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THE MOST FAVOURABLE SELECTION OF THE DISCRETE MODEL IN FEM

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A theorem and proof of the best, in the energetic sense, selection of a discrete model FEM for the linear elasticity problem were presented in the paper. It was proved that the best approximation is obtained if the ratio of the unit elastic strain energy to the density of nodes has a constant value. Proof of the theorem was verified on numerical examples.

1. Introduction

The material elastic body occupies a region $\mathscr{R}$ in n-dimensional Euclidean space ε^n $(n=1, 2, 3)$, approximated by the set of Cartesian coordinates $x=(x^1, ..., x^n)$. Under the term *discrete model*, a division of a region $\mathscr{R}$ on finite elements or topological location of nodes in the region is to be understood. Under specified conditions imposed on the base functions, as the element mesh is refined approximate, monotonics convergent solutions to the true solution are obtained. Proof of this theorem was quoted for instance in the papers [1, 2, 3]. In practice, we are forced to limit the division into a finite number of the elements. The question follows in what manner should the region $\mathscr{R}$ be divided into the number of elements specified in advance in order that this division be the best from all possible ones.

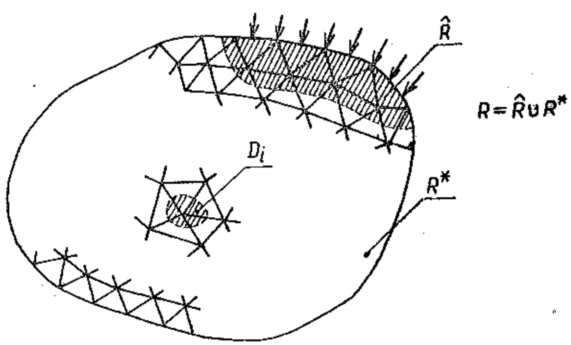


Fig. 1.

The theorem and the proof of the best (in the energetic sense) selection of a discrete model FEM for linear problems of the theory of elasticity were inserted in the paper under the following conditions of the problem:

a) the region $\mathscr{R}$ was divided into simple elements (linear base functions)

b) mechanical loads are limited to an active load at the same time; nodes in which static equivalents caused by this load occur are contained in the bounded subregion $\hat{\mathscr{R}} \subset \mathscr{R}$ (Fig. 1).

2. Theorem and proof

The region $\mathscr{R}$ was divided into simple elements connected in nodes x_i; $i=1, 2, \dots, E$. Then E subregions $\mathscr{D}_i$ were separated in this region so in order to every from theirs contained node i together with corresponding parts of elements adjacent to i-node (Fig. 1), at the same time the following relations are valid: $\mathscr{R}=\bigcup_{i=1}^{E} \mathscr{D}_i$, $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ for $i \neq j$.

Definition 1. *Let $\rho(x)$ be a scalar function in the region $\mathscr{R}$ and stand for the density of subregions $\mathscr{D}_i$ on the region $\mathscr{R}$. The function $\rho(x)$ can be interpreted as a node density in the region $\mathscr{R}$.*

The solution of the theory of the elasticity problem consists in searching a vector function $u(x)$, which belongs to the space $\Phi=\{\varphi_1(x), \dots, \varphi_E(x)\}$, where $\varphi_1(x), \dots$ $\dots, \varphi_E(x)$ are base functions. The space Φ is the Sobolev space $H^1(\mathscr{R})$ with the definite scalar product

$$\langle u(x), v(x) \rangle \equiv \int_{\mathscr{R}} u(x)^T v(x)\, d\mathscr{R} \tag{2.1}$$

and norm

$$\|u(x)\| \equiv \langle u(x)^T, u(x) \rangle^{\frac{1}{2}}. \tag{2.2}$$

The function

$$u(x) = u^i \varphi_i(x) \tag{2.3}$$

minimizes the functional of the form

$$J[u] = V[u] + L[u], \tag{2.4}$$

where $J[u]$, $V[u]$, $L[u]$ are potential energy, elastic strain energy and the variation of external forces energy respectively. For the given approximation (2.3), the basic equations of FEM have the form

$$\begin{aligned} \varepsilon(x) &= \mathscr{B}[u(x)] = \mathscr{B}[\varphi_i(x)]\, u^i = B_i(x)\, u^i, \\ \sigma(x) &= D(x)\, \varepsilon(x) = B_i(x)\, D(x)\, u^i, \end{aligned} \tag{2.5}$$

where $\varepsilon(x)$ — the strain vector, $\sigma(x)$ — the stress vector, $B_i(x)$ — the matrix of geometrical relations, $D(x)$ — the elasticity matrix.

