac{1}{\varphi} - 1.5\right), \qquad \varphi \ll 1, \tag{A.2}$$

φ being a volume fraction of rods in the bundle.

It may be shown that in a limiting case of a one row bundle Eqs. (A.1) and (2.2) are equivalent. Taking one row of cylindrical rods as a bundle of a thickness l we have

$$S = l^2. \tag{A.3}$$

Approximating then a pressure gradient by a proper finite difference relation

(A.4) $$\frac{\partial p}{\partial x} \approx \frac{p_+ - p_-}{l}$$

and inserting Eqs. (A.3) and (A.4) into Eq. (A.1) we obtain

(A.5) $$U = -lF_\perp \frac{p_+ - p_-}{\mu}.$$

It is easy to notice that Eqs. (A.5) and (2.2) are identical, provided the coefficients $F_\perp$ (A.2) and k_x (2.2) are equivalent. To compare Eqs. (A.2) and (2.2) we express a volume fraction φ in terms of one row geometry with the aid of Eqs. (A.3) and (2.1)

(A.6) $$\varphi = \frac{\pi d^2}{4S} = \frac{\pi \delta^2}{4}$$

and insert (A.6) into (A.2). As a result, we obtain Eq. (A.7)

(A.7) $$F_\perp = \frac{1}{4\pi}\left(\ln \frac{1}{\pi\delta} + 0.515\right),$$

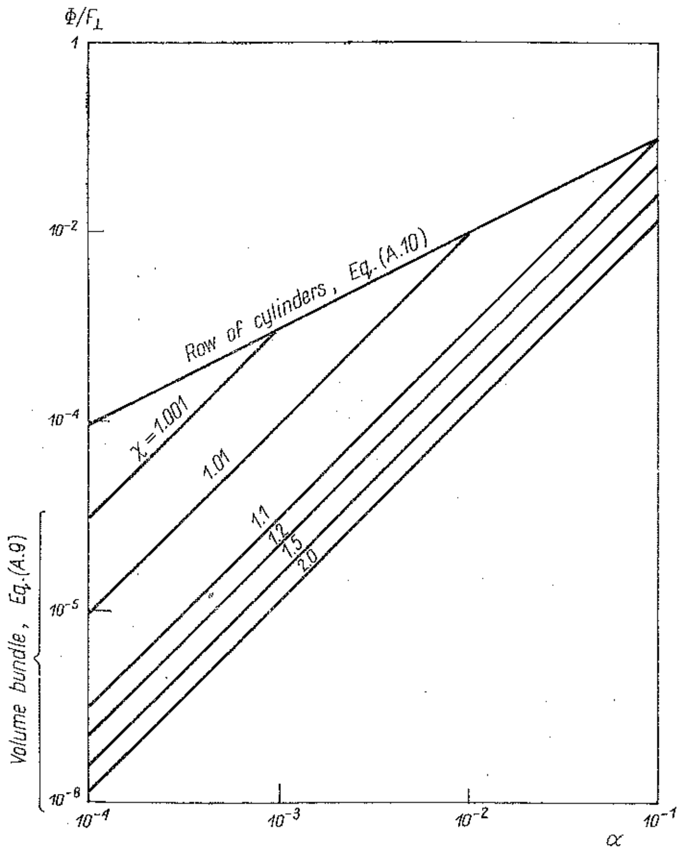

FIG. 10.

which only slightly differs from Eq. (2.2) in the numerical value of the additive constant. We can then take

$$F_{\perp} \approx k_x . \tag{A.8}$$

Now we can examine the influence of the thickness of the channel wall on its permeability. To this aim after [6] (Eqs. (2.7) and (3.3)) we introduce the nondimensional filtration coefficient of the volume bundle Φ_v, which with the aid of Eq. (A.3) takes the form

$$\Phi_v = a^2 F_{\perp} \frac{1}{\chi - 1}, \tag{A.9}$$

where χ is the ratio of the outer l and inner a width of the channel, $\chi = b/a$.

The filtration coefficient of the row of cylinders on the base of Eqs. (2.2) and (A.8) reads

$$\Phi_s = \frac{lk_x}{a} = \alpha F_{\perp} . \tag{A.10}$$

Equations (A.9) and (A.10) have been plotted against α in Fig. 10 for several values of χ. It is seen how permeability of the wall decreases with the growth of the wall thickness. For the limiting case when a volume bundle reduces to a single row cylinders, the lines corresponding to both systems intersect in the points $\alpha = \chi - 1$.

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STRESZCZENIE

PRZEPŁYW CIECZY LEPKIEJ PRZEZ UKŁAD PORUSZAJĄCYCH SIĘ PRĘTÓW

Przedmiotem pracy jest ustalony przepływ cieczy lepkiej nieściśliwej przez układ dwóch równoległych rzędów cylindrycznych prętów, poruszających się wzdłuż swoich osi. Rząd cylindrów potraktowany został jako ciągła powierzchnia, na której zarówno normalna jak styczna składowa prędkości mogą przyjmować wartości różne od zera. Obie powierzchnie potraktowano jako ruchome ścianki kanału równoległego. Przepływ wewnętrzny i zewnętrzny zostały wyznaczone za pomocą metody zaburzeń zastosowanej do odpowiednich ścisłych rozwiązań równań Naviera–Stokesa. Oba rozwiązania zostały skojarzone przy użyciu warunków zgodności na ściance kanału, rozdzielającej oba przepływy, dzięki czemu uzyskano ciągły opis przepływu w całym obszarze. Obliczono rozkłady ciśnień, prędkości i linii prądu. Oszacowano błąd zastosowanej aproksymacji.

Резюме

ТЕЧЕНИЕ ВЯЗКОЙ ЖИДКОСТИ ЧЕРЕЗ СИСТЕМУ ДВИЖУЩИХСЯ СТЕРЖНЕЙ

В настоящей работе представлено установившееся, ламинарное плоское течение несжимаемой жидкости через систему двух параллельных плоских решеток цилиндрических стержней, движущихся вдоль своих осей с данной скоростью. Решетка стержней рассматривается как плоская, подвижная, сплошная поверхность, на которой величины нормальной и касательной составляющих скорости отличны от нуля. Течение внутри и вне канала, сформированного этими поверхностями, вычислено методом возмущения соответствующих точных решений уравнений Навье–Стокса. Используя условия совместности на стенках канала, получено непрерывное решение в целой области течения. Вычислены распределения давления, скоростей и линий тока для внутреннего и внешнего течений. Обсуждены условия допустимости принятой аппроксимации.

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