Substituting Eqs. (2.5) into Eq. (2.4), we have

$$\begin{aligned} J[u] &= \tfrac{1}{2}\langle [\varepsilon(x)]^T, \sigma(x) \rangle - \langle [u(x)]^T, f(x) \rangle = \\ &= \tfrac{1}{2}(u^i)^T \langle [B_i(x)]^T, D(x)\, B_j(x) \rangle\, u^j - (u^i)^T \langle \varphi_i(x)^T, f(x) \rangle = \\ &= \tfrac{1}{2}(u^i)^T K_{ij}\, u^j - (u^i)^T f_i. \end{aligned} \tag{2.6}$$

Minimizing the functional $J[u]$, we obtain

$$\frac{\partial J[u]}{\partial u^i}=0 \Rightarrow K_{ij}u^j=f_i \Rightarrow u^j=N^{ji}f_i. \tag{2.7}$$

DEFINITION 2. *The finite dimensional space* $\Phi=\{\varphi_1(x), \varphi_2(x), ..., \varphi_E(x)\}\subset H^1(\mathcal{R})$ *together with the density function of the nodes* $\rho(x)$ *is considered here as a discrete model* $\mathcal{M}$; *this is written as* $\mathcal{M}=\{\Phi, \rho(x)\}$.

DEFINITION 3. *The class of the discrete models* $G_{\mathcal{M}}=\{\mathcal{M}=\{\Phi,\rho\}, \bar{\mathcal{M}}=\{\bar{\Phi},\bar{\rho}\}, \bar{\bar{\mathcal{M}}}=\{\bar{\bar{\Phi}},\bar{\bar{\rho}}\}, ...,\}$ *will be the set of models* $\mathcal{M}, \bar{\mathcal{M}}, \bar{\bar{\mathcal{M}}}, ...$ *with the following properties:*

1) *the space dimension* $\Phi, \bar{\Phi}, \bar{\bar{\Phi}}, ...$ *equal to* E

$$\Phi=\{\varphi_1(x), \varphi_2(x), .., \varphi_E(x)\},$$
$$\bar{\Phi}=\{\bar{\varphi}_1(x), \bar{\varphi}_2(x), ..., \bar{\varphi}_E(x)\},$$
$$\bar{\bar{\Phi}}=\{\bar{\bar{\varphi}}_1(x), \bar{\bar{\varphi}}_2(x), ..., \bar{\bar{\varphi}}_E(x)\}, ...$$

and

$$\int_{\mathcal{R}}\rho(x)\,d\mathcal{R}=\int_{\mathcal{R}}\bar{\rho}(x)\,d\mathcal{R}=\int_{\mathcal{R}}\bar{\bar{\rho}}(x)\,d\mathcal{R}=E;$$

2) *the system of the elements connections with nodes is identical in each of the models* $\mathcal{M}, \bar{\mathcal{M}}, \bar{\bar{\mathcal{M}}}, ..., \in G_{\mathcal{M}}$. *This means that every finite element and the node belonging to* $\mathcal{M}$ *has its own equivalent in the model* $\bar{\mathcal{M}}$, *etc.*;

3) $$\rho(x)=\bar{\rho}(x)=\bar{\bar{\rho}}(x)=... \quad \text{for} \quad x\in\hat{\mathcal{R}};$$

4) *the distance of the models* $\mathcal{M}, \bar{\mathcal{M}}\in G_{\mathcal{M}}$ *is*

$$d[\mathcal{M}, \bar{\mathcal{M}}]\equiv\|\rho(x)-\bar{\rho}(x)\|.$$

THEOREM. *In the class of models* $G_{\mathcal{M}}$ *the model* $\mathcal{M}\in G_{\mathcal{M}}$ *is the best one in the energetic sense if in the whole subregion* $\mathcal{R}^*$ *the ratio of the unit elastic energy and the nodes density* $\rho(x)$ *is constant.*

Proof. Let us assume that in the model $\mathcal{M}=\{\Phi, \rho(x)\}$ the space Φ has been chosen in such a manner that

$$V_i[u]=\text{const} \quad \text{for} \quad x_i\in\mathcal{R}^*, \tag{2.8}$$

where

$$V_i[u]=\int_{\mathcal{D}_i}[\varepsilon(x)]^T\sigma(x)\,d\mathcal{R}. \tag{2.9}$$

Equation (2.8) is an approximate realization of an assumption contained in the proof essence.

Let us next discuss another discrete model $\bar{\mathcal{M}}=\{\bar{\Phi}, \bar{\rho}(x)\}\in G_{\mathcal{M}}$ close in the sense of the norm (2.4) to the model $\mathcal{M}$ where

$$\begin{aligned}\bar{\Phi}&=\{\bar{\varphi}_1(x), \bar{\varphi}_2(x), ..., \bar{\varphi}_E(x)\},\\ \bar{\rho}(x)&=\rho(x)+\Delta\rho(x)\end{aligned} \tag{2.10}$$

at the same time according with the assumption

$$\|\Delta\rho(x)\| \ll \|\rho(x)\|. \tag{2.11}$$

For the new model $\bar{\mathcal{M}}$ we can write similar relations to those written for the model $\mathcal{M}$, so

$$\begin{aligned} \bar{u}(x) &= \bar{u}^i \bar{\varphi}_i(x),\\ \bar{\varepsilon}(x) &= \bar{B}_i(x)\bar{u}^i,\\ \bar{\sigma}(x) &= D(x)\bar{B}_i(x)\bar{u}^i,\\ J[\bar{u}] &= \tfrac{1}{2}(\bar{u}^i)^T \bar{K}_{ij}\bar{u}^j - (\bar{u}^i)^T \bar{f}_i,\\ \frac{\partial J[\bar{u}]}{\partial \bar{u}^i} &= 0 \Rightarrow \bar{u}^j = \bar{N}^{ji}\bar{f}_i. \end{aligned} \tag{2.12}$$

Assuming Eq. (2.11), one can assess the difference $J[\bar{u}]-J[u]$. Isolated parts of a discrete division of the region $\mathcal{R}$ in the i-node surroundings are depicted in Figs. 2a and 2b. The functions $\varphi_i(x)$ and $\bar{\varphi}_i(x)$ differ from each other as a result of a change of the nodes density in the i-node surroundings by the value $\rho_i=\rho(x_i)$.

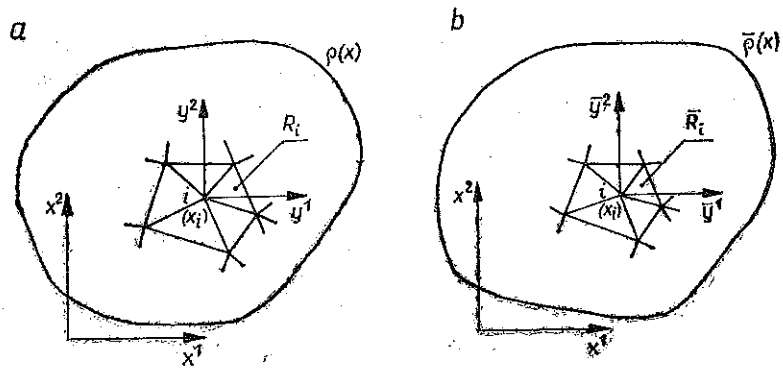


FIG. 2.

In view of the FEM properties the $\varphi_i(x)$ and $\bar{\varphi}_i(x)$ functions have nonzero values in the surroundings of the i-node only ($\mathcal{R}_i$ and $\bar{\mathcal{R}}_i$ regions in Fig. 2). If we assume that $\rho_i=\rho(x_i)$ and $\bar{\rho}_i=\rho(x_i)$ are mean values of the $\rho(x)$ and $\bar{\rho}(x)$ functions in the regions $\mathcal{R}_i$ and $\bar{\mathcal{R}}_i$, then

$$\int_{\mathcal{R}_i} d\mathcal{R} = \gamma_i \int_{\bar{\mathcal{R}}} d\mathcal{R}, \tag{2.13}$$

where $\gamma_i=\bar{\rho}_i/\rho_i$.

Introducing in the regions $\mathcal{R}_i$ and $\bar{\mathcal{R}}_i$ the local coordinate systems

$$y = x - x_i, \quad \bar{y} = \gamma_i^{1/n}(x - x_i), \tag{2.14}$$

we have

$$\varphi_i(x) = \varphi_i(y), \quad \bar{\varphi}_i(x) = \varphi_i(\bar{y}). \tag{2.15}$$

Since $\mathscr{B}\equiv\mathscr{B}_x$ is every time a differential operator of the first order, in that case

$$\mathscr{B}_x=\left[\frac{\partial}{\partial x}\right]=\left[\frac{\partial}{\partial y}\right]=\mathscr{B}_y,$$
(2.16)
$$\mathscr{B}_x=\left[\frac{\partial}{\partial x}\right]=\left[\frac{\partial\bar{y}}{\partial x}\right]\left[\frac{\partial}{\partial\bar{y}}\right]=\gamma^{1/n}\,\mathscr{B}_{\bar{y}}.$$

Availing oneself of the relations (2.16), we have

$$\bar{K}_{ij}=\int_{\bar{\mathscr{R}}_i}\{\mathscr{B}_x[\bar{\varphi}_i(x)]\}^T D(x)\,\mathscr{B}_x[\bar{\varphi}_j(x)]\,dx^1\dots dx^n= \tag{2.17}$$
$$=\gamma_{(i)}^{2/n}\int_{\bar{\bar{\mathscr{R}}}_i}\{\mathscr{B}_{\bar{y}}[\varphi_i(\bar{y})]\}^T D(x)\,\mathscr{B}_{\bar{y}}[\varphi_j(\bar{y})]\,\gamma_{(i)}^{-1}\,d\bar{y}^1\dots d\bar{y}^n.$$

If in the last expression we perform a formal substitution $\bar{y}\to y$, then $\bar{\mathscr{R}}_i\to\mathscr{R}_i$, we have

$$\bar{K}_{ij}=\gamma_{(i)}^{2/n-1}K_{ij},\qquad \bar{N}^{ji}\cong\gamma_{(i)}^{1-2/n}N^{ji}. \tag{2.18}$$

Since $\Delta\rho(x)=0$ for $x\in\hat{\mathscr{R}}$, in that case

$$\bar{f}_i=f_i. \tag{2.19}$$

Availing oneself of the relation (2.19), we have from Eq. (2.12)

$$\bar{u}^j=\bar{N}^{ji}\bar{f}_i\cong\gamma_{(i)}^{1-2/n}N^{ji}f_i. \tag{2.20}$$

Since $f_i=0$ for $x_i\in\mathscr{R}^*$, while $\gamma_i=1$ for $x_i\in\hat{\mathscr{R}}$, then

$$\bar{u}^i\cong u^i. \tag{2.21}$$

Evaluating the functional $J[\bar{u}]$ as a sum over regions lying in the vicinity of the node $\mathscr{R}=\bigcup_{i=1}^{E}\mathscr{D}_i$, we have

$$J[\bar{u}]=\frac{1}{2}\sum_{i=1}^{E}(\bar{u}^i)^T\int_{\bar{\mathscr{D}}_i}\overline{dK_{ij}}\,\bar{u}^j-\bar{u}^i\bar{f}_i. \tag{2.22}$$

Availing oneself of the relations (2.21), (2.17) and (2.18) we have in continuation

$$J[\bar{u}]=\frac{1}{2}\sum_{i=1}^{E}(u^i)^T\gamma_i^{2/n-1}\int_{\mathscr{D}}dK_{ij}\,u^j-u^i f_i. \tag{2.23}$$

Expanding the function $\gamma_i^{2/n-1}=(1+\Delta\rho_i/\rho_i)^{2/n-1}$ in the Taylor series in relation to the increment $\Delta\rho_i$ in the surroundings $\Delta\rho_i=0$, we have

$$\gamma_i^{2/n-1}=1+\left(\frac{2}{n}-1\right)\frac{\Delta\rho_i}{\rho_i}+\frac{1}{2}\left(\frac{2}{n}-1\right)\left(\frac{2}{n}-2\right)\left(\frac{\Delta\rho_i}{\rho_i}\right)^2+O(\Delta\rho_i^3). \tag{2.24}$$

Hence

$$J[\bar{u}]=\sum_{i=1}^{E}V_i-L+\frac{1}{2}\left(\frac{2}{n}-1\right)\sum_{i=1}^{E}\frac{\Delta\rho_i}{\rho_i}V_i+ \tag{2.25}$$
$$+\frac{1}{4}\left(\frac{2}{n}-1\right)\left(\frac{2}{n}-2\right)\sum_{i=1}^{E}\left(\frac{\Delta\rho_i}{\rho_i}\right)^2V_i+\dots.$$

Availing oneself of the relations

$$\sum_{i=1}^{E} \frac{\Delta \rho_i}{\rho_i} \approx \int_{\mathcal{R}} \Delta \rho(x)\, d\mathcal{R},$$

$$V_i[u]=\text{const},$$

we have

$$(2.26)\quad J[\bar{u}]=J[u]+\frac{1}{2}\left(\frac{2}{n}-1\right)V_i[u]\int_{\mathcal{R}} \Delta\rho(x)\, d\mathcal{R}+ \\ +\frac{1}{4}\left(\frac{2}{n}-1\right)\left(\frac{2}{n}-2\right)\sum_{i=1}^{E}\left(\frac{\Delta\rho_i}{\rho_i}\right)^2 V_i[u]+\ldots .$$

Treating the second term from the components as the variation δJ, while the third as $\delta^2 J$ in view of a variation of the function $\rho(x)$, we have

$$(2.27)\quad \begin{aligned} \delta J&=\frac{1}{2}\left(\frac{2}{n}-1\right)V_i[u]\int_{\mathcal{R}} \Delta\rho(x)\, d\mathcal{R}=0, \\ \delta^2 J&=\frac{1}{4}\left(\frac{2}{n}-1\right)\left(\frac{2}{n}-2\right)\sum_{i=1}^{E}\left(\frac{\Delta\rho_i}{\rho_i}\right)^2 V_i[u]\geqslant 0. \end{aligned}$$

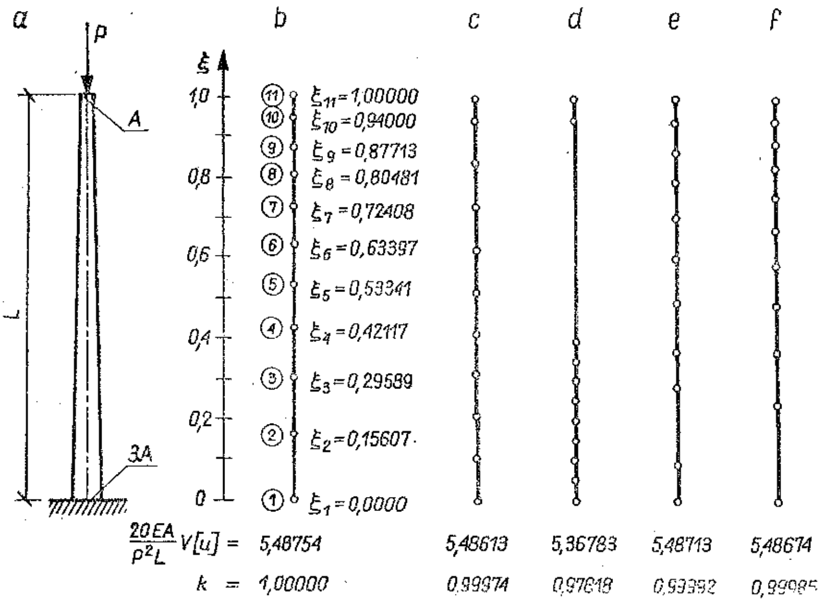
for any increment $\Delta\rho(x)$. Hence the conclusion that the functional $J[u]$ assumes the stationary value and for three-dimensional problems assumes the minimum value.

3. Numerical verification

Pertinence of the theorem already presented was confirmed numerically on some numerical examples for one, two and three-dimensional problems. The optimum model was searched for in the iterative way. For the given space $\mathcal{R}$ and the nodes number E the class of models $G_{\mathcal{M}}$ was assumed. Through the change of the nodes location in the subspace $\mathcal{R}^*$ the optimum model was obtained, for which V_i=const for $x_i \in \mathcal{R}^*$.

Example 1. *One-dimensional problem*

A simple bar of the linear variation of the cross-section area is located by the concentrated force (Fig. 3a). The space was divided into 10 finite elements. The subspace $\hat{\mathcal{R}}$ is comprised in the space of the first upper elements; therefore the optimum model was looked for in the class of models for which location of the nodes 10 and 11 was fixed. Figure 3b presents an optimum discrete model. Figures 3c–3f present an other models for the purpose of comparison. The value of the total elastic energy was determined for every model as well as the comparative coefficient $k=V[u]/V^b[u]$.


a
b
c
d
e
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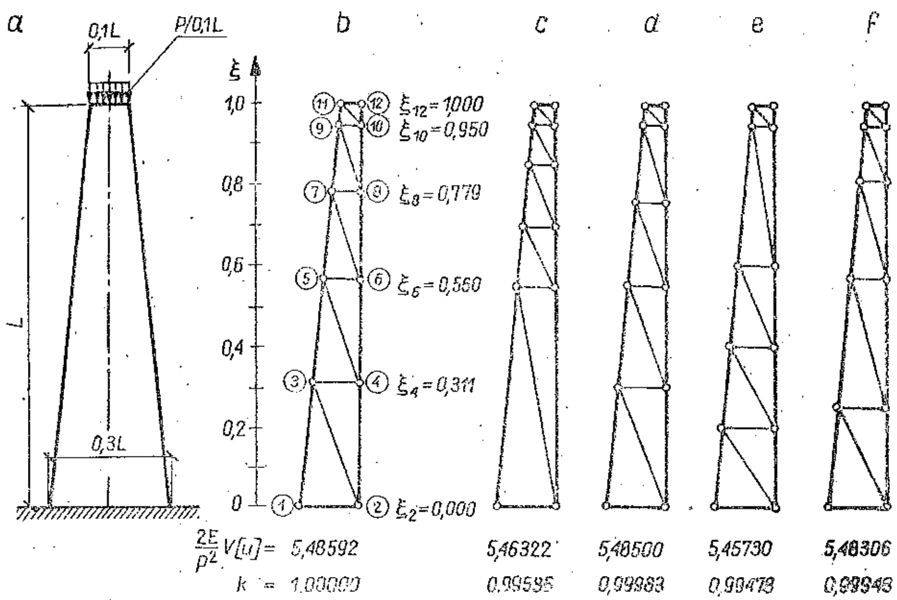
P
A
L
3A
ξ
1,0
0,8
0,6
0,4
0,2
0
ξ11=1,00000
ξ10=0,94000
ξ9=0,87713
ξ8=0,80481
ξ7=0,72408
ξ6=0,63397
ξ5=0,53341
ξ4=0,42117
ξ3=0,29589
ξ2=0,15607
ξ1=0,0000
20EA/P²L V[u] = 5,48754 5,48613 5,36783 5,48713 5,48674
k = 1,00000 0,99974 0,97618 0,99992 0,99985

FIG. 3.


a
b
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f
0,1L
P/0,1L
L
0,3L
ξ
1,0
0,8
0,6
0,4
0,2
0
ξ12=1,000
ξ10=0,950
ξ8=0,779
ξ6=0,560
ξ4=0,311
ξ2=0,000
2E/P² V[u]= 5,48592 5,46322 5,48500 5,45730 5,48306
k = 1,00000 0,99585 0,99983 0,99478 0,99948

FIG. 4.

Example 2. *Two-dimensional problem.*

The bar from example 1 is treated here as a disk with the unit thickness where $A = L/10$ and $\nu = 0.3$ (Fig. 4a). Thanks to the symmetry, half of the disk was divided into 10 triangular elements. An optimum model was searched for in the class of models $G'_{\mathcal{M}} \subset G_{\mathcal{M}}$ for which the location of the 4 upper nodes is fixed, while the rest of nodes exists in pairs on the same level. The results are given in Fig. 4, similarly to the first example.

Example 3. *Three-dimensional problem.*

The bar from example 1 is treated here as a three-dimensional body of the scheme and the load as in Fig. 5, for $\nu = 0.3$. Taking advantage of two symmetry planes, the space was divided into 20 tetrahedral elements. The optimum model was sear-

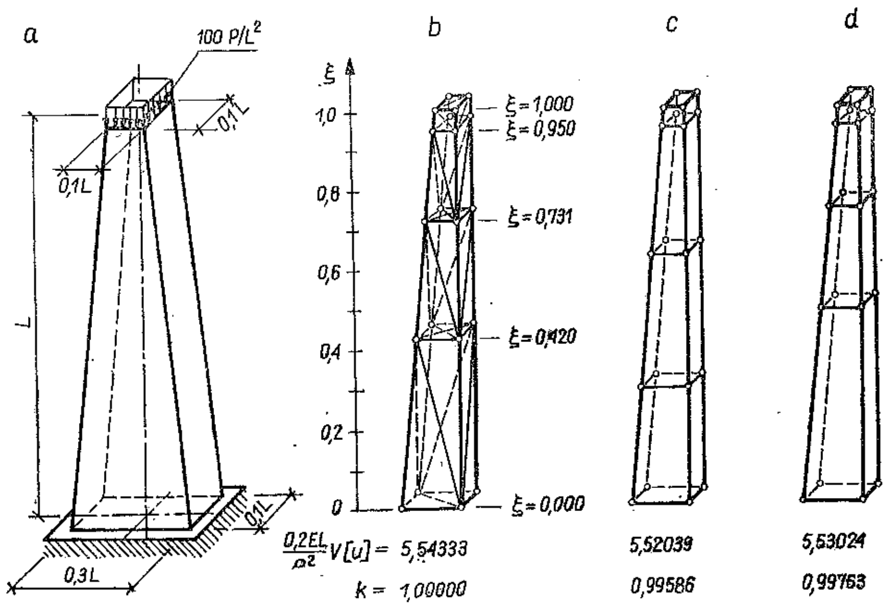

FIG. 5.

ched for in the class of models $G'_{\mathcal{M}} \subset G_{\mathcal{M}}$, for which the location of the 8 upper nodes is fixed, while the rest of the nodes were situated in fours at each level. The results are presented in Fig. 5 in the same way as for examples 1 and 2.

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STRESZCZENIE

OPTYMALNY DOBÓR MODELU DYSKRETNEGO W METODZIE ELEMENTÓW SKOŃCZONYCH

Zamieszczono twierdzenie i dowód najlepszego w sensie energetycznym doboru modelu dyskretnego MES dla zadań liniowej teorii sprężystości. Wykazano, że najlepszą aproksymację otrzymuje się, jeżeli stosunek jednostkowej energii sprężystej do gęstości węzłów ma wartość stałą. Dowód twierdzenia został zweryfikowany na wybranych przykładach liczbowych.

Резюме

ОПТИМАЛЬНЫЙ ПОДБОР ДИСКРЕТНОЙ МОДЕЛИ В МЕТОДЕ КОНЕЧНЫХ ЭЛЕМЕНТОВ

В работе помещены теорема и доказательство наилучшего подбора, в энергетическом смысле, дискретной модели МКЭ для задач линейной теории упругости. Показано, что наилучшую аппроксимацию получается, если отношение единичной упругой-энергии к полотности узлов имеет постоянное значение. Доказательство теоремы проверено на избранных числовых примерах.

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